Informal Seminar on Stanley-Reisner Theory, UMN, Fall 2002
5 December 2002

Local cohomology
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We work with the following objects throughout. Let $\mathbf{k}$ be a field and R a finitely generated, $\mathbb{N}^n$-graded $\mathbf{k}$-algebra, i.e., $R = \bigoplus_{\alpha \in \mathbb{N}^n} R_\alpha$ with $R_\alpha R_\beta \subset R_{\alpha+\beta}$. The motivating example is $R = \mathbf{k}[\Delta]$, the Stanley-Reisner ring of a simplicial complex Δ. Define

$$R_+ := \bigoplus_{\alpha \neq 0} R_\alpha$$

the so-called **irrelevant ideal**. Finally, M will be a $\mathbb{Z}^n$-graded R-module, i.e., $M = \bigoplus_{\alpha \in \mathbb{Z}^n} M_\alpha$ with $R_\alpha M_\beta \subset M_{\alpha+\beta}$.

1. Depth and Cohen-Macaulayness

Definition 1. *An element $a \in R$ is called a* **nonzerodivisor** *(or NZD) on M if $m \in M$, $am = 0$ implies $m = 0$. Equivalently, the map*

$$M \xrightarrow{\cdot a} M$$

given by multiplication by r is one-to-one.

Definition 2. *A sequence of homogeneous elements $\theta_1, \dots, \theta_s$ is a* **regular M-sequence***, or* **M-sequence** *for short, if θ_{i+1} is a NZD on $M/(\theta_1, \dots, \theta_i)M$ for $i = 0, \dots, s-1$.*

Definition 3. *The* **dimension** *of M, denoted $\dim_R M$ or $\dim M$, is the Krull dimension of $R/\operatorname{Ann}_R M$. The* **depth** *of M, denoted $\operatorname{depth}_R M$ or $\operatorname{depth} M$, is the length of a maximal M-sequence. It can be shown that every maximal M-sequence has the same length.*

In general $\operatorname{depth}_R M \leq \dim_R M$ (since any M-sequence of length s generates a height-s ideal of $R/\operatorname{Ann} M$). Equality is an important "niceness" condition which gets its own name:

Definition 4. *M is* **Cohen-Macaulay** *if $\operatorname{depth}_R M = \dim_R M$.*

2. Local Cohomology

Define the **torsion functor** Γ by

$$\Gamma(M) := \left\{ u \in M \;\middle|\; R_+^n u = 0 \text{ for } n \gg 0 \right\}.$$

It is routine to check that Γ is a covariant, left-exact functor. That is, a map $f : M \to N$ of graded R-modules induces a map $\Gamma(f) : \Gamma(M) \to \Gamma(N)$, and if f is injective then $\Gamma(f)$ is injective.

The ith local cohomology functor H^i (more precisely, $H^i_{R_+}$) can now be defined as the ith right derived functor of Γ:

$$H^i(M) = R^i\Gamma(M).$$

That is, one may calculate $H^i(M)$ by taking an injective resolution

$$I^\bullet : \quad 0 \;\to\; M \;\to\; I^0 \;\to\; I^1 \;\to\; \dots,$$

applying Γ, and defining $H^i(M) := H^i(\Gamma I^\bullet)$. (See a textbook on homological algebra for more details.)

Lemma 5. *For all i, the modules $H^i(M)$ are* **R_+-torsion***, i.e., they are killed by some power of R_+.*

Proof. By definition of Γ, every module of the form $\Gamma(N)$ is R_+-torsion. In particular, if $I^\bullet$ is an injective resolution, then every $\Gamma(I^i)$ is R_+-torsion, so the same is true of the cohomology modules of $\Gamma(I^\bullet)$. $\square$

The local cohomology functors are useful because they detect depth and dimension. Specifically, we have the following fact.

Theorem 6. *Let $e = \operatorname{depth}_R M$ and $d = \dim_R M$. Then:*

(1) $H^i(M) = 0$ *unless* $e \le i \le d$.

(2) $H^e(M) \neq 0$ *and* $H^d(M) \neq 0$.

Proof. We'll prove only (1), which is the case we really need. We proceed by induction on e.

If $H^0(M) \neq 0$, then every element of R_+ is a zerodivisor on M, which is exactly the statement that $e = 0$.

If $e = 0$, then

$$R_+ \subset \bigcup_{P \in \operatorname{Ass} M} P,$$

so R_+ is itself an associated prime. It is immediate that $\Gamma(M) = H^0(M) \neq 0$ as desired.

For the inductive step, assume that (1) is true for all R-modules N with $\operatorname{depth} N < e$. Let $a \in R_+$ be a homogeneous NZD on M and let $N = M/aM$. Then $\operatorname{depth}_R N = e - 1$, so by induction $H^i(N) = 0$ for $i < e - 1$ and $H^{e-1}(N) \neq 0$.

