Informal Seminar on Stanley-Reisner Theory, UMN, Fall 2002
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Minimal free resolutions of Stanley-Reisner rings
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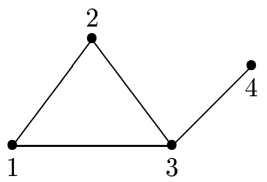
Outline: I. (Minimal) free resolutions and $\mathrm{Tor}^S_\cdot(\mathbf{k}[\Delta], \mathbf{k})$
II. Theorem for $\mathbf{k}[\Delta]$
III. Proof

For more details, see especially section 5.5 of [1].

Let Δ be the simplicial complex

(1)



so that

$$I_\Delta = (x_1x_2x_3,\ x_1x_4,\ x_2x_4)$$

and

$$\mathbf{k}[\Delta] = \underbrace{\mathbf{k}[x_1, \ldots, x_4]}_{S}/I_\Delta.$$

The S-module $\mathbf{k}[\Delta]$ is $\mathbb{Z}^n$-graded; for $\alpha \in \mathbb{Z}^n$, the corresponding graded piece is the linear span of the monomial $\mathbf{x}^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. $\mathbf{k}[\Delta]$ has the following $\mathbb{Z}^n$-graded free resolution as an S-module:

(2)

$$0 \to S^1 \xrightarrow{\begin{array}{cc} & g \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array} & \begin{bmatrix} x_3 \\ 1 \\ -1 \end{bmatrix} \end{array}} S^3 \xrightarrow{\begin{array}{cc} & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} & \begin{bmatrix} 0 & x_4 & x_4 \\ x_2 & -x_2x_3 & 0 \\ -x_1 & 0 & -x_1x_3 \end{bmatrix} \end{array}} S^3 \xrightarrow{\begin{array}{ccc} e_1 & e_2 & e_3 \\ \multicolumn{3}{c}{[x_1x_2x_3 \quad x_1x_4 \quad x_2x_4]} \end{array}} S \to S/I_\Delta \to 0.$$

Here the first copy of S^1 has basis element g of degree $x_1x_2x_3x_4$; the first copy of S^3 has basis $\{f_1, f_2, f_3\}$ of degrees $x_1x_2x_4$, $x_1x_2x_3x_4$, $x_1x_2x_3x_4$ respectively; and the second copy of S^3 has basis $\{e_1, e_2, e_3\}$ of degrees $x_1x_2x_3$, x_1x_4, x_2x_4 respectively.

The resolution (2) is not minimal. In a minimal free resolution (henceforth MFR), we want at each stage that the columns give a minimal generating set of the kernel. The following is an MFR of S/I_Δ:

(3)

$$0 \to S^2 \xrightarrow{\begin{array}{cc} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} & \begin{bmatrix} 0 & x_4 \\ x_2 & -x_2x_3 \\ -x_1 & 0 \end{bmatrix} \end{array}} S^3 \xrightarrow{\begin{array}{ccc} e_1 & e_2 & e_3 \\ \multicolumn{3}{c}{[x_1x_2x_3 \quad x_1x_4 \quad x_2x_4]} \end{array}} S \to S/I_\Delta \to 0.$$

Proposition 1. *Let R be a graded $\mathbf{k}$-algebra and M a graded R-module. Then a graded R-free resolution*

$$\cdots \to R^{\beta_1} \to R^{\beta_0} \to M \to 0$$

is minimal if and only if the matrices only contain elements of

$$\mathfrak{m} := \bigoplus_{n>0} R_n$$

where R_n denotes the nth graded piece of R. (Note: $\mathfrak{m}$ is also denoted R_+ and often called the irrelevant ideal of R.)

For instance, the resolution (2) is not minimal because of the 1 and -1 appearing in the leftmost map.

Note that in this MFR, the free module S^2 has S-basis $\{f_1, f_2\}$ of degrees $x_1x_2x_4, x_1x_2x_3x_4$. It can be shown that the degrees of the basis elements in the terms of an MFR are uniquely determined, even though the maps are not unique.

Corollary 2. *The number of basis elements of R^{β_i} of degree $\mathbf{x}^\alpha$ in any MFR is*

$$\dim_{\mathbf{k}} \operatorname{Tor}_i^S(M, \mathbf{k})_{\mathbf{x}^\alpha},$$

where $\mathbf{k} = R/\mathfrak{m} = R/R_+$.

