

STATISTICS ON ℓ -INTERVAL PARKING FUNCTIONS

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ABSTRACT. The *displacement* of a car with respect to a parking function is the number of spots it must drive past its preferred spot in order to park. An ℓ -interval parking function is one in which each car has displacement at most ℓ . Among our results, we enumerate ℓ -interval parking functions with respect to statistics such as inversion, displacement, and major index. We show that 1-interval parking functions with fixed displacement exhibit a cyclic sieving phenomenon. We give closed formulas for the number of 1-interval parking functions with a fixed number of inversions. We prove that a well-known bijection of Foata preserves the set of ℓ -interval parking functions exactly when $\ell \leq 2$ or $\ell \geq n - 2$, which implies that the inversion and major index statistics are equidistributed in these cases.

CONTENTS

1. Introduction	1
2. Background	3
2.1. Parking functions and ℓ -interval parking functions	3
2.2. Block structure of unit interval parking functions	4
2.3. Permutation statistics	5
3. Enumeration by counting through permutations	5
3.1. A cyclic sieving phenomenon	7
3.2. From $\ell = 1$ to $\ell = 2$	9
4. Enumeration via ciphers	11
4.1. The Avalos–Bly and cipher bijections	12
4.2. Ciphers and Lehmer codes	13
4.3. Unit Fubini rankings	14
5. Foata invariance and inv-maj equidistribution	15
5.1. The Foata transform on words	16
5.2. Permuting parking functions	16
5.3. Proof of Theorem 5.1	18
5.4. Enumeration by major index	22
6. Further Problems	23
Acknowledgments	23
References	24

1. INTRODUCTION

This paper is about enumerating ℓ -interval parking functions with respect to statistics such as inversion, displacement, and major index. Parking functions first appeared in the work of Konheim and Weiss [19] and have become standard objects in combinatorics. Suppose that n cars attempt to park in n spots $1, 2, \dots, n$ on a one-way street. Each car drives to its preferred spot a_i and parks there if possible; if that spot is taken,

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the car continues along the street and parks in the first available spot. If all cars are able to park successfully, the tuple $\alpha = (a_1, a_2, \dots, a_n) \in [n]^n$ is called a **parking function**. In this case, if the i th car parks in spot s_i , then the number $s_i - a_i$ is its **displacement**.

An ℓ -interval parking function, or IPF for short, is a parking function in which each car has displacement less than or equal to some fixed number ℓ ; we write $\text{IPF}_n(\ell)$ for the set of all ℓ -interval parking functions of length n . These objects were first studied by Aguilar-Fraga et al. [2]. (The idea of restricting each car to an interval was considered by Colaric et al. [8], although in that paper the intervals were not required to have the same length.) The case $\ell = 1$ is of particular interest. These **unit interval parking functions**, or UPFs, are enumerated by the Fubini numbers [5, 17], which also count the faces of the permutohedron [7]. UPFs are also related to Boolean intervals in weak Bruhat order on the symmetric group [10]. Enumerative results on ℓ -interval parking functions with $\ell \geq 1$ also give connections to Dyck paths with restricted heights and to preferential arrangements [2].

Bradt et al. [5, Theorem 2.9] proved that unit interval parking functions exhibit a tightly controlled *block structure*, which we describe in Section 2.2. To summarize, if $\alpha = (a_1, \dots, a_n)$ is a UPF with weakly increasing rearrangement $\alpha^\uparrow = (a'_1, \dots, a'_n)$, then $a'_i \in \{i, i-1\}$ for every i , and if α^\uparrow is partitioned into blocks by inserting a separator before every entry with $a'_i = i$, then the elements of each block appear left-to-right in α . This structure is key to many of the enumerative results in this paper. No analogous structure is known for ℓ -interval parking functions with $\ell \geq 2$.

In Section 3, we study the generating function $\Phi_{n,\ell}(q, t)$ for $\text{IPF}_n(\ell)$ by total displacement and inversion number. We use the technique of “counting through permutations”: to count a set X , define an appropriate function $\mathcal{C}: X \rightarrow \mathfrak{S}_n$, count each fiber, and add up the counts. The formula produced by this method cannot always be simplified further, but it is typically much more computationally efficient than brute-force enumeration. We give a general formula for $\Phi_{n,\ell}(q, t)$ in Theorem 3.1, involving the q -analogues of certain numbers $L_\ell(\sigma; i)$ associated with a permutation σ (see eqn. (2)). The case $\ell = n-1$ was previously obtained in [9]. For UPFs, the polynomial $\Phi_{n,1}(q, t)$ has a much simpler expression (Theorem 3.3). In addition, if we write $\Phi_{n,1}(q, t) = \sum_k q^k f_{n,k}(t)$, then each polynomial $f_{n,k}(t)$ exhibits a cyclic sieving phenomenon (Theorem 3.6). There is an analogue of Theorem 3.3 for $\ell = 2$ (Theorem 3.11), obtained by reducing the enumeration of 2-interval parking functions to that of unit interval parking functions. Potentially, this technique could be extended to give (more complicated) formulas for $\Phi_{n,\ell}(q, t)$ for $n \geq 3$.

In Section 4, we present explicit closed formulas for the number of $\text{upf}_{n,k}^{\text{inv}}$ of UPFs of length n with exactly k inversions. The main tool is a bijection of Avalos and Bly [4], which we modify to obtain a bijective labeling of these UPFs by objects we call **ciphers** (Theorem 4.1), which can be counted by elementary “stars and bars” methods. We work out the formulas for $k \leq 3$, which suggest that $\text{upf}_{n,k}^{\text{inv}} = O(2^{n-2k} n^k)$ in general (Conjecture 4.5). As another application of ciphers, we recover the main result of [10], a bijection between unit Fubini rankings (UPFs with all blocks of sizes 1 or 2) and Boolean intervals in weak order (Theorem 4.8).

In Section 5, we prove that the inversion and major index statistics are equidistributed on ℓ -interval parking functions when $\ell \leq 2$ or $\ell \geq n-2$. Recall that the **major index** of $(w_1, w_2, \dots, w_n) \in \mathbb{Z}^n$ is $\text{maj}(w) = \sum_{i \in \text{Des}(w)} i$, where $\text{Des}(w) = \{i \in [n-1] : w_i > w_{i+1}\}$. Foata [11] famously constructed a content-preserving bijection F on the set of words over a fixed alphabet with the property that $\text{inv}(F(w)) = \text{maj}(w)$, thus proving that the two statistics are equidistributed on any set of words invariant under permutation (as proved earlier by MacMahon algebraically [20]). Parking functions are invariant under permutation, but in general, the set of ℓ -interval parking functions is not. Nevertheless, we prove (Theorem 5.1) that it preserves the set $\text{IPF}_n(\ell)$ precisely when $\ell \in \{0, 1, 2, n-2, n-1\}$. The most difficult cases are $\ell = 1$ and (especially) $\ell = 2$. Along the way, we prove several results (Lemma 5.4, Proposition 5.5, Lemma 5.6) to describe how small changes in a parking function (such as transposing two consecutive entries) affect where the cars park; these facts may be of independent interest and utility. In Section 5.4, we apply Theorem 5.1 to the results of Section 3 to enumerate $\text{IPF}_n(\ell)$ by major index.

We conclude in Section 6 with some open problems for future study.

2. BACKGROUND

In this section, we provide background and definitions related to parking functions and ℓ -interval parking functions. A standard source for the basics of parking functions is [26]. For interval and ℓ -interval parking functions, see, e.g., [2, 5, 8].

Remark 2.1. Throughout we interchangeably regard tuples (a_1, a_2, \dots, a_n) and words $a_1 a_2 \cdots a_n$ as identical objects, depending on context.

Throughout, the symbol \mathbb{N} denotes the set of positive integers. For $m, n \in \mathbb{N}$, we set $[n] = \{1, 2, \dots, n\}$ and $[m, n] = \{m, m+1, \dots, n\}$. For a tuple $\alpha = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$, we write α^\uparrow for the tuple obtained by sorting the elements of α in weakly increasing order. The symbol \mathfrak{S}_n denotes the symmetric group of permutations of $[n]$. Throughout, we write permutations in one-line notation.

