Math, Fairness and Social Choice

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Math, Fairness and Social Choice

- **What is the fairest way to conduct an election with multiple candidates?**
- **How can we measure voting power in a system like the Electoral College or UN Security Council?**
- **Is it possible to share indivisible resources fairly?**
- **How can mathematics help answer these questions?**

Part 1: Multi-Candidate Elections

What is the fairest way to conduct an election with multiple candidates?

Election Methods

Standard "plurality" voting: Each voter picks one candidate. The candidate who receives the most votes wins.

Problem: What if "the most" is 20% of the votes? (Or 2%?) How can we be sure that the plurality choice represents the will of the electorate?

Runoff voting: Each voter picks one candidate. The candidate who receives a majority wins. If no candidate receives a majority, the top two vote-getters face each other in another election.

Problem: Inefficient! Requires two separate elections.

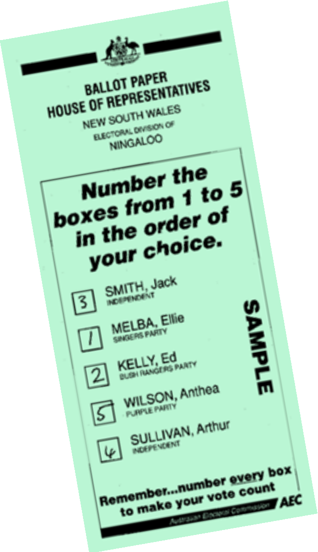
Election Methods: Instant Runoff

Instant runoff (A): Each voter ranks the candidates from most to least favored. The candidate who receives a majority of the first-place votes wins. If no candidate receives a majority, the top two vote-getters are compared head-to-head using the rankings.

Instant runoff (B): Each voter ranks the candidates from most to least favored. The candidate who receives the fewest first-place votes is crossed off all the ballots. This is repeated until some candidate has a majority of first-place votes.

Problem: Complicated! (But that may be an *advantage*. . .)

Election Methods: Instant Runoff



More Election Methods

Point system ("Borda"): Each voter awards 1 point to their favorite candidate, 2 points for their second, ... n points to the nth candidate. Fewest points wins.

Round robin ("Condorcet"): Each voter ranks the candidates from most to least favored. Compare each pair of candidates head-to-head. The candidate who wins the most pairwise comparisons wins the election.

Approval voting: Each voter votes "Yes" or "No" on each candidate. The most "Yes" votes wins.

How can math help decide which method is fairest?

Fairness criteria

What makes an election method "fair"? Some possibilities:

#1. *If* **all** *voters prefer candidate* **A** *to candidate* **B**, *then the method should rank* **A** *higher than* **B**.

(A very mild requirement — all these methods satisfy it.)

#2. *If* **most** *voters prefer candidate* **A** *to candidate* **B**, *then the method should rank* **A** *higher than* **B**.

(A very strong requirement — probably too strong.)

More fairness criteria

#3. *If candidate* **A** *receives a majority of the first-place votes, then the system should declare* **A** *the winner.*

#4. *If one or more voters change their minds and rank* **A** *higher, then* **A***'s final ranking should not be lower.*

#5. *If candidate* **C** *drops out, then the relative order of* **A** *and* **B** *should not change.*

More fairness criteria

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Arrow's Theorem:
No election method can *always* satisfy these three criteria.

In other words, given any election method, there is a potential scenario in which least one of those criteria is violated.

Fairness criteria

Contrast **opinion statements** (open to debate) . . .

- "Criterion X is more important than criterion Y"
- "Election method M is better than method N"

. . . with **mathematical statements** (inarguable facts)

- "Method M satisfies criterion X"
- "No method can always satisfy criteria X, Y, and Z"

Strategic voting

A different kind of fairness criterion:

#6. *Voters should not be able to "game the system" by voting strategically ("insincerely").*

- Mathematics takes no position on whether strategic voting is immoral/unethical/improper.
- However, the possibility of strategic voting means that the election method is not doing its job.

The 2000 US Presidential Election

Candidate	Popular Vote (US)	Popular Vote (FL)	Electoral Votes
George W. Bush	47.87%	48.847%	271
Al Gore	48.38%	48.838%	266
Ralph Nader	2.74%	1.635%	0
All others	1.01%	0.680%	0

"Most Nader supporters probably preferred Gore to Bush. If they had voted for Gore, then Gore might have won Florida."

"Some Nader supporters probably *did* vote for Gore. If they had voted sincerely, Bush might have won Florida easily."

Strategic Voting

A good election method should be immune to insincere voting (i.e., each voter should be best served by being honest).

The Gibbard-Satterthwaite Theorem:

No election method is perfectly immune to manipulation.

On the other hand, the more complicated the system, the harder it may be to figure out how to game it effectively. (This is certainly *not* a mathematical argument!)

Part 2: Weighted Voting Systems

How can we accurately measure voting power?

Weighted Voting Systems

Electoral College: each state gets a number of votes proportional to its population (more or less)

Parliamentary systems with multiple parties (Canada, Great Britain, Italy, Israel, ...)