By general homological nonsense, the short exact sequence of R-modules

$$0 \to M \xrightarrow{\cdot a} M \to N \to 0 \tag{1}$$

induces a long exact sequence on cohomology

$$\ldots \to H^{i-1}(M) \to H^{i-1}(N) \to H^i(M) \xrightarrow{\cdot a} H^i(M) \to \ldots \tag{2}$$

If $i < e$, then $H^{i-1}(N) = 0$, so a is a NZD on $H^i(M)$. But $H^i(M)$ is R_+-torsion and so has depth 0. It follows that $H^i(M) = 0$.

On the other hand, if $i = e$ then the first three terms displayed in (2) are

$$0 \to H^{e-1}(N) \to H^e(M),$$

and $H^e(M) \neq 0$ since $H^{e-1}(N) \neq 0$. □

The local cohomology functors can be computed using the **Čech complex** $\check{C}^\bullet(x_1, \ldots, x_n;\ M)$, which is defined as

$$\check{C}^\bullet(x_1, \ldots, x_n;\ M) := \bigotimes_{i=1}^{n} \big((0 \to R \to R_{x_i} \to 0) \otimes M\big). \tag{3}$$

where $R_+ = \sqrt{(x_1, \ldots, x_n)}$ and $R_{x_i} = R[x_i^{-1}]$. The ith Čech module $\check{C}^i(x_1, \ldots, x_n;\ M)$ may be described explicitly as follows. For $F \subset [n]$, define

$$x_F = \prod_{i \in F} x_i$$

and let

$$R_F = R[x_F^{-1}].$$

Then

$$\check{C}_i^\bullet(x_1, \ldots, x_n;\ M) = \bigoplus_{\substack{F \subset [n] \\ |F| = i}} M_F \tag{4}$$

where $M_F = M \otimes R_F$ and the maps between adjacent terms in the Čech complex are given by the usual Koszul maps (just like simplicial cohomology.)

3. Hochster's Theorem

Let Δ be a simplicial complex on vertices $x_1, \ldots, x_n$, and $R = \mathbf{k}[\Delta] = \mathbf{k}[x_1, \ldots, x_n]/I_\Delta$ its Stanley-Reisner ring. With respect to the obvious $\mathbb{N}^n$-grading, we have $R_+ = (x_1, \ldots, x_n)$.

Definition 7. *Let $F \in \Delta$. The* **star** *of F with respect to Δ is*

$$\operatorname{st}_\Delta F := \{G \in \Delta \mid G \cup F \in \Delta\}$$

and the **link** *of F with respect to Δ is*

$$\operatorname{lk}_\Delta F := \{G \in \Delta \mid G \cup F \in \Delta,\ G \cap F = \emptyset\}.$$

We suppress the subscript when possible.

Note that both $\operatorname{st} F$ and $\operatorname{lk} F$ are simplicial complexes, and that $\operatorname{st} F = \langle F \rangle * \operatorname{lk} F$. For instance, if $\Delta = \langle 123, 14, 24 \rangle$ and $F = 12$, then $\operatorname{lk} F = \langle 3 \rangle$ and $\operatorname{st} F = \langle 123 \rangle$.

Let $q_1, \ldots, q_n$ be indeterminates. Denote by $\operatorname{Hilb}(M;\, q)$ the finely graded Hilbert series of M, i.e.,

$$\operatorname{Hilb}(M;\, q) := \sum_{\alpha \in \mathbb{Z}^n} q^\alpha \dim_{\mathbf{k}} M_\alpha,$$

where $q^\alpha = q_1^{\alpha_1} \ldots q_n^{\alpha_n}$. Also, let $\tilde{H}_i(\Delta;\, \mathbf{k})$ denote the ith reduced simplicial homology of a simplicial complex Δ with coefficients in $\mathbf{k}$.

For $\alpha \in \mathbb{Z}^n$, define

$$\begin{aligned} F(\alpha) &= \{x_i \mid \alpha_i < 0\}, \\ G(\alpha) &= \{x_i \mid \alpha_i > 0\}, \\ \operatorname{supp}(\alpha) &= F(\alpha) \cup G(\alpha) = \{x_i \mid \alpha_i \neq 0\}. \end{aligned}$$

Theorem 8 (Hochster)**.** *We have*

$$\operatorname{Hilb}(H^i(\mathbf{k}[\Delta]);\, q) = \sum_{F \in \Delta} \dim_{\mathbf{k}} \tilde{H}_{i-|F|-1}(\operatorname{lk}_\Delta F;\, \mathbf{k}) \prod_{x_i \in F} \frac{q_i^{-1}}{1 - q_i^{-1}}.$$