2.1. Parking functions and ℓ -interval parking functions. Parking functions are well-studied objects that were defined in the introduction; one survey of their properties is [26].

We write PF_n for the set of parking functions of length n . It is known that $|\text{PF}_n| = (n+1)^{n-1}$ [19]. Moreover, a tuple $\alpha = (a_1, a_2, \dots, a_n) \in [n]^n$ is a parking function if and only if its weakly increasing rearrangement $\alpha^\uparrow = (a'_1, a'_2, \dots, a'_n)$ satisfies $a'_i \leq i$ for all i , or equivalently if there is some permutation $\sigma \in \mathfrak{S}_n$ such that $a_i \leq \sigma_i$ for all $i \in [n]$ [26, p. 4]. We refer to this property as the **rearrangement criterion**. It follows that if $\alpha \in \text{PF}_n$ and $\beta = (b_1, b_2, \dots, b_n)$ satisfy $1 \leq b_i \leq a_i$ for all $i \in [n]$, then $\beta \in \text{PF}_n$.

We write PF_n^\uparrow for the set of weakly increasing parking functions of length n . It is known that $|\text{PF}_n^\uparrow| = \text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}$, the n th Catalan number [23, Exercise 6.19(s)].

Definition 2.2. Let $\alpha = (a_1, a_2, \dots, a_n) \in \text{PF}_n$. The **car permutation** of α is $\text{car}_\alpha = (\text{car}_\alpha(1), \dots, \text{car}_\alpha(n)) \in \mathfrak{S}_n$, where $\text{car}_\alpha(i)$ denotes the car that parks in the i th spot. The **spot permutation** of α is $\text{spot}_\alpha = (\text{spot}_\alpha(1), \dots, \text{spot}_\alpha(n)) \in \mathfrak{S}_n$, where $\text{spot}_\alpha(i)$ denotes the number of spot that the i th car parks in. Note car_α and spot_α are permutations written as tuples and $\text{car}_\alpha = \text{spot}_\alpha^{-1}$.

Example 2.3. If $\alpha = (1, 4, 4, 3, 2, 2) \in \text{PF}_6$, then $\text{car}_\alpha = (1, 5, 4, 2, 3, 6)$ and $\text{spot}_\alpha = (1, 4, 5, 3, 2, 6)$.

Remark 2.4. Much of the parking function literature uses the term “outcome” for what we have called the car permutation of a parking function α . We have adopted the terms “car permutation” and “spot permutation” because we work with both of them in different parts of this paper, and wish to avoid any possible ambiguity in the use of the word “outcome”. In the Sage ParkingFunctions library, the car permutation is computed as `alpha.cars_permutation()` and the spot permutation is `alpha.parking_permutation()`.

Definition 2.5. Let $\alpha = (a_1, a_2, \dots, a_n) \in \text{PF}_n$. The **displacement** of the i th car is $\text{disp}_\alpha(i) = \text{spot}_\alpha(i) - a_i$. That is, $\text{disp}_\alpha(i)$ is the number of additional spaces car i has to drive past its preferred spot in order to park. The **(total) displacement**¹ of α is $\text{disp}(\alpha) = \sum_{i=1}^n \text{disp}_\alpha(i)$. The **maximum displacement** of α is $\text{maxdisp}(\alpha) = \max(\text{disp}_\alpha(1), \text{disp}_\alpha(2), \dots, \text{disp}_\alpha(n))$.

Example 2.6 (Continuing Example 2.3). If $\alpha = (1, 4, 4, 3, 2, 2) \in \text{PF}_6$, then the displacements of cars $1, 2, \dots, 6$ are $0, 0, 1, 0, 0, 4$, respectively.

Definition 2.7. For a positive integer $\ell \leq n$, we say that α is an **ℓ -interval parking function** if $\text{maxdisp}(\alpha) \leq \ell$. The set of ℓ -interval parking functions of length n is denoted $\text{IPF}_n(\ell)$. In the case $\ell = 1$, we call α a **unit interval parking function** and write $\text{UPF}_n = \text{IPF}_n(1)$.

Example 2.8. We can give exact formulas for $|\text{IPF}_n(\ell)|$ in some extreme cases.

$\ell = n - 1$: Every $\alpha \in \text{PF}_n$ satisfies $\text{maxdisp}(\alpha) \leq n - 1$, so

$$|\text{IPF}_n(n-1)| = |\text{PF}_n| = (n+1)^{n-1}.$$

$\ell = n - 2$: Let $\alpha = (a_1, a_2, \dots, a_n) \in \text{PF}_n$. Then $\text{maxdisp}(\alpha) = n - 1$ if and only if the car that parks in spot n preferred spot 1. This can happen only if (i) that car was the last to park and (ii) the first

¹Total displacement is also called **area**, since it is exactly the area under the labeled Dyck path representing α ; see, e.g., [26, pp. 54–55].

$n - 1$ cars parked in the first $n - 1$ spots. Equivalently, $a_n = 1$ and $(a_1, a_2, \dots, a_{n-1}) \in \text{PF}_{n-1}$. Therefore,

$$\begin{aligned} |\text{IPF}_n(n-2)| &= |\text{PF}_n| - |\{\alpha \in \text{PF}_n : \text{maxdisp}(\alpha) = n-1\}| \\ &= (n+1)^{n-1} - n^{n-2}. \end{aligned}$$

$\ell = 1$: The parking functions α with $\text{maxdisp}(\alpha) = 1$ are precisely the unit interval parking functions. Therefore,

$$|\text{IPF}_n(1)| = |\text{UPF}_n| = \text{Fub}_n,$$

where Fub_n is the number of ordered set compositions of $[n]$ (see Remark 2.12 below).

$\ell = 0$: The parking functions α with $\text{maxdisp}(\alpha) = 0$ are precisely the permutations. Therefore,

$$|\text{IPF}_n(0)| = n!.$$

It is immediate from the definition that any rearrangement of a parking function is a parking function. The same is not true for ℓ -parking functions because maximum displacement may change under rearrangement. For instance, $\text{maxdisp}(112) = 1$ but $\text{maxdisp}(121) = 2$. On the other hand, a weaker result holds for all ℓ , as we prove later (Proposition 5.5).

2.2. Block structure of unit interval parking functions. In [5], it was proved that unit interval parking functions are characterized by a tightly controlled **block structure**, as we now explain.

Definition 2.9. A tuple $\beta = (b_1, b_2, \dots, b_n) \in [n]^n$ is called a **block word** if $b_i \in \{i, i-1\}$ for each i (so in particular, $b_1 = 1$). When we write a block word, we place a separator before each b_i such that $b_i = i > 1$. The maximal subwords between separators are called **blocks**. Note that a block word is determined by the ordered list of lengths of its blocks.

For example, the block words of length 3 are 112, 11 | 3, 1 | 22, 1 | 2 | 3.

We now reformulate the characterization of unit interval parking functions based on their block structure.

Theorem 2.10 ([5, Theorem 2.9]). *Let $\alpha = (a_1, a_2, \dots, a_n) \in \text{UPF}_n$ and $\alpha^\uparrow = (a'_1, a'_2, \dots, a'_n)$. Then,*

- (1) *α^\uparrow is a block word. If $\alpha^\uparrow = \pi_1 | \pi_2 | \dots | \pi_m$, we refer to the collection of π_j 's as the **block structure** of α .*
- (2) *The entries in each block appear in increasing order in α . That is, α is a shuffle of its blocks.*
- (3) *Given a block word β , every shuffle of its blocks is a unit interval parking function. Therefore,*

$$|\{\alpha \in \text{UPF}_n : \alpha^\uparrow = \beta\}| = \binom{n}{|\pi_1|, |\pi_2|, \dots, |\pi_m|}.$$

Next we state a bijection between the set of unit interval parking functions of length n with m blocks and the set of surjective functions from $[n]$ to $[m]$.

