Amazing Patterns in the Game of Nim

Jeremy Martin
University of Kansas

Inspired by Math
October 25, 2016

The Game of Nim

Nim is a game for two players, taking turns.
Start with a bunch of piles of “chips” of different colors.

The Game of Nim

Nim is a game for two players, taking turns.
Start with a bunch of piles of “chips” of different colors.

The Game of Nim

Nim is a game for two players, taking turns.
Start with a bunch of piles of “chips” of different colors.

On your turn, you can remove as many chips as you like of any one color.

The Game of Nim

Nim is a game for two players, taking turns.
Start with a bunch of piles of “chips” of different colors.

On your turn, you can remove as many chips as you like of any one color.

like this:

The Game of Nim

Nim is a game for two players, taking turns.
Start with a bunch of piles of “chips” of different colors.

On your turn, you can remove as many chips as you like of any one color.

like this:

or this:

The Game of Nim

Nim is a game for two players, taking turns.
Start with a bunch of piles of "chips" of different colors.

On your turn, you can remove as many chips as you like of any one color.

like this:

---

or this:

---

or this:

The Game of Nim

Now it's your opponent's turn to pick a color and remove as many chips as she likes of that color (again, at least one).

The Game of Nim

Now it's your opponent's turn to pick a color and remove as many chips as she likes of that color (again, at least one).

Then it's your turn again.

The Game of Nim

Now it's your opponent's turn to pick a color and remove as many chips as she likes of that color (again, at least one).

Then it's your turn again.

The goal is to **take the last chip**. Whoever does that **wins the game**.

The Game of Nim

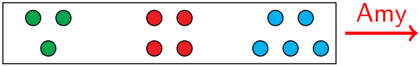
Now it's your opponent's turn to pick a color and remove as many chips as she likes of that color (again, at least one).

Then it's your turn again.

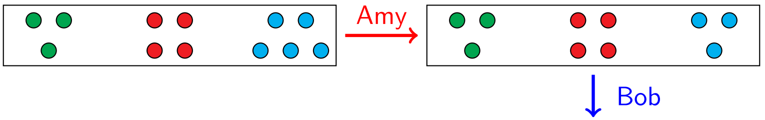
The goal is to **take the last chip**. Whoever does that **wins the game**.

Passing isn't allowed (why not?)