Proof. To compute $\operatorname{Tor}_i^S(M, \mathbf{k})$, we first write down an MFR of M as an S-module:

$$\cdots \to R^{\beta_i} \underset{\phi}{\to} R^{\beta_{i-1}} \to \cdots \to R^{\beta_0} \to M \to 0. \tag{4}$$

Here the matrix ϕ has all entries in $\mathfrak{m}$. Therefore, tensoring the complex (4) with $\mathbf{k}$ over R produces

$$\cdots \to \mathbf{k}^{\beta_i} \xrightarrow{0} R^{\beta_{i-1}} \xrightarrow{0} \ldots \xrightarrow{0} R^{\beta_0} \to 0. \tag{5}$$

The ith homology of this complex is by definition $\operatorname{Tor}_i^S(M, \mathbf{k})$; since all the maps are zero, we have $\operatorname{Tor}_i^S(M, \mathbf{k}) = \mathbf{k}^{\beta_i}$. Note that $\mathbf{k}^{\beta_i}$ still carries a grading, and the $\mathbf{k}$-basis vectors for $\mathbf{k}^{\beta_i} = \operatorname{Tor}_i^S(M, \mathbf{k})$ have the same degrees as the R-basis elements for R^{β_i}. □

Let Δ be a simplicial complex on vertices $[n]$. Recall that the *link* of a face is defined as

$$\operatorname{lk}_\Delta F := \{G \in \Delta \ : \ G \cup F \in \Delta \text{ and } G \cap F = \emptyset\}.$$

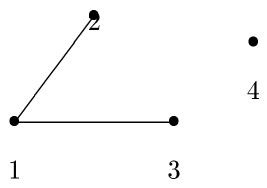
For $S \subset [n]$, we define

$$\Delta|_S := \{F \in \Delta \ : \ F \subset S\}.$$

The *dual complex* of Δ is

$$\Delta^\vee := \{F \in [n] \ : \ [n] - F \notin \Delta\}$$

For instance, if Δ is the simplicial complex of (1), then $\Delta^\vee$ is

(6) 

One way to picture this is as follows. Draw the full Boolean algebra on [4] (I'm not going to bother to put in all the edges):

$$
\begin{array}{ccccccc}
 & & & 1234 & & & \\
 & 123 & & 124 & 134 & & 234 \\
\boxed{12} & \boxed{13} & 14 & \boxed{23} & 24 & \boxed{34} & \\
 & \boxed{1} & & \boxed{2} & \boxed{3} & & \boxed{4} \\
 & & & \boxed{\emptyset} & & &
\end{array}
\tag{7}
$$

The boxes indicate faces of Δ. The unboxed faces are the complements of faces in the dual complex.

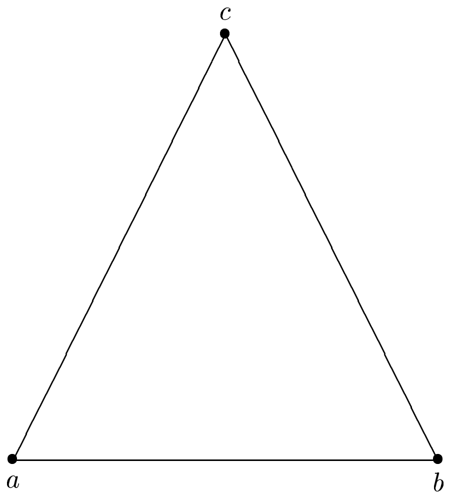
We will also need the notion of the *barycentric subdivision* $\mathrm{Sd}(\Delta)$ of a simplicial complex Δ. Abstractly, this is the simplicial complex whose vertices are the nonempty faces of Δ and whose faces are the *flags* in Δ, that is, strictly increasing sequences

$$\emptyset \neq F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r$$

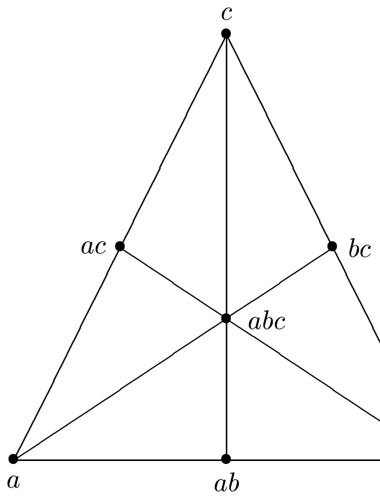
where $F_i \in \Delta$ for all i. (For those familiar with posets, the barycentric subdivision of Δ is the same thing as the order complex of the poset of faces of Δ.)

