Unbounded Matroids

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Matroids

Matroids are combinatorial models of linear (in)dependence.

A matroid M on finite ground set E can be characterized combinatorially in many equivalent ways: basis system, independence system, rank function, lattice of flats, etc., etc.

Our primary point of view is *geometric*.

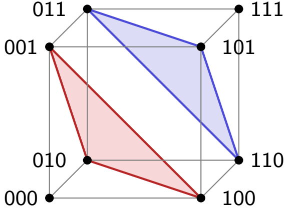
Definition

A **matroid base polytope** is a polytope in $\mathbb{R}^n$ such that

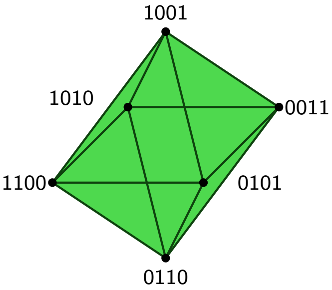
- every vertex lies on the unit cube $\{0, 1\}^n$
- every edge is parallel to the difference of two standard basis vectors

In particular, all vertices have the same coordinate sum.

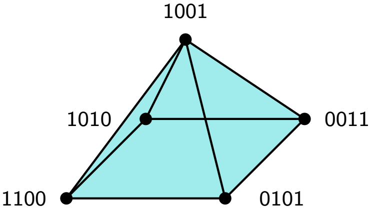
Matroid Base Polytopes

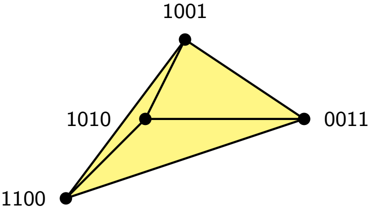


Hypersimplices $P(U_1(3))$ and $P(U_2(3))$

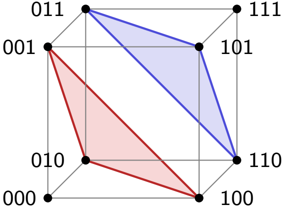


Hypersimplex $P(U_2(4))$

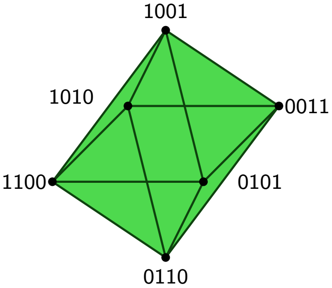




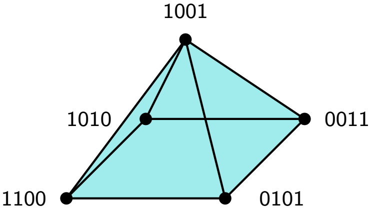
Matroid Base Polytopes

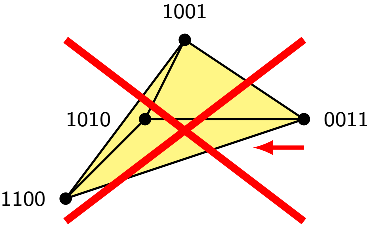


Hypersimplices $P(U_1(3))$ and $P(U_2(3))$



Hypersimplex $P(U_2(4))$





Generalized Permutahedra

The condition on directions of edges says that matroid base polytopes are **generalized permutahedra** [Postnikov '09]:

- All edges are parallel to vectors $\mathbf{e}_i - \mathbf{e}_j$ ($=$ type-A roots)
- Normal fan coarsens the braid fan
- Face maximized by a linear functional $\mathbf{x} \to \mathbf{c} \cdot \mathbf{x}$ depends only on the relative order of $c_1, \ldots, c_n$

Matroid base polytopes are **exactly** the GPs with 0,1-vertices [Gel'fand–Goresky–Macpherson–Serganova '87]

Matroid Basis Systems

Definition

A **matroid basis system** is a nonempty set family $\mathcal{B} \subset 2^E$ with

1. $|B| = |B'| = r$ for all $B, B' \in \mathcal{B}$
2. $\forall e \in B \backslash B' \colon \; \exists e' \in B' \backslash B \colon \; B \backslash e \cup e' \in \mathcal{B}$ (**exchange axiom**)

Canonical example #1: $E =$ vectors, $\mathcal{B} =$ bases of their span
Canonical example #2: $E = E(G)$, $\mathcal{B} =$ spanning trees

- Bases correspond to vertices of the matroid polytope
- Edges correspond to basis exchanges

Matroid Rank Functions

Definition

Let E be a finite set.
A function $\rho : 2^E \to \mathbb{N}_{\geq 0}$ is a **matroid rank function** if

- $\rho(A) \leq |A|$;
- $A \subseteq B \implies \rho(A) \leq \rho(B)$ (**monotonicity**);
- $\rho(A) + \rho(B) \geq \rho(A \cap B) + \rho(A \cup B)$ (**submodularity**).