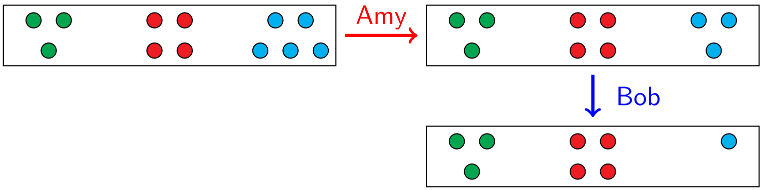
A Sample Game



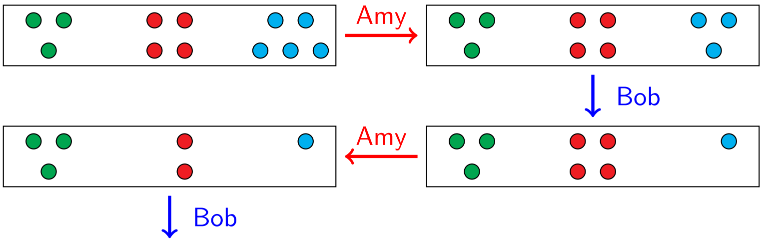
A Sample Game



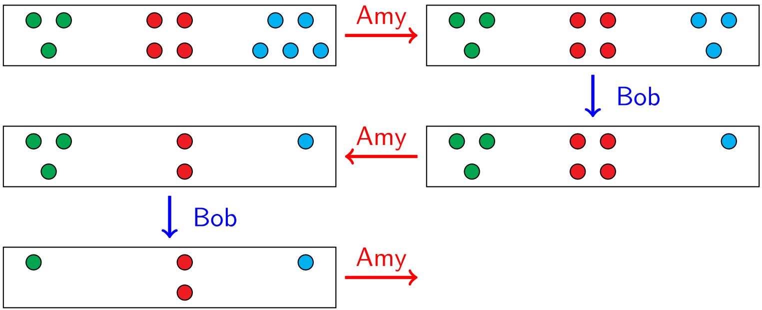
A Sample Game



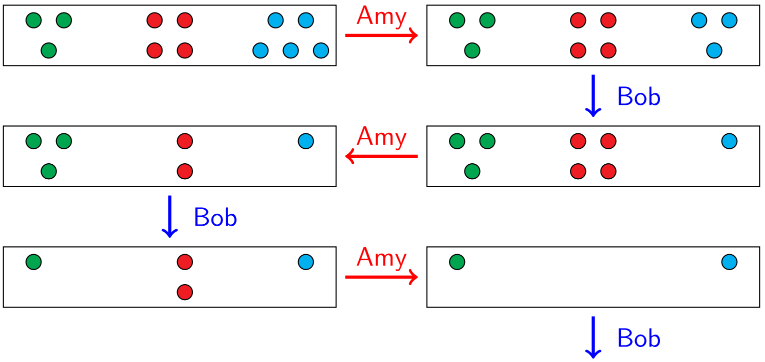
A Sample Game



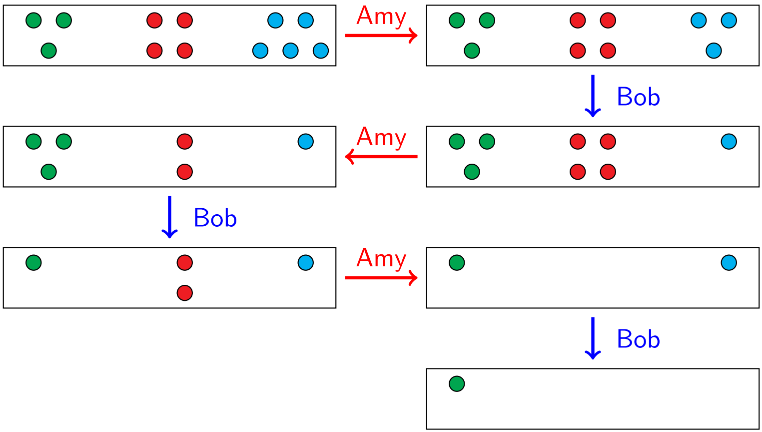
A Sample Game



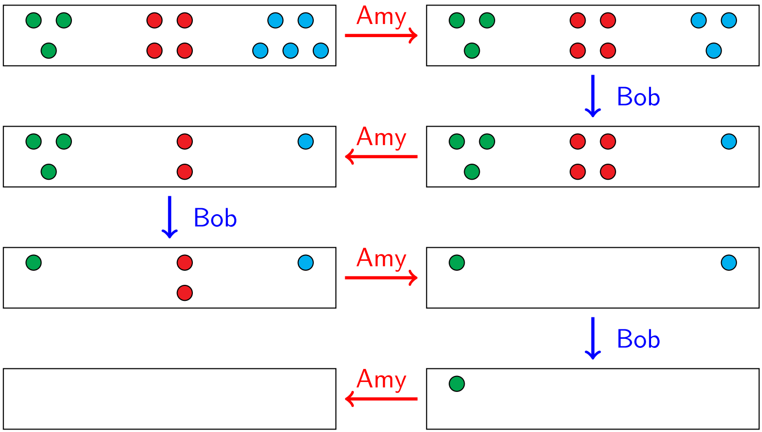
A Sample Game



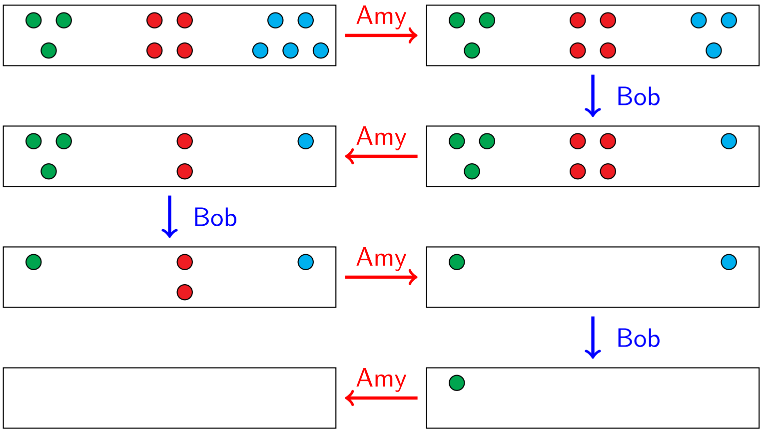
A Sample Game



A Sample Game



A Sample Game



Amy wins!

What's the Best Strategy?

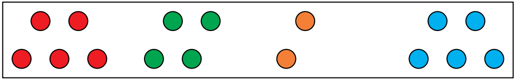
First, let's be more efficient about how we describe Nim positions.

We don't need to keep drawing pictures of colored dots: all we need to know is how many chips there are of each color.

What's the Best Strategy?

First, let's be more efficient about how we describe Nim positions.

We don't need to keep drawing pictures of colored dots: all we need to know is how many chips there are of each color.

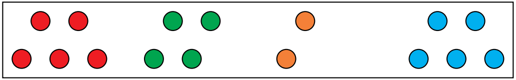


$= \boxed{5\ 4\ 2\ 5}$

What's the Best Strategy?

First, let's be more efficient about how we describe Nim positions.

We don't need to keep drawing pictures of colored dots: all we need to know is how many chips there are of each color.



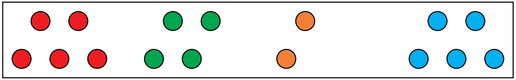
$= \boxed{5\ 4\ 2\ 5}$

In fact, we don't need to distinguish between colors.

What's the Best Strategy?

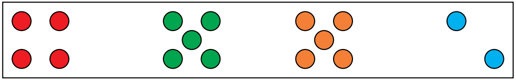
First, let's be more efficient about how we describe Nim positions.

We don't need to keep drawing pictures of colored dots: all we need to know is how many chips there are of each color.



$= \boxed{5\ 4\ 2\ 5}$

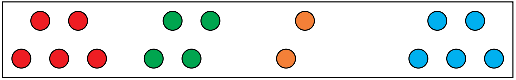
In fact, we don't need to distinguish between colors.



What's the Best Strategy?

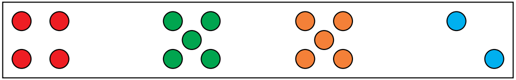
First, let's be more efficient about how we describe Nim positions.

We don't need to keep drawing pictures of colored dots: all we need to know is how many chips there are of each color.



$= \boxed{5\ 4\ 2\ 5}$

In fact, we don't need to distinguish between colors.

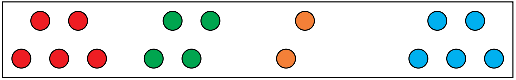


$= \boxed{4\ 5\ 5\ 2}$

What's the Best Strategy?

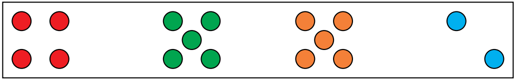
First, let's be more efficient about how we describe Nim positions.

We don't need to keep drawing pictures of colored dots: all we need to know is how many chips there are of each color.



$=$ $\boxed{5\ 4\ 2\ 5}$

In fact, we don't need to distinguish between colors.

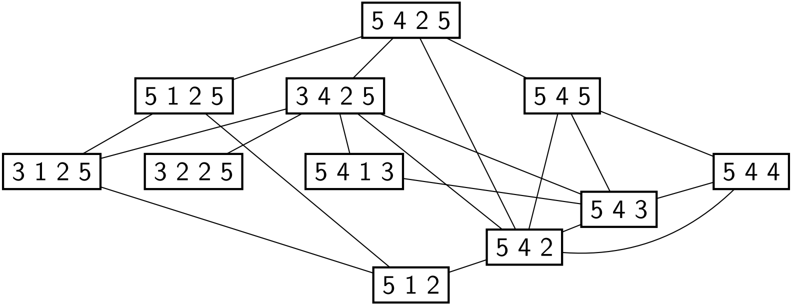


$=$ $\boxed{4\ 5\ 5\ 2}$

$=$ $\boxed{5\ 5\ 4\ 2}$

Game Trees

The game tree keeps track of all the possible moves.



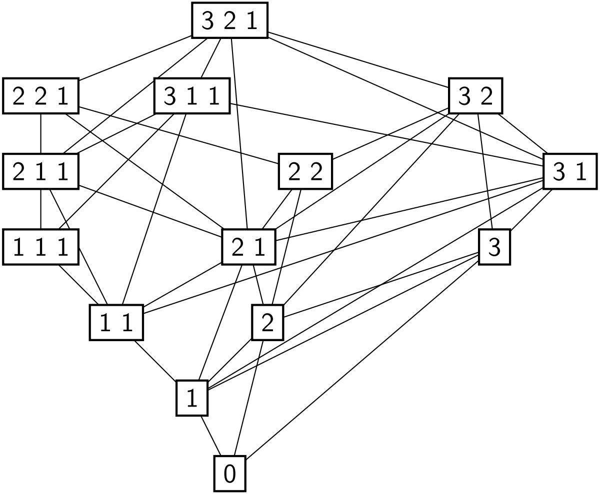
(Maybe “tree” isn’t the best word, since the branches can meet up with each other. . .)

Game Trees

If you know the entire game tree, then you can calculate all possible variations and find the best move.