For example, if Δ is the 2-dimensional simplex on vertices $\{a, b, c\}$, then $\mathrm{Sd}(\Delta)$ has six facets, namely the flags

$$
\begin{array}{lll}
\{a\} \subset \{a,b\} \subset \{a,b,c\}, & \{b\} \subset \{a,b\} \subset \{a,b,c\}, & \{c\} \subset \{a,c\} \subset \{a,b,c\}, \\
\{a\} \subset \{a,c\} \subset \{a,b,c\}, & \{b\} \subset \{b,c\} \subset \{a,b,c\}, & \{c\} \subset \{b,c\} \subset \{a,b,c\}.
\end{array}
$$



Δ



$\mathrm{Sd}(\Delta)$

Proposition 3. *Each of Δ and $\Delta^\vee$ is homotopy equivalent to the complement of the other in $\partial\Delta^{n-1} \cong \mathbb{S}^{n-2}$. Consequently, their (co-)homology groups are related by Alexander duality:*

$$\tilde{H}_i(\Delta^\vee) \cong \tilde{H}^{n-3-i}(\Delta^\vee).$$

Idea of proof: It is possible to embed both barycentric subdivisions $\mathrm{Sd}(\Delta)$ and $\mathrm{Sd}(\Delta^\vee)$ simultaneously and disjointly inside $\mathrm{Sd}(\partial\Delta^{n-1})$. For $\mathrm{Sd}(\Delta^\vee)$ one must apply the antipodal map on the barycentric subdivison before embedding it in the usual way. (This is unLaTeXable, but try it yourself with the complex Δ of (1).) $\square$

For this reason, the complex $\Delta^\vee$ may be called the *canonical Alexander dual* of Δ.

We now state and prove the main result.

Theorem 4. *Let Δ be a simplicial complex on vertices $[n]$, $S = \mathbf{k}[x_1, \ldots, x_n]$, and $\alpha \in \mathbb{N}^n$. Then*

$$\begin{aligned}\mathrm{Tor}_i^S(\mathbf{k}[\Delta], \mathbf{k})_{\mathbf{x}^\alpha} &\cong \begin{cases} \tilde{H}_{|S|-i-1}(\Delta|_S;\ \mathbf{k}) & \text{if } \mathbf{x}^\alpha = \mathbf{x}^S \text{ for some } S \subset [n] \\ 0 & \text{otherwise} \end{cases} \\ &\cong \begin{cases} \tilde{H}_{i-2}(\mathrm{lk}_{\Delta^\vee}(F);\ \mathbf{k}) & \text{if } \mathbf{x}^\alpha = \mathbf{x}^{[n]-F} \text{ for some } F \in \Delta^\vee \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

The first characterization of Tor is due to Hochster [3] and the second to Eagon and Reiner [2]. The equivalence of the two comes from Alexander duality after one checks that $(\Delta|_S)^\vee = \mathrm{lk}_{\Delta^\vee}(F)$ if $S = [n] - F$.

Proof. By general homological nonsense we have

$$\mathrm{Tor}_i^S(\mathbf{k}[\Delta], \mathbf{k}) = \mathrm{Tor}_i^S(S/I_\Delta, \mathbf{k}) \ \cong\ \mathrm{Tor}_{i-1}^S(I_\Delta, \mathbf{k}) \ \cong\ \mathrm{Tor}_{i-1}^S(\mathbf{k}, I_\Delta). \tag{8}$$

We compute the last module via the Koszul resolution of $\mathbf{k}$ as an S-module:

$$0 \to S \otimes \textstyle\bigwedge^n \mathbf{k}^n \to \cdots \to S \otimes \bigwedge^2 \mathbf{k}^n \to S \otimes \bigwedge^1 \mathbf{k}^n \to S \to \mathbf{k} \to 0, \tag{9}$$

where the boundary maps are defined S-linearly by

$$e_{i_1} \wedge \cdots \wedge e_{i_r} \quad \mapsto \quad \sum_{j=1}^{r} (-1)^j x_r e_{i_1} \wedge \cdots \wedge \widehat{e_{i_1}} \wedge \cdots \wedge e_{i_r} \tag{10}$$

(the hat denoting removal). Now, we tensor (9) with I_Δ over S, obtaining

$$\ldots \ \to\ I_\Delta \otimes \textstyle\bigwedge^r \mathbf{k}^n \ \to\ \bigwedge^{r-1} \mathbf{k}^n \ \to\ \ldots \tag{11}$$

where the boundary maps are given by

$$\mathbf{x}^\beta \otimes e_{i_1} \wedge \cdots \wedge e_{i_r} \quad \mapsto \quad \sum_{j=1}^{r} (-1)^j x_r e_{i_1} \wedge \cdots \wedge \widehat{e_{i_1}} \wedge \cdots \wedge e_{i_r}. \tag{12}$$