Corollary 2.11. *Let $\alpha = (a_1, a_2, \dots, a_n) \in \text{UPF}_n$ have block structure $\pi_1 | \pi_2 | \dots | \pi_m$. For each $i \in [n]$, let $s(i)$ be the index of the block to which a_i belongs. Thus, s is a surjective function $[n] \rightarrow [m]$ (equivalently, an ordered partition of $[n]$ into m nonempty blocks). Then the map sending $\alpha \mapsto s$ is a bijection*

$$\{\alpha \in \text{UPF}_n : \alpha \text{ has } m \text{ blocks}\} \rightarrow \{\text{surjective functions } [n] \rightarrow [m]\}.$$

Remark 2.12. A surjective function $[n] \rightarrow [m]$ can be regarded as an (ordered) set composition of $[n]$ with m nonempty blocks. In particular, the number of unit interval parking functions of length n equals the **Fubini number** Fub_n of ordered set compositions of $[n]$, as observed by Hadaway [17]. The Fubini numbers are sequence #A000670 in [18]. We obtain a new proof this fact following Theorem 3.3.

Example 2.13. Let $\alpha = (8, 1, 5, 5, 1, 2, 4, 7) \in \text{UPF}_8$. Then the block structure (as in (2) of Theorem 2.10) is $\alpha^\uparrow = 112 | 4 | 55 | 7 | 8$, and the corresponding element of $[5]^8$, as in Corollary 2.11, is $s = 51331124$.

Block structure appears to be unique to the case $\ell = 1$; no result analogous to Theorem 2.10 is known for ℓ -interval parking functions when $\ell > 1$. We state this as an open problem in Section 6.

2.3. Permutation statistics. Let $w = w_1 \cdots w_n \in \mathbb{N}^n$ be a sequence of positive integers. An **inversion** of w is a pair of indices (i, j) such that $1 \leq i < j \leq n$ and $w_i > w_j$. We let $\text{Inv}(w)$ be the set of inversions of w and set $\text{inv}(w) = |\text{Inv}(w)|$. A **descent** is a position $i \in [n - 1]$ such that $w_i > w_{i+1}$. We let $\text{Des}(w)$ be the set of descents of w and set $\text{des}(w) = |\text{Des}(w)|$. The **major index** of w is

$$\text{maj}(w) = \sum_{i \in \text{Des}(w)} i.$$

By a bijection of Foata [20, 12] that we describe in detail in Section 5, inversions and major index are **equidistributed** as statistics on the set of permutations, meaning that there are as many permutations $w \in \mathfrak{S}_n$ with $\text{inv}(w) = j$ as there are with $\text{maj}(w) = j$, for each nonnegative integer j . More generally, inversions and major index are equidistributed on any set of words $W \subseteq \mathbb{N}^n$ satisfying the property that if $w \in W$ and u is a rearrangement of w , then $u \in W$ [13]. Hence, inversions and major index are equidistributed on the set of parking functions. In Section 5, we completely characterize the pairs (n, ℓ) such that inversions and major index are equidistributed on $\text{IPF}_n(\ell)$.

It is immediate from the definition that $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$ for every permutation σ . The following result asserts that inversions of parking functions can be computed as inversions of permutations.

Lemma 2.14. *Let $\alpha \in \text{UPF}_n$. Then $\text{Inv}(\alpha) = \text{Inv}(\text{spot}_\alpha)$.*

Proof. Let $\alpha = (a_1, a_2, \dots, a_n) \in \text{UPF}_n$ and let (i, j) be an inversion of $\tau = \text{spot}_\alpha$. That is, car i arrives earlier than car j and parks to its right. It follows that $a_i > a_j$, so $(i, j) \in \text{Inv}(\alpha)$.

Now let (i, j) be an inversion of α . Since $a_i > a_j$ and $i < j$, it follows by (3) of Theorem 2.10 that a_i and a_j belong to different blocks. Since cars in earlier blocks have earlier preferences than cars in later blocks and park earlier than cars in later blocks, this means that $\tau_i > \tau_j$. Thus every inversion of α is an inversion of τ . \square

An **ascent** of a word $w \in \mathbb{N}^n$ is a position $i \in [n - 1]$ such that $w_i < w_{i+1}$. We let $\text{Asc}(w)$ be the set of ascents of w and set $\text{asc}(w) = |\text{Asc}(w)|$. It is known [24, Exercise 1.133a] that

$$(1) \quad \sum_{\sigma \in \mathfrak{S}_n} 2^{\text{asc}(\sigma)} = \text{Fub}_n = |\text{UPF}_n|$$

(see Corollary 2.11 and Remark 2.12). We obtain a more refined enumerative result in Theorem 3.3.

3. ENUMERATION BY COUNTING THROUGH PERMUTATIONS

In this section, we study the generating function

$$\Phi_{n,\ell} = \Phi_{n,\ell}(q, t) = \sum_{\alpha \in \text{IPF}_n(\ell)} q^{\text{disp}(\alpha)} t^{\text{inv}(\text{car}_\alpha)}.$$

We give formulas for $\Phi_{n,\ell}$ for various values of ℓ by partitioning $\text{IPF}_n(\ell)$ into the fibers of the function $\mathcal{C}_{n,\ell}: \text{IPF}_n(\ell) \rightarrow \mathfrak{S}_n$ defined by $\mathcal{C}_{n,\ell}(\alpha) = \text{car}_\alpha$.

For a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$ and position $i \in [n]$, define

$$(2) \quad L_\ell(i; \sigma) = \min(\ell + 1, i - t + 1),$$

where $\sigma_t, \sigma_{t+1}, \dots, \sigma_i$ is the longest contiguous subsequence of σ such that $\sigma_k \leq \sigma_i$ for all $k \in [t, i]$. That is, $L_\ell(i; \sigma)$ is the number of possible preferred spots for car σ_i that results in car σ_i parking in spot i , subject to the choices of the previous cars and the constraint that the result must be an ℓ -interval parking function. It follows that for all $\ell \in \mathbb{N}$ and $\sigma \in \mathfrak{S}_n$, we have

$$(3) \quad |\mathcal{C}_{n,\ell}^{-1}(\sigma)| = \prod_{i=1}^n L_\ell(i; \sigma).$$

The $\ell = n - 1$ case of (3) is [19, Eqns. 2.1, 2.2]; see also [9, Proposition 3.1] and [23, Exercise 5.49d]. Our $L_{n-1}(i; \sigma)$ corresponds to $\tau_{i,n}(\sigma)$ in [19] and to $\ell(i; \sigma)$ in [9].

In the case that $\sigma = \text{id}$ is the identity permutation, the number $L_\ell(i, \text{id})$ is either i (if $i \leq \ell$) or $\ell + 1$ (if $\ell + 1 \leq i \leq n$), so $|\mathcal{C}_{n,\ell}^{-1}(\text{id})| = (\ell + 1)^{n-\ell} \cdot \ell!$. These numbers occur as sequence #A299504 in [18], where they count certain permutations; here, they count certain parking functions.

It follows from (3) that

$$(4) \quad |\text{IPF}_n(\ell)| = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n L_\ell(i; \sigma).$$

This is more efficient for explicit computation than iterating over all parking functions and picking out those with maximum displacement at most ℓ . In general, we do not have a closed formula for $|\text{IPF}_n(\ell)|$ except in extreme cases; see Example 2.8.

In [9, Props. 5.2, 5.3], the authors provide q -analogues of equations (3) and (4) for parking functions using total displacement. They extend this to a qt -analogue in [9, Cor. 5.3] using permutation statistics. We can obtain a similar analogue for ℓ -interval parking functions.