Proof. We compute $H^i(R)$ explicitly as the ith cohomology of the Čech complex $\check{C}^\bullet = \check{C}^\bullet(x_1, \ldots, x_n;\, M)$. If $F \notin \Delta$, then the ring R_F is zero, because $x_F = 0$ in R. On the other hand, if $F \in \Delta$, then the variables in F become units in R_F, and those not in $\operatorname{st}_\Delta F$ get killed (since they annihilate the unit x_F). That is,

$$R_F = \mathbf{k}\left[\{x_i, x_i^{-1} : i \in F\} \cup \{x_j : x_j \in \operatorname{lk} F\}\right] \otimes R.$$

Let $\alpha \in \mathbb{Z}^n$. We will compute the αth graded piece $\check{C}^\bullet_\alpha$ of the Čech complex. If $\operatorname{supp}(\alpha) \notin \Delta$, then $\check{C}^\bullet_\alpha = 0$, because $R_\alpha = 0$ and adjoining inverses doesn't change this. So suppose that $\operatorname{supp}(\alpha) \in \Delta$. Let $F = F(\alpha)$, $j = |F|$, and $G = G(\alpha)$. A priori, we have

$$\check{C}^r_\alpha = \left[\bigoplus_{|F'|=r} R_{F'}\right]_\alpha = \bigoplus_{|F'|=r} [R_{F'}]_\alpha. \tag{5}$$

A whole bunch of these summands are zero. Specifically, for $R_{F'}$ to be nonzero, we must have $F' \in \Delta$ (as previously noted), $F' \supset F$ (since the variables in F must be units in the αth graded piece of the Čech complex), and $F' \cup G \in \Delta$ (so that x^α itself is nonzero). This is all equivalent to the condition that $F'' = F' \setminus F$ belong to $\operatorname{lk}_{\operatorname{st} G} F$, so we may write

$$\check{C}^r_\alpha = \left[\bigoplus_{\substack{F'' \in \operatorname{lk}_{\operatorname{st} G} F \\ |F''| = r-j}} R_{F \cup F''}\right]_\alpha. \tag{6}$$

The maps in $\check{C}^r_\alpha$ correspond to the usual coboundary maps of the simplicial cochain complex of $\mathrm{lk}_{\mathrm{st}\,G} F$, shifted by $j+1$. That is,

$$[H^i(R)]_\alpha = \tilde{H}^{i-j-1}(\mathrm{lk}_{\mathrm{st}\,G} F;\ \mathbf{k}) \tag{7}$$

$$\cong \tilde{H}_{i-j-1}(\mathrm{lk}_{\mathrm{st}\,G} F;\ \mathbf{k}) \tag{8}$$

since this is a finite-dimensional $\mathbf{k}$-vector space, hence isomorphic to its dual. (The isomorphism is not canonical, but we don't care because we're really only interested in its dimension.)

If $G \neq \emptyset$, then $\mathrm{lk}_{\mathrm{st}\,G} F$ is a cone over G. In particular it is contractible, so $\tilde{H}_\bullet(\mathrm{lk}_{\mathrm{st}\,G} F;\ \mathbf{k}) = 0$. Therefore we only have nonzero terms when $G = \emptyset$, so $\mathrm{lk}_{\mathrm{st}\,G} F = \mathrm{lk}_\Delta F$ and the last equation becomes

$$[H^i(R)]_\alpha \cong \tilde{H}_{i-j-1}(\mathrm{lk}\, F;\ \mathbf{k}). \tag{9}$$

Therefore

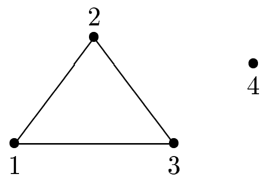
$$\mathrm{Hilb}(H^i(\mathbf{k}[\Delta]);\ q) = \sum_{F\in\Delta} \sum_{\substack{\alpha:\\ \mathrm{supp}(\alpha)=F(\alpha)=F}} \dim_{\mathbf{k}} \tilde{H}_{i-|F|-1}(\mathrm{lk}\, F;\ \mathbf{k}) q^\alpha \tag{10}$$

$$= \sum_{F\in\Delta} \dim_{\mathbf{k}} \tilde{H}_{i-|F|-1}(\mathrm{lk}\, F;\ \mathbf{k}) \sum_{\substack{\alpha:\\ \mathrm{supp}(\alpha)=F(\alpha)=F}} q^\alpha \tag{11}$$

$$= \sum_{F\in\Delta} \dim_{\mathbf{k}} \tilde{H}_{i-|F|-1}(\mathrm{lk}\, F;\ \mathbf{k}) \prod_{x_i\in F} \frac{q_i^{-1}}{1-q_i^{-1}} \tag{12}$$

as desired. □

Here's an example (from Stanley) of computing a Čech complex. Let Δ be the complex $\langle 12, 13, 23, 4\rangle =$

(13) 

and $R = \mathbf{k}[\Delta] = \mathbf{k}[x_1, x_2, x_3, x_4]/(x_1x_4, x_2x_4, x_3x_4, x_1x_2x_3)$. Then the Čech complex is

$$0 \to \underbrace{R}_{\check{C}^0} \to \underbrace{R_1 \oplus R_2 \oplus R_2 \oplus R_3 \oplus R_4}_{\check{C}^1} \to \underbrace{R_{12} \oplus R_{13} \oplus R_{23}}_{\check{C}^2} \to 0,$$

where $R_1 = R[x_1^{-1}]$, $R_{12} = R[x_1^{-1}, x_2^{-1}]$, etc.