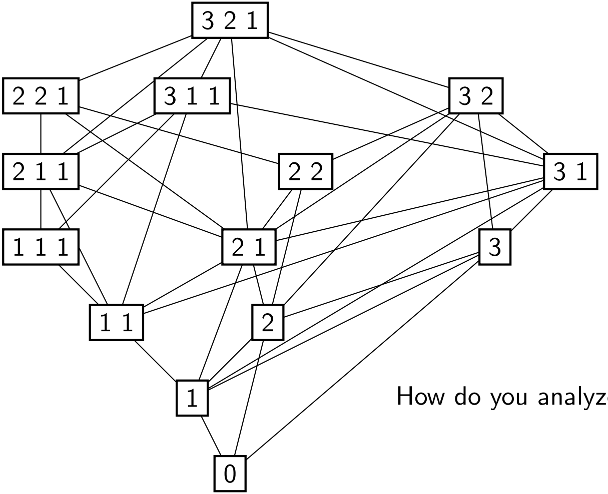
Game Trees

If you know the entire game tree, then you can calculate all possible variations and find the best move.



Game Trees

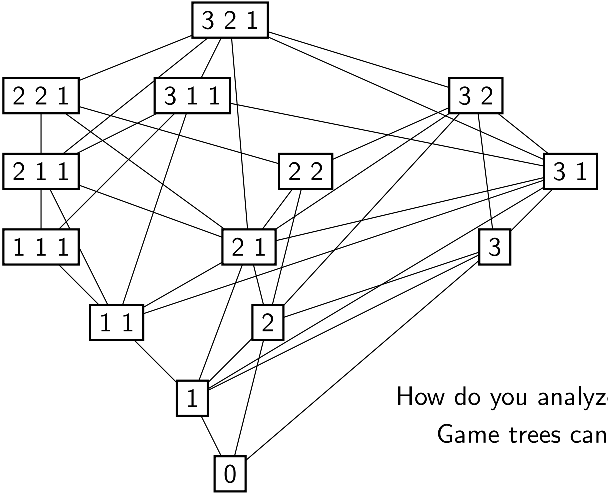
If you know the entire game tree, then you can calculate all possible variations and find the best move.



How do you analyze this thing?

Game Trees

If you know the entire game tree, then you can calculate all possible variations and find the best move.



How do you analyze this thing?

Game trees can get **big**.

Amazing Pattern #1

Starting position	Size of game tree
0	1
1	2
1 1	3
1 1 1	4
2	3
2 2	6
2 2 2	10
3	4
3 3	10
3 3 3	20
3 3 3 3	35

What are these numbers anyway?

Amazing Pattern #1

Starting position	**Size of game tree**
0	1
1	2
2	3
3	4
4	5

Amazing Pattern #1

Starting position	**Size of game tree**
0	1
1	2
2	3
3	4
4	5

Amazing Pattern #1

Starting position	Size of game tree
0	1
1	2
11 2	3
22 3	4
33 4	5

Amazing Pattern #1

Starting position

0				
	1			
11		2		
	22		3	
		33		4

Size of game tree

1				
	2			
3		3		
	6		4	
		10		5

Amazing Pattern #1

Starting position

	0				
0		1			
	11		2		
		22		3	
			33		4

Size of game tree

	1				
1		2			
	3		3		
		6		4	
			10		5

Amazing Pattern #1

Starting position

		0					
	0		0				
0		1		0			
	11		2		0		
		22		3		0	
			33		4		0

Size of game tree

		1					
	1		1				
1		2		1			
	3		3		1		
		6		4		1	
			10		5		1

Amazing Pattern #1

Starting position

			0						
		0		0					
	0		1		0				
0		11		2		0			
	111		22		3		0		
		222		33		4		0	

Size of game tree

			1						
		1		1					
	1		2		1				
1		3		3		1			
	4		6		4		1		
		10		10		5		1	

Amazing Pattern #1

Starting position

0

0 0

0 1 0

0 11 2 0

0 111 22 3 0

1111 222 33 4 0

Size of game tree

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

5 10 10 5 1

Amazing Pattern #1

Starting position

0

0 0

0 1 0

0 11 2 0

0 111 22 3 0

0 1111 222 33 4 0

Size of game tree

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Amazing Pattern #2

Starting position	**Size of game tree**
0	1
1	2
2 1	5
3 2 1	14
4 3 2 1	42
5 4 3 2 1	132
$\vdots$	$\vdots$
10 9 8 7 6 5 4 3 2 1	16796
$\vdots$	$\vdots$

(To find out what is so amazing about these numbers, come to Dr. Jennifer Wagner's Inspired by Math talk this coming spring!)

Game Trees Can Get Really Big

For a game like chess, where each player might have 20 different available moves on his or her turn, the number of different things that could happen over the next 6 moves is

$$20 \times 20 \times 20 \times 20 \times 20 \times 20 \;=\; 20^6 \;=\; 64{,}000{,}000.$$

For a game that lasts 30 moves (which is a very short game!), the number of possibilities would be

$$20^{30} \;=\; 1{,}073{,}741{,}824{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000.$$

Just making a list of all these games would require about a million million million Internets (give or take).[1]

---

[1] Source: http://www.sciencefocus.com/qa/how-many-terabytes-data-are-internet. This article was published in 2013. Maybe in 2016 it would only take a thousand million million Internets.

What's the Best Strategy?

Trying to understand the game tree for Nim looks utterly hopeless.

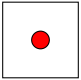
But there is good news: You can play Nim perfectly without ever thinking about the game tree!

The key is to start small. Let's look at some really simple games of Nim and work our way up to more complicated ones.

In all of these games, **Amy** will go first and **Bob** will go second.

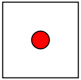
Simple Nim Game #1

How about the starting position **1**?



Simple Nim Game #1

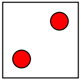
How about the starting position $\mathbf{1}$?



Here Amy has only one move — and it's a winning one.

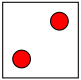
Simple Nim Game #2

How about the starting position **2**?



Simple Nim Game #2

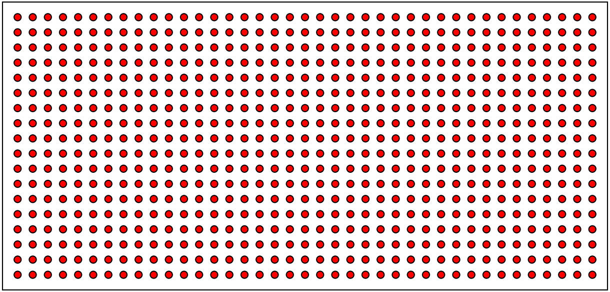
How about the starting position **2**?



Now Amy has two moves. One is bad and one is good. With best play, she can win.

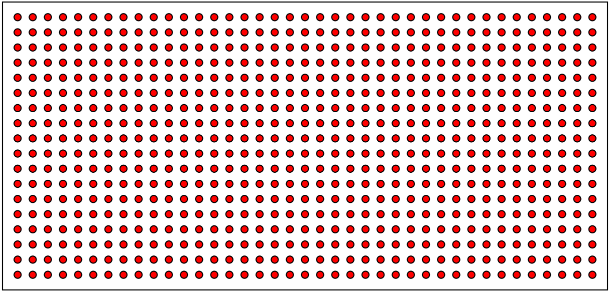
Simple Nim Game #702

How about the starting position **702**?