Stockholder votes: number of shares = number of votes ("51% control")

The fraction of votes controlled is generally NOT a good measure of power exerted.

Nassau County, NY Board of Supervisors, 1964

Each district of Nassau County elects one supervisor, who controls a number of votes proportional to the number of voters in the district.

District	**Votes**
Hempstead #1	31
Hempstead #2	31
Oyster Bay	28
North Hempstead	21
Long Beach	2
Glen Cove	2
Total	**115**

Why is this unfair?

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Any two of the three biggest districts together control a majority of the votes.

The other three districts **have zero real power!**

The Penrose-Banzhaf-Coleman Power Index

One way to measure power mathematically:

- List all **winning coalitions** (blocs of voters who collectively control enough votes to pass a motion).
- For each winning coalition, identify which voters are **critical** for the coalition's success. Award each critical voter one "point."

$$\boxed{\text{Power of voter V} = \frac{\text{number of points scored by V}}{\text{total number of points scored}}}$$

The Penrose-Banzhaf-Coleman Power Index

Example: A has 3 votes, B has 2 votes, C has two votes.

Winning coalitions: AB, AC, BC, ABC

Voter	# votes	Critical	Points	Power Index
A	3	AB, AC	2	33.33%
B	2	AB, BC	2	33.33%
C	2	AC, BC	2	33.33%
		Total	6	

The Penrose-Banzhaf-Coleman Power Index

Example: A has 5 votes, B has 2 votes, C has two votes.

Winning coalitions: A, AB, AC, ABC

Voter	# votes	Critical	Points	Power Index
A	5	A, AB, AC, ABC	4	100%
B	2	None	2	0%
C	2	None	2	0%
		Total	4	

The Penrose-Banzhaf-Coleman Power Index

- This measure of power detects when a voter is a "dummy" (has no real power) and when two voters with unequal numbers of votes actually have equal power.
- Works equally well for systems in which if passing a motion requires a supermajority (e.g., 60% or 75%)
- How reliable are these numbers? There are other ways of measuring power that produce different numbers, but are equally good at detecting dummies and equality.

Power at the UN

The UN Security Council consists of

- 5 permanent members with veto power (US, Russia, China, France, Great Britain)
- 10 rotating members (currently Argentina, Australia, Chad, Chile, Jordan, Lithuania, Luxembourg, Nigeria, Rwanda, South Korea).

Passing a motion requires 9 votes including all permanent members.

How much is veto power worth?

Example: The UN Security Council

Observations:

- We don't need to assign numerical weights — all we need to do is figure out which coalitions are winning and which ones are losing.
- Each winning coalition must contain all five permanent members, along with at least four rotating members.
- All permanent members have equal power.
- All rotating members have equal power.

Power at the UN

Points for each permanent member $= 848$	$=$	# of coalitions including all five permanent members and **at least** four rotating members
Points for each rotating member $= 84$	$=$	number of coalitions including all five permanent members and **exactly three other** rotating members

Total number of points $= 5 \times 848 + 10 \times 84 = 5080$

Power at the UN

5 permanent members, each with critical count 848
10 rotating members, each with critical count 84

Banzhaf power index of each permanent member:

$$\frac{848}{(5 \times 848) + (10 \times 84)} = \frac{848}{5080} \approx 0.1669 = \mathbf{16.69\%}$$

Banzhaf power index of each rotating member:

$$\frac{84}{(5 \times 848) + (10 \times 84)} = \frac{84}{5080} \approx 0.0165 = \mathbf{1.65\%}$$

Part 3: Fair Division

Is it possible to share indivisible resources fairly?

Fair Division

Four squabbling siblings (Pauline, Quentin, Roberta, and Severus) are joint heirs to an estate consisting of a castle in Spain, a 75-foot sailboat, and a replica of the Statue of Liberty made entirely from chocolate.

How can they divide the estate so that each person gets what s/he considers to be a fair share?

Fair-division methods exploit sharers' different valuations, reward honesty, and typically produce a surplus.

Fair Division by Sealed Bids

1. Each person submits a sealed bid on each item.
   Bids are revealed simultaneously.
2. Highest bidder on each item gets to purchase it at $3/4$ of their bid.
3. Everyone else receives $1/4$ of their bid in cash.
4. Leftover cash is then divided equally.

The system **rewards honesty**. You don't know whether you are buying or selling, so in order to guarantee getting a fair share (in your own mind), you must bid exactly what you think it is worth!

Thanks for listening! Please ask lots of questions!

Extra Slide 1: Shapley-Shubik Power Index

Idea: Regard coalitions as groups that voters join **one at a time** ("sequential coalitions" / "sequences").

Example: Player A has 2 votes; Players B and C have 1 vote each.

Sequence	Pivotal player
A, B, C	B
A, C, B	C
B, A, C	A
B, C, A	A
C, A, B	A
C, B, A	A

Player	SSPI
A	$4/6 \approx 67\%$
B	$1/6 \approx 17\%$
C	$1/6 \approx 17\%$