Theorem 3.1. *Let $0 \leq \ell \leq n - 1$. For all $\sigma \in \mathfrak{S}_n$, we have*

$$(5) \quad \sum_{\alpha \in \mathcal{C}_{n,\ell}^{-1}(\sigma)} q^{\text{disp}(\alpha)} = \prod_{i=1}^n [L_\ell(i; \sigma)]_q,$$

and therefore,

$$(6) \quad \Phi_{n,\ell}(q, t) = \sum_{\alpha \in \text{IPF}_n(\ell)} q^{\text{disp}(\alpha)} t^{\text{inv}(\text{car}_\alpha)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{inv}(\sigma)} \prod_{i=1}^{\ell} [L_\ell(\sigma; i)]_q.$$

Proof. Let $\sigma \in \mathfrak{S}_n$ and $\alpha = (a_1, a_2, \dots, a_\ell) \in \mathcal{C}_{n,\ell}^{-1}(\sigma)$. For each i , if car σ_i wants to park in spot i , then a_i must be selected from $[i - L_\ell(i; \sigma) + 1, i]$. Hence, for each $j \in \{0, \dots, L_\ell(i; \sigma) - 1\}$, the selection of $a_i = i - j \in [i - L_\ell(i; \sigma) + 1, i]$ contributes j to the total displacement. Since $[L_\ell(i; \sigma)]_q = 1 + q + \dots + q^{L_\ell(i; \sigma) - 1}$, we obtain (5). For (6), observe that

$$\Phi_{n,\ell}(q, t) = \sum_{\sigma \in \mathfrak{S}_n} \left(\sum_{\alpha \in \mathcal{C}_{n,\ell}^{-1}(\sigma)} q^{\text{disp}(\alpha)} \right) t^{\text{inv}(\sigma)}$$

and then apply (5). \square

The $\ell = n - 1$ case is [9, Cor. 5.3]. As noted there, one can replace inversion number in (6) with *any* statistic on permutations. For the case $\ell = n - 2$, we can rewrite the formulas in Theorem 3.1 in terms of displacement enumerators for (classical) parking functions. This also follows from the description of $\text{PF}_n \setminus \text{IPF}_n(n - 2)$ in Example 2.8.

Corollary 3.2. *For all $n \geq 2$,*

$$\Phi_{n,n-2}(q, t) = \sum_{\alpha \in \text{PF}_n} q^{\text{disp}(\alpha)} t^{\text{inv}(\alpha)} - (qt)^{n-1} \sum_{\beta \in \text{PF}_{n-1}} q^{\text{disp}(\beta)} t^{\text{inv}(\beta) - \text{ones}(\beta)},$$

where $\text{ones}(\beta) = |\{i \in [n-1] : b_i = 1\}|$.

In the case $\ell = 1$ (unit interval parking functions), the formula of Theorem 3.1 can be written in a very explicit form as a generating function for permutations.

Theorem 3.3. *For all $n \geq 1$,*

$$\Phi_{n,1}(q, t) = \sum_{\sigma \in \mathfrak{S}_n} (1 + q)^{\text{asc}(\sigma)} t^{\text{inv}(\sigma)}.$$

Proof. Taking $\ell = 1$ in (2), we see that

$$L_1(i; \sigma) = \begin{cases} 2 & \text{if } i - 1 \text{ is an ascent of } \sigma, \\ 1 & \text{otherwise.} \end{cases}$$

The result now follows from Lemma 2.14 and Theorem 3.1, together with the identity $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$. \square

Setting $q = t = 1$ in Theorem 3.3 recovers (1). More generally, setting $t = 1$ produces

$$(7) \quad \sum_{\alpha \in \text{IPF}_n(1)} q^{\text{disp}(\alpha)} = \sum_{k=1}^n k!S(n, k)q^{n-k} = \sum_{\sigma \in \mathfrak{S}_n} (1+q)^{\text{asc}(\sigma)},$$

where $S(n, k)$ denotes the Stirling number of the second kind; the second equality is [16, p. 269, eqn. (6.39)]. Note that $k!S(n, k)$ is the number of unit interval parking functions with displacement $n - k$. Extracting the $q = 1$ coefficient from the first and third expressions in (7), we recover the enumerative result [3, Thm. 4] that $|\{\alpha \in \text{UPF}_n : \text{disp}(\alpha) = 1\}|$ is the n th Lah number (#A001286 of [18]).

Remark 3.4. Setting $q = -1$ in Theorem 3.3 yields

$$(8) \quad \sum_{\alpha \in \text{UPF}_n} (-1)^{\text{disp}(\alpha)} t^{\text{inv}(\alpha)} = t^{\binom{n}{2}}$$

since the permutation $w_0(i) = n - i + 1$ is the only one with no ascents. (Further specializing $t = 1$ recovers (1).) This formula can also be proven combinatorially by constructing a sign-reversing, inversion-preserving involution on unit interval parking functions. Given $\alpha \in \text{UPF}_n \setminus \{w_0\}$, let $\pi_1 \mid \pi_2 \mid \dots \mid \pi_m$ be its block structure (see Theorem 2.10), and let k be the smallest index such that either (i) $|\pi_k| \geq 2$, or (ii) $|\pi_k| = 1$ and π_k appears earlier in α than every member of π_{k+1} . (The condition $\alpha \neq w_0$ implies that at least one of conditions (i), (ii) must be satisfied.) Then, define $\phi(\alpha)$ by modifying α as follows:

- (1) If $\pi_k = (j, j, j+1, j+2, \dots)$, then replace the second occurrence of j by $j+1$.
- (2) If $\pi_k = (j)$, then replace the first occurrence of $j+1$ in α by j .

The map ϕ is a fixed-point-free involution on $\text{UPF}_n \setminus \{w_0\}$ satisfying $\text{disp}(\phi(\alpha)) = \text{disp}(\alpha) \pm 1$ and $\text{inv}(\phi(\alpha)) = \text{inv}(\alpha)$ (we omit the proof). If one associates unit interval parking functions with faces of the permutohedron, as done in [7], the function ϕ appears to give an acyclic matching in the sense of discrete Morse theory [14, 15].

Remark 3.5. We discuss a broader implication of Lemma 2.14. Let $\alpha \in \text{UPF}_n$ and set $\sigma = \text{car}_\alpha$. Then $\text{Inv}(\alpha) = \text{Inv}(\sigma^{-1})$ (Lemma 2.14), so $\text{Des}(\alpha) = \text{Des}(\sigma^{-1})$, and in particular

$$\text{des}(\alpha) = \text{des}(\sigma^{-1}) \text{ and } \text{maj}(\alpha) = \text{maj}(\sigma^{-1}).$$

Then the argument of Theorem 3.3 implies that

$$(9) \quad \sum_{\alpha \in \text{UPF}_n} q^{\text{disp}(\alpha)} t^{\text{inv}(\alpha)} x^{\text{des}(\alpha)} y^{\text{maj}(\alpha)} = \sum_{\sigma \in \mathfrak{S}_n} (1+q)^{\text{asc}(\sigma)} t^{\text{inv}(\sigma)} x^{\text{des}(\sigma^{-1})} y^{\text{maj}(\sigma^{-1})}.$$

An analogous formula holds for any other statistic that depends only on the inversion set.

3.1. A cyclic sieving phenomenon. Let X be a finite set, let $X(q)$ be a polynomial in q such that $X(1) = |X|$, let $C = \langle g \rangle$ be a cyclic group of order n acting on X , and let ω be a primitive n th root of unity. The triple $(X, X(q), C)$ is said to exhibit the **cyclic sieving phenomenon** [21] if $X(\omega^j) = |\{x \in X : c^j(x) = x\}|$ for every $c \in C$.

We use the following facts (setting $a = 1$ in [21, Equation (4.5)]). Let ω be a primitive n th root of unity as before, let d be a divisor of n , and let $c = (c_1, c_2, \dots, c_k) \models n$. Then

$$(10) \quad \left[\begin{matrix} n \\ c_1, c_2, \dots, c_\ell \end{matrix} \right]_{t=\omega^{n/d}} = \left(\frac{n}{d}, \frac{c_1}{d}, \frac{c_2}{d}, \dots, \frac{c_k}{d} \right),$$

where we adopt the convention that any multinomial expression involving one or more non-integers is zero.

As shown in [7, Lemma 3.12, Proposition 3.13, Corollary 3.14], there is an \mathfrak{S}_n -action on UPF_n , in which α and β belong to the same orbit if and only if $\alpha^\uparrow = \beta^\uparrow$; that is, α and β have the same block structure. In particular, UPF_n^\uparrow is a system of representatives for the \mathfrak{S}_n -orbits of UPF , and total displacement is constant on each orbit.