Basis systems and rank functions provide the same information (they are “cryptomorphic”):

$$\begin{aligned} \mathcal{B} &= \{B \subseteq E \colon \rho(B) = |B| = \rho(E)\} \\ \rho(A) &= \max\{|A \cap B| \colon B \in \mathcal{B}\} \end{aligned}$$

Polymatroids

Definition (Edmonds '70)

A **polymatroid rank function** is a submodular rank function $\rho : 2^E \to \mathbb{R}$ that is

- *calibrated*: $\rho(\emptyset) = 0$,
- *monotone:* $S \subseteq T \implies \rho(S) \leq \rho(T)$,
- but does not necessarily satisfy $\rho(S) \leq |S|$.

The **base polytope** of ρ is

$$P(M) = \left\{ \mathbf{x} \in \mathbb{R}^E \mid \mathbf{x}(A) \leq \rho(A) \ \forall A \subseteq E, \ \ \mathbf{x}(E) = \rho(E) \right\}.$$

This construction gives a bijection between generalized permutahedra and polymatroids.

Submodular Systems

Definition (Fujishige '05)

A **submodular system** is a triple $M = (E, \mathcal{D}, \rho)$, where

- $\mathcal{D}$ is a distributive sublattice of 2^E; and
- $\rho : \mathcal{D} \to \mathbb{R}$ is a calibrated submodular rank function.

(Or: $\rho : 2^E \to \mathbb{R} \cup \{\infty\}$ and $\mathcal{D} = \{A \subseteq E \mid \rho(A) < \infty\}$.)

The corresponding **base polyhedron** is (again)

$$P(M) = \left\{ \mathbf{x} \in \mathbb{R}^E \mid \mathbf{x}(A) \leq \rho(A) \ \forall A \in \mathcal{D}, \ \ \mathbf{x}(E) = \rho(E) \right\}.$$

This polyhedron is unbounded iff $\mathcal{D} \neq 2^E$. It is a generalized permutahedron: all edges and rays are parallel to roots of type A.

Polymatroids correspond to submodular systems whose base polyhedra are bounded.

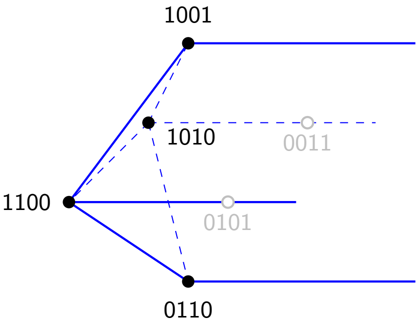
Matroid polytopes correspond to submodular systems whose base polyhedra are bounded and have 0,1-vertices.

What happens if we require 0,1-vertices, but drop the boundedness requirement?

Example: The Stalactite

The **stalactite** is the polyhedron

$$Q = \{\mathbf{x} \in \mathbb{R}^4 \colon x_1 + x_2 + x_3 + x_4 = 2,\ x_2, x_3, x_4 \geq 0,\ x_1, x_2, x_3 \leq 1\}.$$



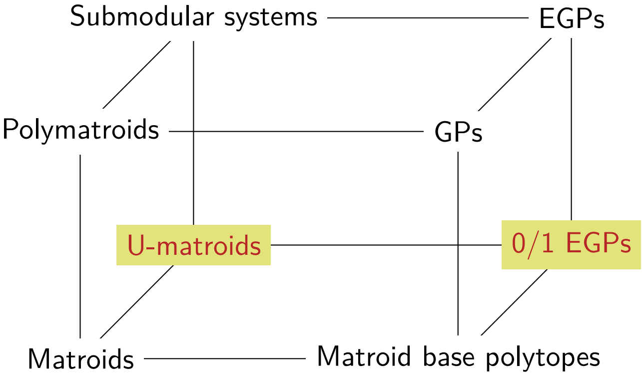
- Recession cone: ray spanned by $\langle -1, 0, 0, 1 \rangle$
- Vertices $\{12, 13, 14, 23\}$ do not form a matroid basis system

U-Matroids

Definition

An **unbounded matroid** or **U-matroid** is a submodular system $M = (E, \mathcal{D}, \rho)$, where $\mathcal{D} \subseteq 2^E$ is a distributive lattice and $\rho : \mathcal{D} \to \mathbb{N}$ is **integral**, **monotone**, and **unit-increase** (as well as calibrated and submodular).