Simple Nim Game #702

How about the starting position **702**?



Amy can win by taking all the chips. (There are also 701 bad moves that let Bob win, but all you need is one good move.) Any one-pile game is a

win for Amy.

Simple Nim Game #0

How about the starting position $\mathbf{0}$?

Simple Nim Game #0

How about the starting position $\mathbf{0}$?

This is a game that someone just won.

More precisely, it's a game that Bob just won.

Simple Nim Game #0

How about the starting position $\mathbf{0}$?

This is a game that someone just won.

More precisely, it's a game that Bob just won.

This may look silly, but actually it's a very important game. (After all, every game reaches this point eventually!)

One-Pile Nim: Summary

What can we learn from the one-pile game?

One-Pile Nim: Summary

What can we learn from the one-pile game?

- If only one pile of chips is left, then the first player can win.

One-Pile Nim: Summary

What can we learn from the one-pile game?

- If only one pile of chips is left, then the first player can win.
- Every game will eventually get to the point where there is only one pile of chips left.

One-Pile Nim: Summary

What can we learn from the one-pile game?

- If only one pile of chips is left, then the first player can win.
- Every game will eventually get to the point where there is only one pile of chips left.

  Your goal is to make sure it is your move when that happens.

One-Pile Nim: Summary

What can we learn from the one-pile game?

- If only one pile of chips is left, then the first player can win.
- Every game will eventually get to the point where there is only one pile of chips left.

  Your goal is to make sure it is your move when that happens.
- In other words, the winning strategy is to force your opponent to take the last chip of the second-to-last color.

One-Pile Nim: Summary

What can we learn from the one-pile game?

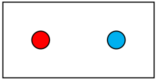
- If only one pile of chips is left, then the first player can win.
- Every game will eventually get to the point where there is only one pile of chips left.

  Your goal is to make sure it is your move when that happens.
- In other words, the winning strategy is to force your opponent to take the last chip of the second-to-last color.

Let's look at some two-pile games.

Two-Pile Nim: Example #1

The game **1 1**:



Now Amy has 2 moves... but they're both equivalent, and they're both losing moves.

Amy has no winning move, so **1 1 is a win for Bob.**

Two-Pile Nim: Example #2

The game **2 1**:

What should Amy do?

Two-Pile Nim: Example #2

The game **2 1**:

Two-Pile Nim: Example #2

The game **2 1**:

2 1 is a win for Amy.

Two-Pile Nim: Example #3

The game **2 2**:

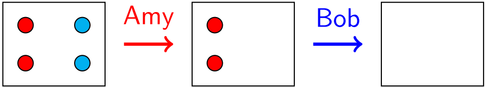
What should Amy do?

Two-Pile Nim: Example #3

The game **2 2**:

What should Amy do?

Taking two chips is a bad idea.

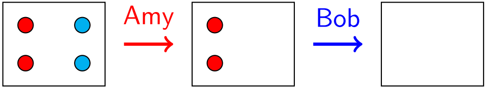


Two-Pile Nim: Example #3

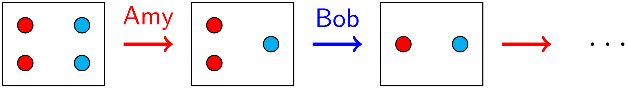
The game **2 2**:

What should Amy do?

Taking two chips is a bad idea.



But taking one chip is no better.

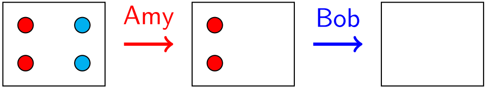


Two-Pile Nim: Example #3

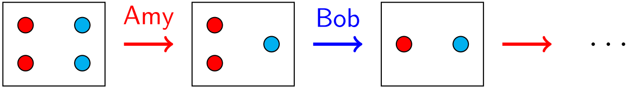
The game **2 2**:

What should Amy do?

Taking two chips is a bad idea.



But taking one chip is no better.



2 2 is a win for Bob.

A-Positions and B-Positions

- In some positions, the first player can ensure a win with best play, no matter what the second player does. We'll call these **A-positions**.

  For example: 1, 2, 702, 2 1, ...

- In some other positions, the roles are reversed — the second player can ensure a win with best play. We'll call these **B-positions**.

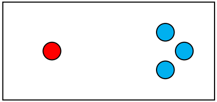
  For example: 1 1, 2 2, 0, ...

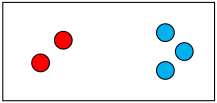
> **Important Fact:**
> Every Nim position is either an A-position or a B-position.

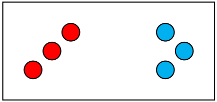
This may or may not be clear to you right now, but I promise to explain it soon!

Two-Pile Nim: More Examples

What about these positions?

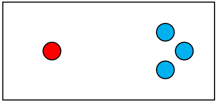


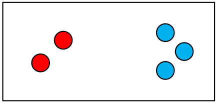




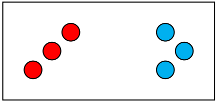
Two-Pile Nim: More Examples

What about these positions?



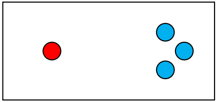


Try them yourself — I'll wait.

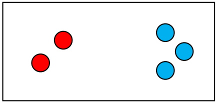


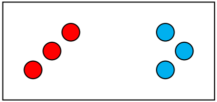
Two-Pile Nim: More Examples

What about these positions?



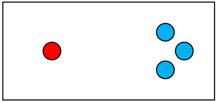
A-position



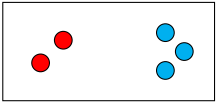


Two-Pile Nim: More Examples

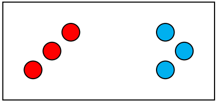
What about these positions?



A-position

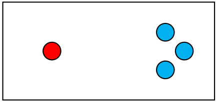


A-position

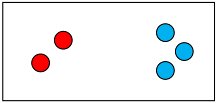


Two-Pile Nim: More Examples

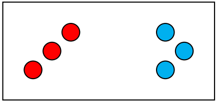
What about these positions?



A-position



A-position



B-position

Two-Pile Nim: A Theorem

Theorem: A Nim game with **two piles** is. . .

a **B-position** (second-player win) if the piles have **the same size**, and an **A-position** (first-player win) if the piles have **different sizes**.