The cyclic subgroup C_n of \mathfrak{S}_n generated by the n -cycle $g = (1 \ 2 \ \dots \ n) \in \mathfrak{S}_n$ acts on UPF_n by restricting the \mathfrak{S}_n -action on UPF_n to C_n . For $k \in \mathbb{N}$, let $\text{UPF}_{n,k}^{\text{disp}}$ be the set of unit interval parking functions with total displacement k and set

$$f_{n,k}^{\text{disp}}(t) = \sum_{\alpha \in \text{UPF}_{n,k}^{\text{disp}}} t^{\text{inv}(\alpha)},$$

so that $\Phi_{n,1}(q, t) = \sum_k q^k f_{n,k}^{\text{disp}}(t)$.

Theorem 3.6. *The triple $(\text{UPF}_{n,k}^{\text{disp}}, f_{n,k}^{\text{disp}}(t), C_n)$ satisfies the cyclic sieving phenomenon for each n and k . That is, if ω is a primitive n th root of unity, then, for all n, k, j ,*

$$f_{n,k}^{\text{disp}}(\omega^j) = \sum_{\alpha \in \text{UPF}_{n,k}^{\text{disp}}} (\omega^j)^{\text{inv}(\alpha)} = |\{\alpha \in \text{UPF}_{n,k}^{\text{disp}} : g^j \alpha = \alpha\}|.$$

Proof. Let ω be a primitive n th root of unity. It suffices to consider the case that $n = dj$ for some positive integers d, j , so that ω^j is a primitive d th root of unity. For each $\beta \in \text{UPF}_n^{\uparrow}$, let $O(\beta)$ be its \mathfrak{S}_n -orbit, and write its block structure as $\pi_1^{\beta} \mid \pi_2^{\beta} \mid \dots \mid \pi_{m_{\beta}}^{\beta}$.

Recall that the elements of each block of $\alpha \in \text{UPF}_n$ occur in weakly increasing order in α , left to right. Therefore, the number of inversions of α is unchanged by replacing each element in its j th block with the number j . Accordingly,

$$\begin{aligned} \sum_{\alpha \in O(\beta)} t^{\text{inv}(\alpha)} &= \sum_{u \in \mathfrak{S}_n \cdot w_{\beta}} t^{\text{inv}(u)} \\ (11) \quad &= \begin{bmatrix} n \\ |\pi_1^{\beta}|, |\pi_2^{\beta}|, \dots, |\pi_{m_{\beta}}^{\beta}| \end{bmatrix}_t, \end{aligned}$$

where $w_{\beta} = 1 \dots 12 \dots 2 \dots m_{\beta} \dots m_{\beta}$ is a word with $|\pi_1^{\beta}|$ 1's, $|\pi_2^{\beta}|$ 2's, etc, written in weakly increasing order. The equality in (11) is [24, Prop. 1.7.1], originally due to MacMahon [20, pp. 314-315]. Now

$$\begin{aligned} \sum_{\alpha \in \text{UPF}_n} q^{\text{disp}(\alpha)} (\omega^j)^{\text{inv}(\alpha)} &= \sum_{\beta \in \text{UPF}_n^{\uparrow}} q^{\text{disp}(\beta)} \sum_{\alpha \in O(\beta)} (\omega^j)^{\text{inv}(\alpha)} \\ &= \sum_{\beta \in \text{UPF}_n^{\uparrow}} q^{\text{disp}(\beta)} \begin{bmatrix} n \\ |\pi_1^{\beta}|, |\pi_2^{\beta}|, \dots, |\pi_{m_{\beta}}^{\beta}| \end{bmatrix}_{t=\omega^j} \\ (12) \quad &= \sum_{\beta \in \text{UPF}_n^{\uparrow}} q^{\text{disp}(\beta)} \binom{n/d}{|\pi_1^{\beta}|/d, |\pi_2^{\beta}|/d, \dots, |\pi_{m_{\beta}}^{\beta}|/d}, \end{aligned}$$

where the equality in (12) follows from (10).

We can interpret the multinomial coefficient in (12) combinatorially. By [7, Prop. 3.13], an element $\alpha = (a_1, a_2, \dots, a_n) \in \text{UPF}_n$ is fixed by g^j if and only if, for every $i \in [j]$, all entries $a_i, a_{i+j}, \dots, a_{i+(d-1)j}$ whose indices form a congruence class modulo j belong to a common block. Hence, if α has block structure $\pi_1 \mid \pi_2 \mid \dots \mid \pi_m$, then there are

$$\binom{n/d}{|\pi_1|/d, |\pi_2|/d, \dots, |\pi_m|/d}$$

elements in the \mathfrak{S}_n -orbit of α that are fixed by g^j . (The elements of each block have to appear in increasing order, so the only choice to make is how to apportion the congruence classes among the blocks.) Thus, (12) becomes

$$(13) \quad \sum_{\alpha \in \text{UPF}_n} q^{\text{disp}(\alpha)} (\omega^j)^{\text{inv}(\alpha)} = \sum_{\beta \in \text{UPF}_n^{\uparrow}} q^{\text{disp}(\beta)} |\{\alpha \in O(\beta) : g^j \alpha = \alpha\}|$$

and grouping together unit interval parking functions with the same total displacement, we have

$$\sum_{\alpha \in \text{UPF}_{n,k}^{\text{disp}}} (\omega^j)^{\text{inv}(\alpha)} = |\{\alpha \in \text{UPF}_{n,k}^{\text{disp}} : g^j \alpha = \alpha\}|,$$

as desired. \square

Corollary 3.7. *Let n be a positive integer.*

(1) *If ω is a primitive n th root of unity, then*

$$\sum_{\alpha \in \text{UPF}_n} q^{\text{disp}(\alpha)} \omega^{\text{inv}(\alpha)} = q^{n-1}.$$

(2) If n is even, then

$$\sum_{\alpha \in \text{UPF}_n} q^{\text{disp}(\alpha)} (-1)^{\text{inv}(\alpha)} = q^{n/2} \sum_{\alpha \in \text{UPF}_{n/2}} q^{\text{disp}(\alpha)}.$$

Proof. (1) This is the case $j = 1$ of Theorem 3.6. A UPF is fixed by g if and only if it has a single block of size n . There is exactly one such UPF, namely $\alpha = (1, 1, 2, \dots, n-1)$, which has total displacement $n-1$.

(2) Let UPF_n^2 be the set of unit interval parking functions whose block sizes are all divisible by 2 and let $\text{UPF}_n^{2\uparrow} = \text{UPF}_n^2 \cap \text{UPF}_n^\uparrow$. Since an increasing UPF is determined by its list of block sizes, in order, there is a bijection $\phi: \text{UPF}_n^{2\uparrow} \rightarrow \text{UPF}_{n/2}^\uparrow$ given by dividing all block sizes by 2. (For instance, $\phi(1123 \mid 55 \mid 7789) = 11 \mid 3 \mid 44$, because these are the unique increasing UPFs with block sizes 4, 2, 4 and 2, 1, 2 respectively.) Note also that $\text{disp}(\beta) = \text{disp}(\phi(\beta)) + n/2$. Now substituting $j = n/2$ in (12) yields

$$\begin{aligned} \sum_{\alpha \in \text{UPF}_n} q^{\text{disp}(\alpha)} (-1)^{\text{inv}(\alpha)} &= \sum_{\beta \in \text{UPF}_n^{2\uparrow}} q^{\text{disp}(\beta)} \binom{n/2}{|\pi_1^\beta|/2, |\pi_2^\beta|/2, \dots, |\pi_{m_\beta}^\beta|/2} \\ &= q^{n/2} \sum_{\alpha \in \text{UPF}_{n/2}^\uparrow} q^{\text{disp}(\alpha)} \binom{n/2}{|\pi_1^\alpha|, |\pi_2^\alpha|, \dots, |\pi_{m_\alpha}^\alpha|} \quad (\text{setting } \alpha = \phi(\beta)) \\ &= q^{n/2} \sum_{\alpha \in \text{UPF}_{n/2}} q^{\text{disp}(\alpha)}, \end{aligned}$$

where the last step follows from (12), replacing n and j with $n/2$ and d with 1. \square

3.2. From $\ell = 1$ to $\ell = 2$. Theorem 3.3 can be viewed as reducing the enumeration of 1-interval parking functions to the enumeration of 0-interval parking functions, which are just permutations. We now describe a similar reduction from 2-interval parking functions to 1-interval parking functions, although it is more complicated.