- U-matroids are essentially identical to the *pregeometries* of [Faigle 1980]. However, Faigle defined bases differently (and purely combinatorially).
- A U-matroid is a matroid precisely when $\mathcal{D} = 2^E$.
- U-matroids form a Hopf monoid [Castillo–JLM–Samper 2024]

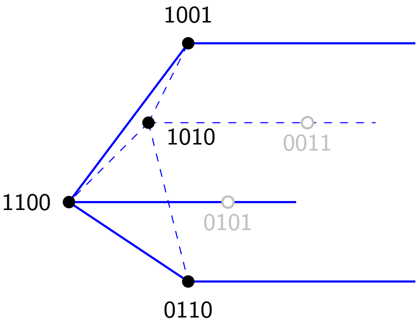


- Horizontal lines are bijections; others are inclusions
- Left/right = combinatorial/geometric
- Bottom/top = integer/real
- Front/back = bounded/possibly unbounded

Example: The Stalactite

The **stalactite** is the polyhedron

$$Q = \{\mathbf{x} \in \mathbb{R}^4 \colon x_1 + x_2 + x_3 + x_4 = 2,\ x_2, x_3, x_4 \geq 0,\ x_1, x_2, x_3 \leq 1\}.$$

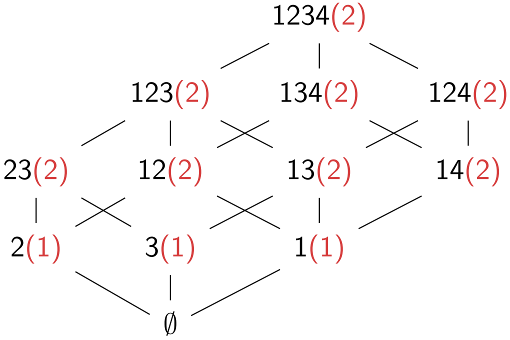


- Recession cone: ray spanned by $\mathbf{e}_4 - \mathbf{e}_1$
- Lattice: $\mathcal{D}_{\mathrm{stal}} = \{A \subseteq [r] \colon 4 \in A \implies 1 \in A\}$

Example: The Stalactite

$\mathcal{D} = \mathcal{D}_{\text{stal}}$

$\rho_{\mathcal{D}}(A) = \min(|A|, 2)$

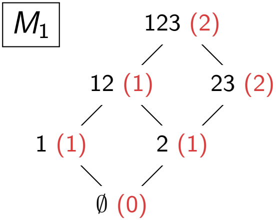


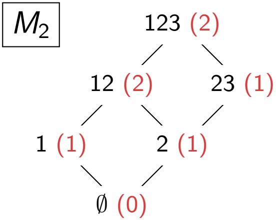
Maximal chain in $\mathcal{D}$	$\rightsquigarrow$	**Vertex** $\mathbf{x} = (x_1, \dots, x_n)$
$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = E$		$x_{A_i \setminus A_{i-1}} = \rho(A_i) - \rho(A_{i-1})$

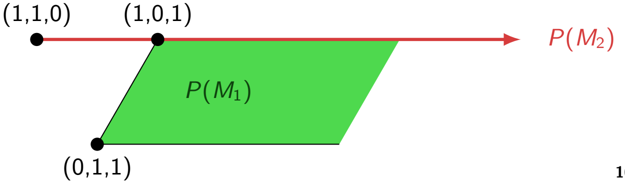
- Vertices $\{12, 13, 14, 23\}$ do not form a matroid basis system

U-Matroids are Weirder than Matroids

Dimension of a U-matroid polyhedron is not easy to read from the rank function.