Proof:

- If the piles have the same size, then Bob can win with a "copycat" strategy, forcing Amy to be the first player to remove a pile entirely.
- On the other hand, if the piles have different sizes, then Amy can win by using her first move to equalize them, producing a B-position. $\square$

Summary: Two Or Fewer Piles

Number of piles	Winner (assuming best play)
0	Bob
1	Amy
2 (equal)	Bob
2 (unequal)	Amy

Summary: Two Or Fewer Piles

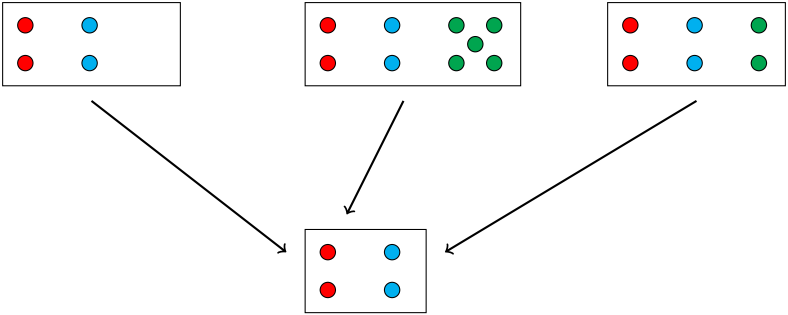
Number of piles	**Winner (assuming best play)**
0	Bob
1	Amy
2 (equal)	Bob
2 (unequal)	Amy

Okay, what about three piles?

Three Piles: An Easy Case

Theorem: Every Nim position with **three piles, including two of the same size**, is an A-position.

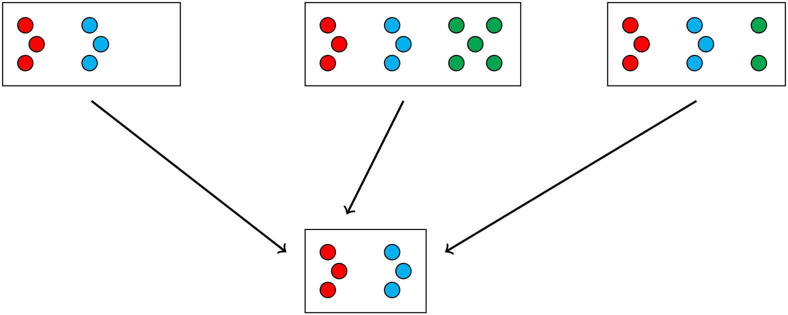
Proof: Amy can win by removing an entire pile, leaving two of the same size.



Three Piles: An Easy Case

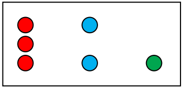
Theorem: Every Nim position with **three piles, including two of the same size**, is an A-position.

Proof: Amy can win by removing an entire pile, leaving two of the same size.



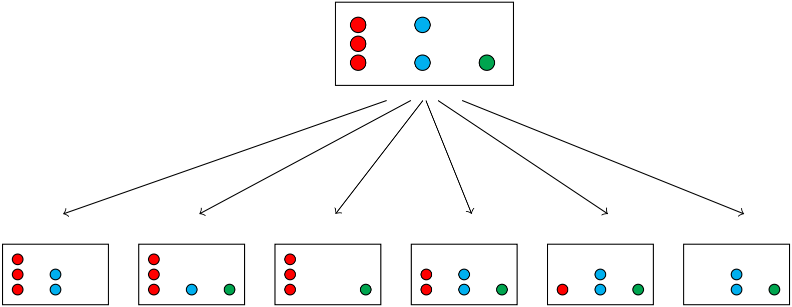
Three Unequal Piles

What about **three piles of all different sizes**? For example: 3 2 1.



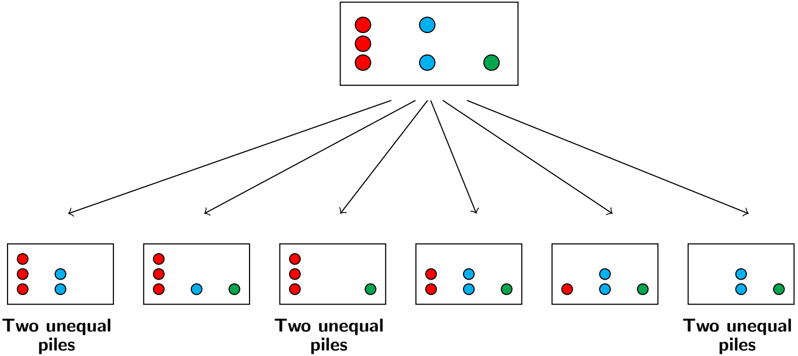
Three Unequal Piles

What about **three piles of all different sizes**? For example: 3 2 1.



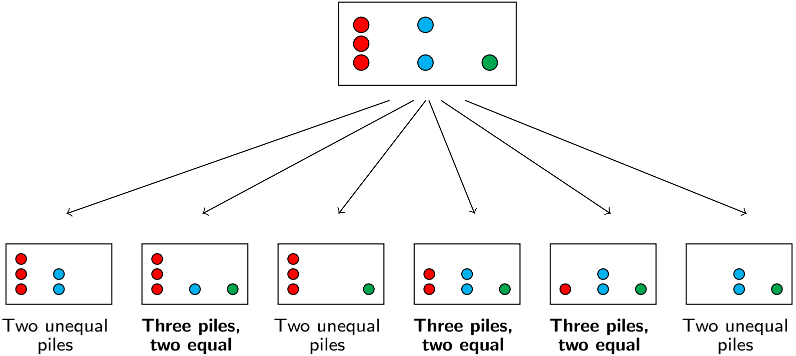
Three Unequal Piles

What about **three piles of all different sizes**? For example: 3 2 1.



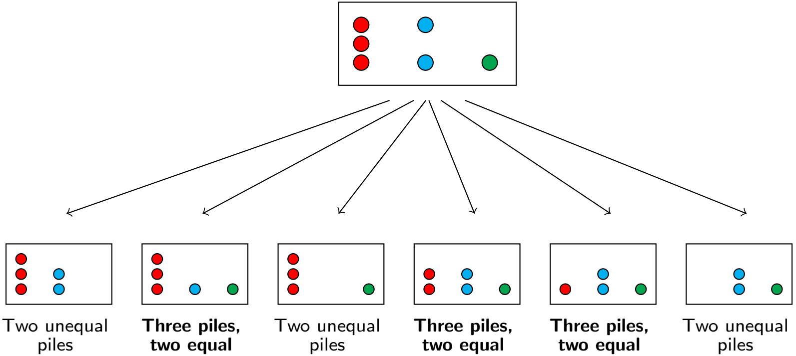
Three Unequal Piles

What about **three piles of all different sizes**? For example: 3 2 1.



Three Unequal Piles

What about **three piles of all different sizes**? For example: 3 2 1.



3 2 1 is a B-position.

Three Unequal Piles Including a 1

- 3 2 1 is a B-position.

Three Unequal Piles Including a 1

- 3 2 1 is a B-position.
  - Therefore 4 2 1 is an A-position, since Amy can move to 3 2 1.

Three Unequal Piles Including a 1

- 3 2 1 is a B-position.
  - Therefore 4 2 1 is an A-position, since Amy can move to 3 2 1.
  - So are 5 2 1 and 6 2 1 and 7 2 1 and. . .