Let $\beta = (b_1, b_2, \dots, b_n) \in \text{UPF}_n$ have block structure $\pi_1 \mid \pi_2 \mid \dots \mid \pi_m$. Let $\mathcal{R}(\beta)$ be the set of numbers $i \in [n]$ such that b_i is the second entry in its block π_j , where $j > 1$, and b_i occurs after the last entry of block π_{j-1} . Let $\mathcal{S}(\beta)$ be the set of numbers $i \in [n]$ such that b_i is the third or later entry in its block π_j . Then

$$(14) \quad |\mathcal{S}(\beta)| = \sum_{i=1}^m \max(|\pi_i| - 2, 0).$$

Example 3.8. If $\beta = 7511278935 \in \text{UPF}_{10}$, then its block structure is $\pi_1 \mid \pi_2 \mid \pi_3 = 1123 \mid 55 \mid 7789$. The second entry in block π_2 is $b_{10} = 5$ and appears after the last entry of π_1 , which is $b_9 = 3$. Hence, $10 \in \mathcal{R}(\beta)$. The second entry in block π_3 is $b_6 = 7$ and does not appear after the last entry of π_2 , which is $b_8 = 7$. Hence, $6 \notin \mathcal{R}(\beta)$. Thus, we have $\mathcal{R}(\beta) = \{10\}$. Lastly, in block $\pi_1 = 1123$ the third or later entries are $b_5 = 2$ and $b_9 = 3$, while in block $\pi_3 = 7789$ the third or later entries are $b_7 = 8$ and $b_8 = 9$. Hence, $\mathcal{S}(7511278935) = \{5, 7, 8, 9\}$. As expected, by (14), $|\mathcal{S}(\beta)| = 2 + 0 + 2 = 4$.

Observe that

$$(15) \quad \mathcal{R}(\beta) \cap \mathcal{S}(\beta) = \emptyset, \text{ and}$$

$$(16) \quad \mathcal{R}(\beta) \cup \mathcal{S}(\beta) = \{i \in [2, n]: \text{spots } b_{i-1} \text{ and } b_i \text{ are both occupied when car } i \text{ parks}\}.$$

In particular, if $i \in \mathcal{R}(\beta)$, then spot b_{i-1} is occupied by the last car of block π_{j-1} , and spot b_i by the first car of block π_j .

Example 3.9. Continuing Example 3.8, let $\beta = 7511278935$ and we can confirm $\mathcal{R}(\beta) \cap \mathcal{S}(\beta) = \emptyset$. One can readily confirm that the only indices $i \in [2, n]$ such that spots b_{i-1} and b_i are occupied when car i parks are precisely the indices $i \in \{5, 7, 8, 9, 10\} = \mathcal{R}(\beta) \cup \mathcal{S}(\beta)$. For example, when car 7 parks, spots $b_7 - 1 = 8 - 1 = 7$ and $b_7 = 8$ are occupied by cars 1 and 6, respectively.

Define a map $\eta: \text{IPF}_n(2) \rightarrow \text{UPF}_n$ by

$$\eta(\alpha)_i = \begin{cases} a_i + 1 & \text{disp}_\alpha(i) = 2, \\ a_i & \text{disp}_\alpha(i) \leq 1. \end{cases}$$

Informally, $\eta(\alpha)$ is the result of asking each car with displacement 2, “Please lower your expectations by one spot.” It is unsurprising, though not entirely obvious, that the result is a unit interval parking function with the same spot permutation, as we now prove.

Lemma 3.10. *Let $\alpha \in \text{IPF}_n(2)$ and $\beta = \eta(\alpha)$. Then $\beta \in \text{UPF}_n$, and $\text{spot}_\beta = \text{spot}_\alpha$.*

Proof. We show by induction that $\text{spot}_\beta(i) = \text{spot}_\alpha(i)$ and $\text{disp}_\beta(i) \leq 1$ for all $i \in [n]$. For $i = 1$, we have $\text{disp}_\alpha(1) = 0$, so $b_1 = a_1 = \text{spot}_\beta(1) = \text{spot}_\alpha(1)$. For the inductive step, suppose that $\text{spot}_\alpha(j) = \text{spot}_\beta(j)$ for all $j < i$.

- If $\text{disp}_\alpha(i) \leq 1$, then $b_i = a_i$ and $\text{spot}_\beta(i) = \text{spot}_\alpha(i)$; in particular, $\text{disp}_\beta(i) \leq 1$.
- If $\text{disp}_\alpha(i) = 2$, then $\text{spot}_\alpha(i) = a_i + 2$. In particular, spots a_i and $a_i + 1$ are full when car i parks. Changing the preference of the i th car from a_i to $b_i = a_i + 1$ does not affect where car i parks, but then $\text{disp}_\beta(i) = a_i + 2 - b_i = 1$. \square

We now prove our main enumeration result on 2-interval parking functions.

Theorem 3.11. *For all $n \geq 1$,*

$$\sum_{\alpha \in \text{IPF}_n(2)} q^{\text{disp}(\alpha)} t^{\text{inv}(\alpha)} = \sum_{\beta \in \text{UPF}_n} q^{\text{disp}(\beta)} t^{\text{inv}(\beta)} (1+q)^{|\mathcal{S}(\beta)|} (1+qt)^{|\mathcal{R}(\beta)|}.$$

In particular,

$$|\text{IPF}_n(2)| = \sum_{\beta \in \text{UPF}_n} 2^{|\mathcal{S}(\beta)| + |\mathcal{R}(\beta)|}.$$

Proof. Let $\beta = (b_1, b_2, \dots, b_n) \in \text{UPF}_n$ have block structure $\pi_1 \mid \pi_2 \mid \dots \mid \pi_m$. Let $\mathcal{R} = \mathcal{R}(\beta)$ and $\mathcal{S} = \mathcal{S}(\beta)$. For $R \subseteq \mathcal{R}$ and $S \subseteq \mathcal{S}$, define $\alpha = (a_1, a_2, \dots, a_n) = g_\beta(R, S)$ by

$$a_i = \begin{cases} b_i - 1 & \text{if } i \in R \cup S, \\ b_i & \text{if } i \notin R \cup S. \end{cases}$$

Then α is a parking function by the rearrangement criterion. In light of the description of $\mathcal{R} \cup \mathcal{S}$ in (16), it follows that for all $i \in [n]$,

$$(17a) \quad i \in R \cup S \text{ if and only if } \text{disp}_\alpha(i) = 2;$$

$$(17b) \quad i \notin R \cup S \text{ if and only if } \text{disp}_\alpha(i) \leq 1;$$

$$(17c) \quad \text{spot}_\alpha(i) = \text{spot}_\beta(i).$$

Therefore, $\alpha \in \eta^{-1}(\beta) \subseteq \text{IPF}_n(2)$. That is, we have a function

$$g_\beta: 2^{\mathcal{R}} \times 2^{\mathcal{S}} \rightarrow \eta^{-1}(\beta).$$

We claim that g_β is a bijection. It is injective by definition (note that $\mathcal{R} \cap \mathcal{S} = \emptyset$). To show that g_β is surjective, it suffices to show that if $\alpha \in \eta^{-1}(\beta)$ and $\text{disp}_\alpha(i) = 2$, then $i \in \mathcal{R} \cup \mathcal{S}$. Indeed, $b_i = a_i + 1$ by definition of η . Since $\text{disp}_\alpha(i) = 2$, when car i parks, spots $a_i = b_i - 1$ and $a_i + 1 = b_i$ are both occupied, and since $\text{spot}_\alpha = \text{spot}_\beta$ by Lemma 3.10, we have $i \in \mathcal{R} \cup \mathcal{S}$ by (16).