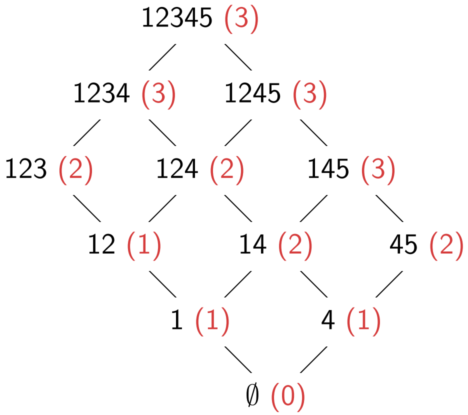






U-Matroids are Weirder than Matroids

Minimal sets of maximal ρ do not behave like bases.



- Minimal sets of max rank vary in cardinality: 1234, 1245, 145
- Vertices of polyhedra are 134, 145 — note that $134 \notin \mathcal{D}$.

Lattice Extensions and Sheared Polyhedra

Definition

Let $M = (E, \mathcal{D}, \rho)$ be a U-matroid.
Let $\mathcal{D}'$ be a distributive lattice with $\mathcal{D} \subseteq \mathcal{D}' \subseteq 2^E$.

A **lattice extension** of M is a U-matroid $M' = (E, \mathcal{D}', \rho')$ such that $\rho'|_{\mathcal{D}} = \rho$.

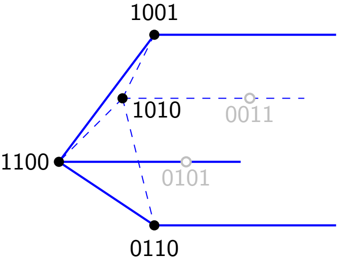
Theorem (Berggren–JLM–Samper)

M' is a lattice extension of M if and only if

- $P(M') \subseteq P(M)$ *and*
- $V(P(M')) \supseteq V(P(M))$.

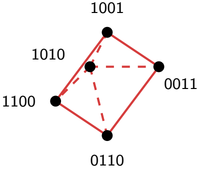
In this case we say that $P(M')$ is a **shearing** of $P(M)$.

Example: Shearing the Stalactite

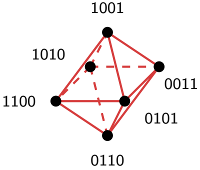


$$\mathcal{D} = \mathcal{D}_{\mathrm{stal}}$$
$$\rho_{\mathcal{D}}(A) = \min(|A|, 2)$$

Lattice extensions of $\rho_{\mathcal{D}}$ to $\mathcal{D}' = 2^{[4]}$:



$\rho(A) = \min(|A|, 2)$
except $\rho(24) = 1$



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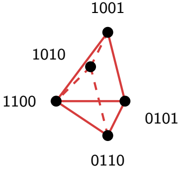


$\rho(A) = \min(|A|, 2)$
except $\rho(34) = 1$

Generous Extensions

Theorem (Berggren–JLM–Samper)

Let $M = (E, \mathcal{D}, \rho)$ be a U-matroid and $\mathcal{D} \subseteq \mathcal{D}' \subseteq 2^E$.

*Then there exists a U-matroid $M' = (E, \mathcal{D}', \rho_g)$ (the **generous extension of M to $\mathcal{D}'$**) such that:*

1. *M' is a lattice extension of M to $\mathcal{D}'$.*
2. *If $(E, \mathcal{D}', \rho')$ is any lattice extension of M to $\mathcal{D}'$, then $\rho_g(A) \geq \rho'(A)$ for all $A \subseteq E$.*

Corollary

1. *$P(M')$ is the unique largest sheared polyhedron of $P(M)$ with recession cone $R(\mathcal{D}')$.*
2. *$P(M)$ contains a unique largest matroid base polytope.*

Generous Extensions

Sketch of proof: Enough to consider $\mathcal{D}' = \mathcal{D}[a]$ = smallest distributive lattice containing $\mathcal{D} \cup \{\{a\}\}$, where $a \in E$. Define

$$\rho_g(S) = \begin{cases} \rho(S) & \text{if } S \in \mathcal{D} \\ \rho(S - a) & \text{if } S \notin \mathcal{D} \text{ and } \rho(S - a) = \rho(\sup_{\mathcal{D}}(S)) \\ \rho(S - a) + 1 & \text{if } S \notin \mathcal{D} \text{ and } \rho(S - a) < \rho(\sup_{\mathcal{D}}(S)) \end{cases}$$

where $\sup_{\mathcal{D}}(S)$ = smallest element of $\mathcal{D}$ containing S.