Three Unequal Piles Including a 1

- 3 2 1 is a B-position.
  - Therefore 4 2 1 is an A-position, since Amy can move to 3 2 1.
  - So are 5 2 1 and 6 2 1 and 7 2 1 and...
  - So are 4 3 1 and 5 3 1 and 6 3 1 and...

Three Unequal Piles Including a 1

- 3 2 1 is a B-position.
  - Therefore 4 2 1 is an A-position, since Amy can move to 3 2 1.
  - So are 5 2 1 and 6 2 1 and 7 2 1 and...
  - So are 4 3 1 and 5 3 1 and 6 3 1 and...
- 5 4 1 is a win for Bob.

Three Unequal Piles Including a 1

- 3 2 1 is a B-position.
  - Therefore 4 2 1 is an A-position, since Amy can move to 3 2 1.
  - So are 5 2 1 and 6 2 1 and 7 2 1 and...
  - So are 4 3 1 and 5 3 1 and 6 3 1 and...
- 5 4 1 is a win for Bob.
  - Therefore 6 4 1 and 7 4 1 and 8 4 1 are A-positions.

Three Unequal Piles Including a 1

- 3 2 1 is a B-position.
  - Therefore 4 2 1 is an A-position, since Amy can move to 3 2 1.
  - So are 5 2 1 and 6 2 1 and 7 2 1 and...
  - So are 4 3 1 and 5 3 1 and 6 3 1 and...
- 5 4 1 is a win for Bob.
  - Therefore 6 4 1 and 7 4 1 and 8 4 1 are A-positions.
  - And so are 6 5 1 and 7 5 1 and 8 5 1, etc.

Three Unequal Piles Including a 1

- 3 2 1 is a B-position.
  - Therefore 4 2 1 is an A-position, since Amy can move to 3 2 1.
  - So are 5 2 1 and 6 2 1 and 7 2 1 and...
  - So are 4 3 1 and 5 3 1 and 6 3 1 and...
- 5 4 1 is a win for Bob.
  - Therefore 6 4 1 and 7 4 1 and 8 4 1 are A-positions.
  - And so are 6 5 1 and 7 5 1 and 8 5 1, etc.
- But 7 6 1 is a B-position

Three Unequal Piles Including a 1

- 3 2 1 is a B-position.
  - Therefore 4 2 1 is an A-position, since Amy can move to 3 2 1.
  - So are 5 2 1 and 6 2 1 and 7 2 1 and...
  - So are 4 3 1 and 5 3 1 and 6 3 1 and...
- 5 4 1 is a win for Bob.
  - Therefore 6 4 1 and 7 4 1 and 8 4 1 are A-positions.
  - And so are 6 5 1 and 7 5 1 and 8 5 1, etc.
- But 7 6 1 is a B-position
- So are 9 8 1 and 11 10 1 and...

The upshot: If $x > y$, then $x\ y\ 1$ is a B-position if and only if y is even and $x = y + 1$.

Reminder: Why Are We Doing This?

What we are doing is sorting Nim games into two types: **A-positions** and **B-positions**.

We are working our way up from simple games to more complex ones.

- If there is some way to move from the current position to a B-position, then the current position is an A-position. (Amy has a winning move.)

- Otherwise, the current position is a B-position. (Amy has nothing but losing moves.)

(Does this make more sense now?)

Three Unequal Piles

Here is what we know so far:

- A-positions: 1 pile; 2 unequal piles; 3 piles with at least two equal
- B-positions: 0; 2 equal piles

What about three unequal piles?
Most are A-positions, but here are some that are B-positions...

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	...
2 4 6	2 5 7	2 8 10	2 9 11	2 10 12	2 13 15	...
3 4 7	3 5 6	3 8 11	3 9 10	3 12 15	3 13 14	...
4 8 12	4 9 13	4 10 14	4 11 15	4 16 20	4 17 21	...
5 8 13	5 9 12	5 10 15	5 11 14	5 16 21	5 17 20	...
6 8 14	6 9 15	6 10 12	6 11 13	6 16 22	6 17 23	...
7 8 15	7 9 14	7 10 13	7 11 12	7 16 23	7 17 22	...
8 16 24	8 17 25	8 18 26	8 19 27	8 20 28	8 21 29	...
9 16 25	9 17 24	9 18 27	9 19 26	9 20 29	9 21 28	...

Binary Numbers

Reminder: Every number can be written in **binary**.

Decimal (base ten): 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, . . . , 99, 100, 101, . . .
Binary (base two): 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010,. . .

$$\begin{aligned} 101011_{\text{bin}} &= 2^5 + 2^3 + 2^1 + 2^0 \\ &= 32 + 8 + 2 + 1 \\ &= 43_{\text{dec}} \end{aligned}$$

Binary Numbers

Reminder: Every number can be written in **binary**.

Decimal (base ten): 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, . . . , 99, 100, 101, . . .
Binary (base two): 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010,. . .

$$\begin{aligned} 101011_{\mathrm{bin}} &= 2^5 + 2^3 + 2^1 + 2^0 \\ &= 32 + 8 + 2 + 1 \\ &= 43_{\mathrm{dec}} \end{aligned}$$

"There are 10 kinds of people in the world. Those who can count in binary and those who can't."

Three-Pile Nim and Binary Numbers

Let's look at some three-pile Nim games — but write the pile sizes in **binary** instead of decimal, and stack them on top of each other.

112	**122**	**123**	**124**	**125**	**134**
$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$
$1 = 0\ 0\ 1$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$3 = 0\ 1\ 1$
$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$3 = 0\ 1\ 1$	$4 = 1\ 0\ 0$	$5 = 1\ 0\ 1$	$4 = 1\ 0\ 0$

135	**145**	**235**	**246**	**247**	**257**
$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$
$3 = 0\ 1\ 1$	$4 = 1\ 0\ 0$	$3 = 0\ 1\ 1$	$4 = 1\ 0\ 0$	$4 = 1\ 0\ 0$	$5 = 1\ 0\ 1$
$5 = 1\ 0\ 1$	$5 = 1\ 0\ 1$	$5 = 1\ 0\ 1$	$6 = 1\ 1\ 0$	$7 = 1\ 1\ 1$	$7 = 1\ 1\ 1$

What's the pattern?

Three-Pile Nim and Binary Numbers

Let's look at some three-pile Nim games — but write the pile sizes in **binary** instead of decimal, and stack them on top of each other.

112	122	123	124	125	134
$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$
$1 = 0\ 0\ 1$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$3 = 0\ 1\ 1$
$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$3 = 0\ 1\ 1$	$4 = 1\ 0\ 0$	$5 = 1\ 0\ 1$	$4 = 1\ 0\ 0$

135	145	235	246	247	257
$1 = 0\ 0\ 1$	$1 = 0\ 0\ 1$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$	$2 = 0\ 1\ 0$
$3 = 0\ 1\ 1$	$4 = 1\ 0\ 0$	$3 = 0\ 1\ 1$	$4 = 1\ 0\ 0$	$4 = 1\ 0\ 0$	$5 = 1\ 0\ 1$
$5 = 1\ 0\ 1$	$5 = 1\ 0\ 1$	$5 = 1\ 0\ 1$	$6 = 1\ 1\ 0$	$7 = 1\ 1\ 1$	$7 = 1\ 1\ 1$

What's the pattern?