Note that the inverse map $f_\beta = g_\beta^{-1}$ is given by $f_\beta(\alpha) = (R, S)$, where

$$R = \mathcal{R} \cap \{i \in [n]: \text{disp}_\alpha(i) = 2\}, \text{ and } S = \mathcal{S} \cap \{i \in [n]: \text{disp}_\alpha(i) = 2\}.$$

Now, let $\beta \in \text{UPF}_n$, and let $\alpha = g_\beta(R, S)$ for some $R \subseteq \mathcal{R}(\beta)$ and $S \subseteq \mathcal{S}(\beta)$. Then

$$\text{disp}_\alpha(i) = \begin{cases} \text{disp}_\beta(i) & \text{if } i \notin R \cup S, \\ \text{disp}_\beta(i) + 1 & \text{if } i \in R \cup S, \end{cases}$$

so

$$(18) \quad \text{disp}(\alpha) = \text{disp}(\beta) + |R| + |S|.$$

We now describe $\text{inv}(\alpha)$ in terms of $\text{inv}(\beta)$. Suppose for contradiction that (i, j) is an inversion of β but not of α . Then, by definition of g_β , $i \in R \cup S$, $j \notin R \cup S$, and $b_i = b_j + 1$. Hence, $a_i = b_i - 1 = b_j = a_j$. Since β is a unit interval parking function, it does not have inversions within blocks (Theorem 2.10, part (2));

hence, b_i and b_j must be in different blocks. Thus, b_i is either the first or second element of some block π_{k+1} and b_j is the last element of block π_k . If b_i is the first element of π_{k+1} , then $i \notin \mathcal{R} \cup \mathcal{S}$. If b_i is the second element of π_{k+1} then again $i \notin \mathcal{R} \cup \mathcal{S}$ because b_i occurs before the last entry of π_k , namely b_j . Hence, $i \notin \mathcal{R} \cup \mathcal{S}$, which is a contradiction. We conclude that $\text{Inv}(\beta) \subseteq \text{Inv}(\alpha)$.

Now suppose that (i, j) is an inversion of α but not of β . Then $i \notin R \cup S$, $j \in R \cup S$ and $b_i = b_j$ by definition of g_β . Hence, $j \in R$ because it is the second occurrence of the same number (since $i < j$) and thus is the second entry of its block. Conversely, whenever $b_i = b_j$ with $i < j$, the pair (i, j) is an inversion of α but not of β . Therefore,

$$(19) \quad \text{inv}(\alpha) = \text{inv}(\beta) + |R|.$$

and combining (18) and (19) we obtain

$$(20) \quad q^{\text{disp}(\alpha)} t^{\text{inv}(\alpha)} = q^{\text{disp}(\beta) + |R| + |S|} t^{\text{inv}(\beta) + |R|}.$$

Consequently,

$$\begin{aligned} \sum_{\alpha \in \eta^{-1}(\beta)} q^{\text{disp}(\alpha)} t^{\text{inv}(\alpha)} &= \sum_{R \subseteq \mathcal{R}} \sum_{S \subseteq \mathcal{S}} q^{\text{disp}(\beta) + |R| + |S|} t^{\text{inv}(\beta) + |R|} \\ &= q^{\text{disp}(\beta)} t^{\text{inv}(\beta)} \left(\sum_{R \subseteq \mathcal{R}} (qt)^{|R|} \right) \left(\sum_{S \subseteq \mathcal{S}} q^{|S|} \right) \\ &= q^{\text{disp}(\beta)} t^{\text{inv}(\beta)} (1 + q)^{|\mathcal{S}(\beta)|} (1 + qt)^{|\mathcal{R}(\beta)|}, \end{aligned}$$

and summing over all $\beta \in \text{UPF}_n$ yields the theorem. \square

Example 3.12. Let $\beta = 6411624 \in \text{UPF}_7$, which has block structure $\beta^\uparrow = 112|44|66$. Positions 4,5,7 contain the second elements of the blocks. Note that $\mathcal{R}(\beta) = \{7\}$, because:

- $4 \notin \mathcal{R}(\beta)$ because it is the second element of the *first* block.
- $5 \notin \mathcal{R}(\beta)$: the 6 in position 5 is the second element of π_3 , but it occurs to the left of the largest element of π_2 , namely the 4 in position 7.
- $7 \in \mathcal{R}(\beta)$: the 4 in position 7 is the second element of π_2 , and it occurs to the right of the largest element of π_1 , namely the 2 in position 5.

Meanwhile, $\mathcal{S}(\beta) = \{6\}$. We have

$$\text{spot}_\beta = 6412735, \quad \text{disp}(\beta) = 0+0+0+1+1+1+1 = 4, \quad \text{inv}(\beta) = |\{12, 13, 14, 16, 17, 23, 24, 26, 56, 57\}| = 10.$$

The possibilities for R and S and the corresponding 2-interval parking functions $\alpha = g_\beta(R, S)$ are as follows.

R	S	α	spot_α	$\text{disp}(\alpha)$
\emptyset	\emptyset	6411624	6412735	$0 + 0 + 0 + 1 + 1 + 1 + 1 = 4$
7	\emptyset	<u>6411623</u>	6412735	$0 + 0 + 0 + 1 + 1 + 1 + 2 = 5$
\emptyset	6	6411614	6412735	$0 + 0 + 0 + 1 + 1 + 2 + 1 = 5$
7	6	<u>6411613</u>	6412735	$0 + 0 + 0 + 1 + 1 + 2 + 2 = 6$

In the cases that R is nonempty, the new inversion is underlined.

4. ENUMERATION VIA CIPHERS

For positive integer n and nonnegative integer k , define

$$\text{UPF}_{n,k}^{\text{inv}} = \{\alpha \in \text{UPF}_n : \text{inv}(\alpha) = k\}, \quad \text{upf}_{n,k}^{\text{inv}} = |\text{UPF}_{n,k}^{\text{inv}}|.$$

Setting $q = 1$ in Theorem 3.3 and extracting the coefficient of t^k yields

$$(21) \quad \text{upf}_{n,k}^{\text{inv}} = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{inv}(\sigma) = k}} 2^{\text{asc}(\sigma)}$$

although it is not clear how to convert this expression to a closed formula. In this section, we describe how work of Avalos and Bly [4], together with block structure theory (Theorem 2.10), can be used to determine the numbers $\text{upf}_{n,k}^{\text{inv}}$ through the use of **ciphers**, an analogue of Lehmer codes for unit interval parking

functions. We then use ciphers to provide a new proof of [10, Theorem 1.2], which counts Boolean intervals of \mathfrak{S}_n under the right weak order by certain unit interval parking functions.

4.1. The Avalos–Bly and cipher bijections. Let m and n be nonnegative integers. In what follows, $T = (T_1, T_2, \dots, T_m)$ always denotes an ordered list of multisets of nonnegative integers. Let $\text{type}(T) = (|T_1|, |T_2|, \dots, |T_m|)$. Avalos and Bly [4, Defns. 1.8, 1.10] defined

$$\begin{aligned}\mathcal{T}_n^m &= \{T = (T_1, T_2, \dots, T_m) : |T_1| + |T_2| + \dots + |T_m| = n \text{ and } 0 \leq x \leq \sum_{j=1}^{i-1} |T_j| \text{ for all } x \in T_i\}, \\ \mathcal{T}_n^m(k) &= \{T = (T_1, T_2, \dots, T_m) \in \mathcal{T}_n^m : \sum_i \sum_{x \in T_i} x = k\}.\end{aligned}$$

They proved [4, Proposition 1.9] that there is a bijection

$$\varphi: [m]^n \rightarrow \mathcal{T}_n^m$$

defined as follows. Given $w = (w_1, w_2, \dots, w_n) \in [m]^n$ and $j \in [n]$, define

$$(22) \quad \text{inv}(w, j) = |\{\ell \in [j+1, n] : w_j > w_\ell\}|$$

(the number of inversions with w_j as the left element).