("Adjoining a increments rank except when it obviously can't.")

Every extension to $\mathcal{D}'$ is bounded by ρ_g.
As a consequence, the order of adjoining atoms does not matter.

Generous Extensions

Corollary

1. *Every U-matroid base polyhedron is the Minkowski sum of its recession cone with a matroid base polytope.*
2. *The base polytope of the generous matroid extension of a U-matroid M is*
   - *the convex hull of all 0,1-vectors in $P(M)$ (not just vertices!);*
   - *the intersection of $P(M)$ with the appropriate hypersimplex;*
   - *the union of all sheared matroid polytopes in $P(M)$.*

Remark

- We do not know a closed formula for the rank function of the generous matroid extension.
- The construction fails entirely without the $0/1$-condition!

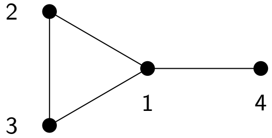
Bases of U-Matroids

Definition

The **pseudo-independence complex** of $M = (E, \mathcal{D}, \rho)$ is the simplicial complex $\Delta(M)$ on E generated by the bases (vertices).

Example

The stalactite has $\mathcal{B} = \{12, 13, 14, 23\}$ and $\Delta = \langle 12, 13, 14, 23 \rangle$.



This pseudo-independence complex is **not** a matroid complex.
(Here, $\Delta|_{\{2,3,4\}}$ is not pure.)

The Pseudo-Independence Complex

Theorem

Every U-matroid pseudo-independence complex $\Delta(M)$ *is shellable.*

In fact, every generic linear functional ℓ in the interior of $R(P(M))^*$ defines a linear order on vertices of $P(M)$ that is a shelling order on $\Delta(M)$.

Proof uses a polyhedral result of Heaton and Samper.

Questions

- What characterizes these complexes?
- What do their h-numbers count?

U-Matroids and Subspace Arrangements

$\mathcal{A} = (V_1, \ldots, V_n) =$ arrangement of linear subspaces in $\Bbbk^d$
$c_i = \operatorname{codim} V_i$

Here is a U-matroid $M(\mathcal{A})$ that **represents** $\mathcal{A}$:

$$\mathcal{D} = \{\mathbf{a} = (a_1, \ldots, a_n) \mid 0 \leq a_i \leq c_i \quad \forall i\}$$

$$\rho(\mathbf{a}) = \max \left\{ \operatorname{codim}(W_1 \cap \cdots \cap W_n) \;\middle|\; \begin{array}{l} V_i \subseteq W_i \subseteq \Bbbk^d \\ \operatorname{codim} W_i = a_i \end{array} \quad \forall i \right\}$$

- This construction is due to [Barnabei–Nicoletti–Pezzoli '98].
- $M(\mathcal{A})$ is a *poset matroid* in their sense (a U-matroid such that every vertex is in $\mathcal{D}$).

U-Matroids and Subspace Arrangements

Theorem

1. *Suppose that* $c_k = \operatorname{codim} V_k \geq 2$ *for some* $k \in [n]$*. Let* a *be the atom corresponding to the top element of the chain* $[0, c_k]$*. Then the generous extension of* $M(\mathcal{A})$ *to* $\mathcal{D}[a]$ *represents*

$$\mathcal{A} \setminus \{V_k\} \cup \{V_k', V_k''\}$$

   where V_k', V_k'' *are generic linear spaces containing* V_k*, of codimensions 1 and* $c_k - 1$ *respectively.*

2. *The generous matroid extension of* $M(\mathcal{A})$ *represents any hyperplane arrangement formed by replacing every* V_i *with* c_i *generic hyperplanes containing* V_i*.*

In particular, these generous matroid extensions are **multisymmetric** in the sense of Crowley–Huh–Larson–Simpson–Wang.

Further Questions

1. Analogues of other matroid axiomatizations? (For example, what is a circuit in a U-matroid?)
   - Some analogues were proposed by Faigle and others circa 1980, but we want a definition motivated by geometry.
2. Combinatorial formula for the rank function of a generous matroid extension? (This may be hopeless.)
3. Is the generous matroid of a subspace arrangement useful in understanding its topology?
   - For *hyperplane* arrangements, the topology of the complement is governed strongly by combinatorics (Orlik–Solomon). For more general arrangements we do not have strong control.

¡Muchas gracias por escucharme!

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