Three-Pile Nim and Binary Numbers

Let's look at some three-pile Nim games — but write the pile sizes in **binary** instead of decimal, and stack them on top of each other.

112	**122**	**123**	**124**	**125**	**134**
1 = 0 0 1	1 = 0 0 1	1 = 0 0 1	1 = 0 0 1	1 = 0 0 1	1 = 0 0 1
1 = 0 0 1	2 = 0 1 0	2 = 0 1 0	2 = 0 1 0	2 = 0 1 0	3 = 0 1 1
2 = 0 1 0	2 = 0 1 0	3 = 0 1 1	4 = 1 0 0	5 = 1 0 1	4 = 1 0 0
0 1 2	0 2 1	**0 2 2**	1 1 1	1 1 2	1 1 2

135	**145**	**235**	**246**	**247**	**257**
1 = 0 0 1	1 = 0 0 1	2 = 0 1 0	2 = 0 1 0	2 = 0 1 0	2 = 0 1 0
3 = 0 1 1	4 = 1 0 0	3 = 0 1 1	4 = 1 0 0	4 = 1 0 0	5 = 1 0 1
5 = 1 0 1	5 = 1 0 1	5 = 1 0 1	6 = 1 1 0	7 = 1 1 1	7 = 1 1 1
1 1 3	**2 0 2**	1 2 2	**2 2 0**	2 2 1	**2 2 2**

What's the pattern?

Look at the **"Nimbers."**

Nimbers

Observation: In a **B-position**, all Nimbers are even.

Having made this observation, what do we do next?

Nimbers

Observation: In a **B-position**, all Nimbers are even.

Having made this observation, what do we do next?

1. Check that it works for other cases that we know about.

Nimbers

Observation: In a **B-position**, all Nimbers are even.

Having made this observation, what do we do next?

1. Check that it works for other cases that we know about.
2. See whether it works in other cases (and refine it if necessary).

Nimbers

Observation: In a **B-position**, all Nimbers are even.

Having made this observation, what do we do next?

1. Check that it works for other cases that we know about.
2. See whether it works in other cases (and refine it if necessary).
3. Figure out why it works.

Nimbers

Observation: In a **B-position**, all Nimbers are even.

Having made this observation, what do we do next?

1. Check that it works for other cases that we know about.
2. See whether it works in other cases (and refine it if necessary).
3. Figure out why it works.

These steps are the same as those we would carry out in any other science (chemistry, physics, biology, . . .) . . . but in mathematics there's an additional step:

Nimbers

Observation: In a **B-position**, all Nimbers are even.

Having made this observation, what do we do next?

1. Check that it works for other cases that we know about.
2. See whether it works in other cases (and refine it if necessary).
3. Figure out why it works.

These steps are the same as those we would carry out in any other science (chemistry, physics, biology, ...) ...but in mathematics there's an additional step:

4. **Prove** that it **always** works!

Nimbers

Guess: In a **B-position**, all Nimbers are even.

Does this pattern work in the 2-pile game?

Nimbers

Guess: In a **B-position**, all Nimbers are even.

Does this pattern work in the 2-pile game? Sure!

Nimbers

Guess: In a **B-position**, all Nimbers are even.

Does this pattern work in the 2-pile game? Sure!

- If the two piles are equal then every Nimber is certainly even.
- If they are different then there is some digit that occurs in one pile but not the other, so one of the Nimbers is 1.

Nimbers

Guess: In a **B-position**, all Nimbers are even.

Does this pattern work in the 2-pile game? Sure!

- If the two piles are equal then every Nimber is certainly even.
- If they are different then there is some digit that occurs in one pile but not the other, so one of the Nimbers is 1.

Does this pattern work in the 1- and 0-pile games?

Nimbers

Guess: In a **B-position**, all Nimbers are even.

Does this pattern work in the 2-pile game? Sure!

- If the two piles are equal then every Nimber is certainly even.
- If they are different then there is some digit that occurs in one pile but not the other, so one of the Nimbers is 1.

Does this pattern work in the 1- and 0-pile games? Sure!

Nimbers

Guess: In a **B-position**, all Nimbers are even.

Does this pattern work in the 2-pile game? Sure!

- If the two piles are equal then every Nimber is certainly even.
- If they are different then there is some digit that occurs in one pile but not the other, so one of the Nimbers is 1.

Does this pattern work in the 1- and 0-pile games? Sure!

- If there is one pile then at least one digit (possibly several) occurs once.
- If there are no piles then there aren't any digits.

Nimbers

Guess: In a **B-position**, all Nimbers are even.

Does this pattern work in the 2-pile game? Sure!

- If the two piles are equal then every Nimber is certainly even.
- If they are different then there is some digit that occurs in one pile but not the other, so one of the Nimbers is 1.

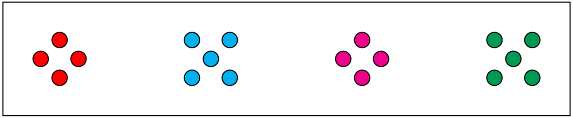
Does this pattern work in the 1- and 0-pile games? Sure!

- If there is one pile then at least one digit (possibly several) occurs once.
- If there are no piles then there aren't any digits.

Does this pattern work in the 4-pile game?

Nimbers

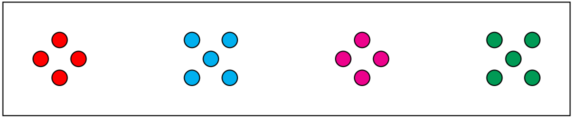
Conjecture: In a **B-position**, all Nimbers are even.



		4	*2*	*1*
4	=	1	0	0
5	=	1	0	1
4	=	1	0	0
5	=	1	0	1
		4	**0**	**2**

Nimbers

Conjecture: In a **B-position**, all Nimbers are even.

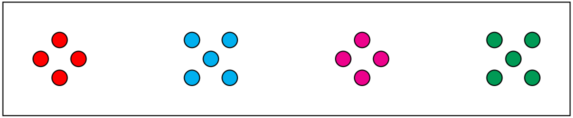


		4	*2*	*1*
4	=	1	0	0
5	=	1	0	1
4	=	1	0	0
5	=	1	0	1
		4	**0**	**2**

Who wins with best play?

Nimbers

Conjecture: In a **B-position**, all Nimbers are even.



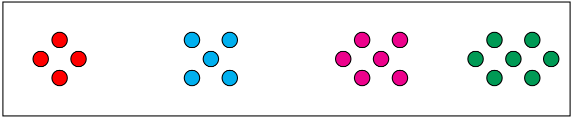
		4	*2*	*1*
4	=	1	0	0
5	=	1	0	1
4	=	1	0	0
5	=	1	0	1
		4	**0**	**2**

Who wins with best play?