- For $\alpha = (0,0,0,0)$, so $F(\alpha) = G(\alpha) = \emptyset$, we have
$$\mathrm{lk}_{\mathrm{st}\,G} F = \mathrm{lk}_\Delta \emptyset = \Delta,$$
so $[H^i(R)]_\alpha = \tilde{H}_{i-1}(\Delta;\ \mathbf{k})$.
- For $\alpha = (-2,3,0,0)$, we have
$$F = \{x_1\}, \qquad G = \{x_2\}, \qquad \mathrm{lk}_{\mathrm{st}\,G} F = \langle x_2\rangle \text{ (i.e., a point)}$$
so $[H^i(R)]_\alpha = \tilde{H}_{i-2}(\text{point}) = 0$.

4. Reisner's Theorem

Let Δ be a simplicial complex and $R = -\mathbf{k}[Delta]$. Let $d = \dim R = 1 + \dim \Delta$. We will say that Δ satisfies **Reisner's criterion** if for all $F \in \Delta$, and $i < \dim(\mathrm{lk}\, F)$. we have

$$\tilde{H}_i(\mathrm{lk}\, F;\ \mathbf{k}) = 0.$$

Theorem 9. *Δ is Cohen-Macaulay if and only if it satisfies Reisner's criterion.*

Remark: Δ is **Gorenstein** (a stronger condition than Cohen-Macaulayness) if in addition $\tilde{H}_i(\operatorname{lk} F;\ \mathbf{k}) \cong \mathbf{k}$ for $i = \dim(\operatorname{lk} F)$.

Proof. First, we show that a Cohen-Macaulay complex is pure (i.e., all maximal faces have the same dimension). Indeed, if Δ is Cohen-Macaulay of dimension $d-1$ and $\dim F < d-1$, then $\tilde{H}_{-1}(\operatorname{lk} F) = 0$ by Hochster's theorem, so $\operatorname{lk} F \neq 0$ and F is not maximal. (This can also be shown without Hochster's theorem; see Bruns and Herzog, p. 210.)

Next, we show that a complex Δ satisfying Reisner's criterion is pure. If $\dim F = 0$ then there is nothing to show. Otherwise, we induct on dimension. Reisner's criterion gives $\tilde{H}_0(\Delta) = \tilde{H}_0(\operatorname{lk} \emptyset) = 0$, so Δ is connected. Moreover, for every vertex v, the subcomplex $\operatorname{lk}\{v\}$ of Δ satisfies Reisner's criterion and has dimension less than that of Δ, so it is pure by induction. Now, for any maximal face F, let $v, w \in F$; we have

$$\dim \operatorname{lk}\{v\} = |F - v| - 1 = |F - w| - 1 = \dim \operatorname{lk}\{w\};$$

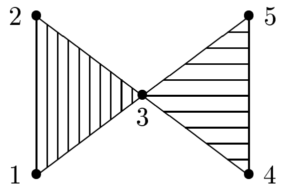
by connectedness all links of vertices must have the same dimension, and the same equation implies that Δ is pure.

By these two observations together, we may assume that Δ is pure, so $|F| = d$ for all maximal faces and $\dim(\operatorname{lk} F) = d - |F| - 1$ for all faces. So Cohen-Macaulayness

$$\begin{aligned} F \text{ is Cohen-Macaulay} \quad &\iff \quad \mathbf{k}[\Delta] \text{ is Cohen-Macaulay} \\ &\iff \quad \tilde{H}_{j-|F|-1}(\operatorname{lk} F;\ \mathbf{k}) = 0 \text{ for } j < d,\ F \in \Delta \end{aligned}$$

which is exactly Reisner's criterion (set $j = i + |F| + 1$). $\square$

By the way, a pure connected simplicial complex Δ certainly need not be Cohen-Macaulay (unless $\dim \Delta \leq 1$). The "minimal" example is the complex $\langle 123, 345 \rangle =$



which is not Cohen-Macaulay because $\operatorname{lk}(3) = \langle 12, 45 \rangle$ is disconnected, so has $\tilde{H}_0 \neq 0$. (Also, the h-vector of this complex can be computed as $(1, 2, -1)$. In a Cohen-Macaulay complex, every entry of the h-vector is nonnegative.)