Denoting $e_{i_1} \wedge \cdots \wedge e_{i_r}$ by e_G if $G = \{i_1, \ldots, i_r\}$, we see that a $\mathbf{k}$-basis in degree $\mathbf{x}^\alpha$ for the complex (11) is

$$\left\{ \mathbf{x}^\beta \otimes e_G \ :\ G \subset [n],\ \mathbf{x}^\beta \in I_\Delta,\ \mathbf{x}^\beta \mathbf{x}^G = \mathbf{x}^\alpha \right\} . \tag{13}$$

This shows that in degree $\mathbf{x}^\alpha$, this complex coincides (up to shift in homological degree by 1) with the usual augmented chain complex for the simplicial complex

$$\Delta_\alpha := \left\{ G \subset [n] \ :\ \frac{\mathbf{x}^\alpha}{\mathbf{x}^G} \in I_\Delta \right\} = \left\{ G \subset [n] \ :\ \mathrm{supp}\left(\frac{\mathbf{x}^\alpha}{\mathbf{x}^G} \right) \notin \Delta \right\}. \tag{14}$$

Thus we have

$$\mathrm{Tor}_i^S(\mathbf{k}[\Delta], \mathbf{k})_{\mathbf{x}^\alpha} \ \cong\ \mathrm{Tor}_{i-1}^S(\mathbf{k}, I_\Delta)_{\mathbf{x}^\alpha} \ \cong\ \tilde{H}_{i-2}(\Delta_\alpha; \mathbf{k}). \tag{15}$$

Note that if $\mathbf{x}^\alpha$ is divisible by x_i^2, then

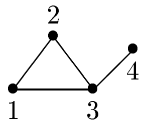
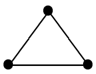
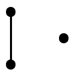
$$\mathrm{supp}\left(\frac{\mathbf{x}^\alpha}{\mathbf{x}^G} \right) \ =\ \mathrm{supp}\left(\frac{\mathbf{x}^\alpha}{\mathbf{x}^{G \cup \{i\}}} \right)$$

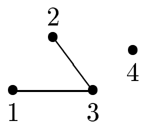
so i will be a cone vertex for Δ_α. Hence without loss of generality $\mathbf{x}^\alpha = \mathbf{x}^S$ for some subset $S \subset [n]$. Let $F = [n] - S$; then

$$\begin{aligned} \frac{\mathbf{x}^S}{\mathbf{x}^G} \in I_\Delta \quad &\iff \quad G \subset S \text{ and } S - G \notin \Delta \\ &\iff \quad G \cap F = \emptyset \text{ and } [n] - (G \cup F) \notin \Delta \\ &\iff \quad G \in \mathrm{lk}_{\Delta^\vee}(F). \end{aligned}$$

Therefore $\Delta_\alpha = \mathrm{lk}_{\Delta^\vee}(F)$. □

As an illustration, we can now go back and explain the degrees of the basis elements for the terms in the MFR:

Restricted complex	Homological observations
$\Delta\vert_{1234} \;=$ 	$\tilde{H}_1 \neq 0$
$\Delta\vert_{123} \;\cong$ 	$\tilde{H}_1 \neq 0$
$\Delta\vert_{14} \;\cong\; \Delta\vert_{23} \;\cong$ 	$\tilde{H}_0 \neq 0$
$\Delta\vert_{124} \;\cong$ 	$\tilde{H}_0 \neq 0$

Link	Homological observations
$\Delta^\vee = \mathrm{lk}_{\Delta^\vee}(\emptyset) \;=$ 	$\tilde{H}_0 \neq 0$
$\mathrm{lk}_{\Delta^\vee}(3) \;\cong$ 	$\tilde{H}_0 \neq 0$
$\mathrm{lk}_{\Delta^\vee}(4), \mathrm{lk}_{\Delta^\vee}(23), \mathrm{lk}_{\Delta^\vee}(13) = \{\emptyset\}$	$\tilde{H}_{-1} \neq 0$

References

[1] W. Bruns and J. Herzog. *Cohen-Macaulay rings.* Cambridge U. Press, 1993

[2] J. Eagon and V. Reiner. *Resolutions of Stanley-Reisner rings and Alexander duality.* J. of Pure and Applied Algebra **130** (1998), 265–275.

[3] M. Hochster. *Cohen-Macaulay rings, combintorics, and simplicial complexes.* In B.R. McDonald and R.A. Morris, eds. *Ring theory II*, Lecture Notes in Pure and Applied Mathematics **26**, M. Dekker, 1977, pp. 171–223.