Bob can win by following a copycat strategy.

Nimbers

Conjecture: In a **B-position**, all Nimbers are even.

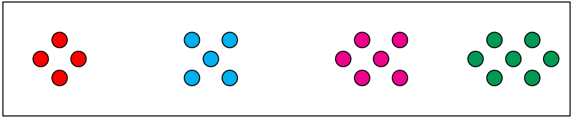


		4	*2*	*1*
4	=	1	0	0
5	=	1	0	1
6	=	1	1	0
7	=	1	1	1
		4	**2**	**2**

Who wins with best play?

Nimbers

Conjecture: In a **B-position**, all Nimbers are even.



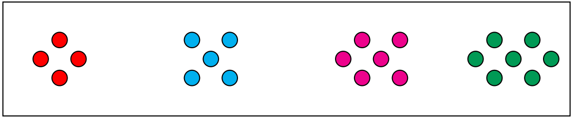
		4	*2*	*1*
4	=	1	0	0
5	=	1	0	1
6	=	1	1	0
7	=	1	1	1
		4	**2**	**2**

Who wins with best play?

Bob can win (details left to the reader).

Nimbers

Conjecture: In a **B-position**, all Nimbers are even.



		4	*2*	*1*
4	=	1	0	0
5	=	1	0	1
6	=	1	1	0
7	=	1	1	1
		4	**2**	**2**

Who wins with best play?

Bob can win (details left to the reader).

Idea: Amy must always make some Nimber odd and Bob can always make them all even.

The Universal Foolproof Strategy For Winning At Nim

The Universal Foolproof Strategy For Winning At Nim

Make sure that after every move you make, all Nimbers are even.

The Universal Foolproof Strategy For Winning At Nim

For example, suppose the starting position is **5 4 3 2 1**.

		4	*2*	*1*
5	=	1	0	1
4	=	1	0	0
3	=	0	1	1
2	=	0	1	0
1	=	0	0	1
		2	**2**	**3**

The Universal Foolproof Strategy For Winning At Nim

For example, suppose the starting position is **5 4 3 2 1**.

		4	*2*	*1*
5	=	1	0	1
4	=	1	0	0
3	=	0	1	1
2	=	0	1	0
1	=	0	0	1
		2	**2**	**3**

What move(s) will keep the 4-Nimber and the 2-Nimber even and change the 1-Nimber from odd to even?

The Universal Foolproof Strategy For Winning At Nim

For example, suppose the starting position is **5 4 3 2 1**.

		4	*2*	*1*
5	=	1	0	1
4	=	1	0	0
3	=	0	1	1
2	=	0	1	0
1	=	0	0	1
		2	**2**	**3**

What move(s) will keep the 4-Nimber and the 2-Nimber even and change the 1-Nimber from odd to even?

There are two possibilities.

The Universal Foolproof Strategy For Winning At Nim

For example, suppose the starting position is **5 4 3 2 1**.

		4	*2*	*1*
5	=	1	0	1
4	=	1	0	0
3	=	0	1	1
2	=	0	1	0
1	=	0	0	1
		2	**2**	**3**

What move(s) will keep the 4-Nimber and the 2-Nimber even and change the 1-Nimber from odd to even?

There are two possibilities.

		4	*2*	*1*
4	=	1	0	0
4	=	1	0	0
3	=	0	1	1
2	=	0	1	0
1	=	0	0	1
		2	**2**	**2**

		4	*2*	*1*
5	=	1	0	1
4	=	1	0	0
3	=	0	1	1
2	=	0	1	0
0	=	0	0	0
		2	**2**	**2**

The Universal Foolproof Strategy For Winning At Nim

Theorem: A Nim position is a **B-position** if every Nimber is even, and a **A-position** if at least one Nimber is odd.

Proof: I have to convince you of three things:

1. If every Nimber is even, then **every** move Amy can possibly make will turn some Nimber odd.
2. If at least one Nimber is odd, then Amy has **some** move that will turn all Nimbers even.

- For #1, every move changes one pile, therefore changes at least one Nimber from even to odd.
- For #2, look for a pile that contributes to the biggest Nimber.

Applying The Strategy

For example, if the starting position is **52 34 25 23 17 4**, then. . .

		32	*16*	*8*	*4*	*2*	*1*
52	=	1	1	0	1	0	0
34	=	1	0	0	0	1	0
25	=	0	1	1	0	0	1
23	=	0	1	0	1	1	1
17	=	0	1	0	0	0	1
4	=	0	0	0	1	0	0
		2	**4**	**1**	**3**	**2**	**3**

Applying The Strategy

For example, if the starting position is **52 34 25 23 17 4**, then. . .

		32	*16*	*8*	*4*	*2*	*1*
52	=	1	1	0	1	0	0
34	=	1	0	0	0	1	0
25	=	0	1	1	0	0	1
23	=	0	1	0	1	1	1
17	=	0	1	0	0	0	1
4	=	0	0	0	1	0	0
		2	**4**	**1**	**3**	**2**	**3**

Biggest Nimber: **8**

Applying The Strategy

For example, if the starting position is **52 34 25 23 17 4**, then...

		32	*16*	*8*	*4*	*2*	*1*
52	=	1	1	0	1	0	0
34	=	1	0	0	0	1	0
25	=	0	1	**1**	0	0	1
23	=	0	1	0	1	1	1
17	=	0	1	0	0	0	1
4	=	0	0	0	1	0	0
		2	**4**	**1**	**3**	**2**	**3**

Biggest Nimber: **8**

All odd Nimbers: *8*, *4*, *1*

Applying The Strategy

For example, if the starting position is **52 34 25 23 17 4**, then. . .

		32	*16*	*8*	*4*	*2*	*1*
52	=	1	1	0	1	0	0
34	=	1	0	0	0	1	0
25	=	0	1	**1**	0	0	1
23	=	0	1	0	1	1	1
17	=	0	1	0	0	0	1
4	=	0	0	0	1	0	0
		2	**4**	**1**	**3**	**2**	**3**

Biggest Nimber: **8**

All odd Nimbers: *8*, *4*, *1*

Winning move: Change 0 1 $\dot{1}$ $\dot{0}$ 0 $\dot{1}$
to 0 1 0 1 0 0.

What Next?

- What about **Misère Nim**? ("Misère" means that whoever takes the last chip loses.)
  - The strategy is actually very similar to regular Nim until you get down to two small piles.

- What about **other games**?
  - The Sprague-Grundy Theorem says that many other games can be modeled using Nim!
  - (Specifically: all <u>two-player, finite, impartial</u> games.)
  - More complex games require more complex mathematics. . .

Thank you very much!

Contact me:	jlmartin@ku.edu
These slides and more:	http://www.math.ku.edu/~jmartin