They then define multisets $T_i = \{\text{inv}(w, j) : w_j = i\}$ and set $\varphi(w) = (T_1, T_2, \dots, T_m)$. In particular, since $\text{inv}(w) = \sum_{j=1}^m \text{inv}(w, j)$, it follows that φ restricts to a bijection

$$\{w \in [m]^n : \text{inv}(w) = k\} \rightarrow \mathcal{T}_n^m(k),$$

which is [4, Proposition 1.11]. If we define the **content** of $w \in [m]^n$ as $\text{cont}(w) = (|w^{-1}(1)|, \dots, |w^{-1}(m)|)$ (thinking of w as a map $w: [n] \rightarrow [m]$), then the construction implies that $\text{type}(T) = \text{cont}(w)$. Therefore, for each weak composition c of n , the map φ restricts to bijections

$$(23) \quad \begin{aligned}\{w \in [m]^n : \text{cont}(w) = c\} &\rightarrow \{T \in \mathcal{T}_n^m : \text{type}(T) = c\}, \\ \{w \in [m]^n : \text{cont}(w) = c, \text{inv}(w) = k\} &\rightarrow \{T \in \mathcal{T}_n^m(k) : \text{type}(T) = c\}.\end{aligned}$$

Combining the Avalos–Bly bijections (23) with that of Corollary 2.11, we obtain bijections

$$\begin{aligned}\{\alpha \in \text{UPF}_n : (|\pi_1|, \dots, |\pi_m|) = c\} &\rightarrow \{T \in \mathcal{T}_n^m : \text{type}(T) = c\}, \\ \{\alpha \in \text{UPF}_{n,k}^{\text{inv}} : (|\pi_1|, \dots, |\pi_m|) = c\} &\rightarrow \{T \in \mathcal{T}_n^m(k) : \text{type}(T) = c\},\end{aligned}$$

for each (strict) composition c . We can now state the main result of this section, which follows from gluing these last bijections together over all compositions c .

Theorem 4.1 (Cipher Bijection). *Let n and m be positive integers and let k be a nonnegative integer. Then*

$$(24) \quad \psi: \text{UPF}_{n,k}^{\text{inv}} \rightarrow \bigcup_{m=1}^n \mathcal{G}_n^m(k),$$

where

$$\mathcal{G}_n^m(k) = \{(T_1, T_2, \dots, T_m) \in \mathcal{T}_n^m(k) : |T_i| > 0 \text{ for all } i\}.$$

We call $\psi(\alpha)$ the **cipher** of α . We represent ciphers by writing each T_i as a string of boldface numbers in weakly decreasing order, separated by bars. For instance, **00|110|31|1** is an element of $\mathcal{G}_8^4(7)$.

Example 4.2 (One inversion). Let $n \geq 2$. We calculate $\text{upf}_{n,1}^{\text{inv}}$. By (24), we have a bijection

$$\text{UPF}_{n,1}^{\text{inv}} \rightarrow \bigcup_{m=1}^n \mathcal{G}_n^m(1).$$

The right-hand side consists of sequences (T_1, T_2, \dots, T_m) where each T_i is a nonempty multiset of 0's and 1's, containing a total of $n-1$ 0's and one 1, with the 1 not appearing in T_1 (so in particular $m \geq 2$). The ciphers for such a sequence are precisely those words that start with **0**, followed by some permutation of the following substrings: one copy of **|1**, and $n-2$ more strings each of which can be either **0** or **|0**. (The number of **|0**'s used is thus $m-1$.) The “regular expression” for all such ciphers is

$$\mathbf{0} \cdot (\{\mathbf{|1}\}^1, \{\mathbf{0}, \mathbf{|0}\}^{n-2})$$

(we use this notation in the next example as well) and it is now elementary that

$$(25) \quad \text{upf}_{n,1}^{\text{inv}} = 2^{n-2}(n-1).$$

Example 4.3 (Two inversions). Let $n \geq 3$. We calculate $\text{upf}_{n,2}^{\text{inv}}$. Here are the possibilities:

- (1) The cipher has one **2** and $n-1$ **0**'s. The regular expression is $(\mathbf{00}, \mathbf{0}|\mathbf{0}) \cdot (\{\mathbf{2}\}^1, \{\mathbf{0}, \mathbf{0}\}^{n-3})$. There are 2^{n-2} such words.
- (2) The cipher has two **1**'s and $n-2$ **0**'s. There are two subcases:
 - (a) The **1**'s are in different blocks.
Regular expression: $\mathbf{0} \cdot (\{\mathbf{1}\}^2, \{\mathbf{0}, \mathbf{0}\}^{n-3})$
Count: $2^{n-3} \binom{n-1}{2}$
 - (b) The **1**'s are in the same block.
Regular expression: $\mathbf{0} \cdot (\{\mathbf{11}\}^1, \{\mathbf{0}, \mathbf{0}\}^{n-3})$
Count: $2^{n-3}(n-2)$

Adding these counts, we obtain

$$(26) \quad \text{upf}_{n,2}^{\text{inv}} = 2^{n-3} \left(3(n-2) + \binom{n-1}{2} \right) = 2^{n-4}(n-2)(n+5).$$

Example 4.4 (Three inversions). Let $n \geq 3$. We calculate $\text{upf}_{n,3}^{\text{inv}}$. The regular expressions and their counts are as follows:

$$\begin{array}{ll} \mathbf{0} \cdot (\{\mathbf{0}, \mathbf{0}\})^2 \cdot (\{\mathbf{3}\}^1, \{\mathbf{0}, \mathbf{0}\}^{n-4}) & 2^2 \cdot 2^{n-4} \cdot (n-3) \\ \{\mathbf{00}, \mathbf{0}|\mathbf{0}\}^1 \cdot (\{\mathbf{21}\}^1, \{\mathbf{0}, \mathbf{0}\}^{n-4}) & 2 \cdot 2^{n-4} \cdot (n-3) \\ \mathbf{0}|\mathbf{1} \cdot (\{\mathbf{2}\}^1, \{\mathbf{0}, \mathbf{0}\}^{n-3}) & 2^{n-3} \cdot (n-2) \\ \{\mathbf{00}, \mathbf{0}|\mathbf{0}\}^1 \cdot (\{\mathbf{2}\}^1, \{\mathbf{1}\}^1, \{\mathbf{0}, \mathbf{0}\}^{n-4}) & 2 \cdot 2^{n-4} \cdot (n-3)(n-2) \\ \mathbf{0} \cdot (\{\mathbf{1}\}^3, \{\mathbf{0}, \mathbf{0}\}^{n-4}) & 2^{n-4} \cdot \binom{n-1}{3} \\ \mathbf{0} \cdot (\{\mathbf{11}\}^1, \{\mathbf{1}\}^1, \{\mathbf{0}, \mathbf{0}\}^{n-4}) & 2^{n-4} \cdot (n-3)(n-2) \\ \mathbf{0} \cdot (\{\mathbf{111}\}^1, \{\mathbf{0}, \mathbf{0}\}^{n-4}) & 2^{n-4} \cdot (n-3) \end{array}$$

Adding up the counts, we obtain

$$(27) \quad \text{upf}_{n,3}^{\text{inv}} = 2^{n-4} \left(\frac{1}{6}n^3 + 2n^2 - \frac{25}{6}n - 8 \right).$$

Comparing the formulas (25), (26), and (27), we are led to the following general conjecture, which we leave as an open problem.

Conjecture 4.5. Let $n \geq 1$ and $k \in \{0, 1, \dots, n(n-1)/2\}$. Then $\text{upf}_{n,k}^{\text{inv}} = O(2^{n-2k}n^k)$.

4.2. Ciphers and Lehmer codes. We describe another way to construct ciphers by way of Lehmer codes for permutations. Let $E_{n,k}$ be the set of words $w \in \{0, 1, \dots, n-1\}^n$ satisfying

- (1) $w_i \leq i-1$ for all $i \in [n]$, and
- (2) $\sum_{i=1}^n w_i = k$.

Then a cipher $T \in \mathcal{G}_n(k)$ can be constructed from a word $w \in E_{n,k}$ by inserting bars. If $w_i < w_{i+1}$, we *must* insert a bar between w_i and w_{i+1} ; otherwise, we *may* insert a bar. Hence, we can construct $2^{n-1-\text{asc}(w)}$ ciphers from w , and every cipher arises in this way. It follows that

$$(28) \quad \text{upf}_{n,k}^{\text{inv}} = \sum_{w \in E_{n,k}} 2^{n-1-\text{asc}(w)}.$$

Define

$$E_n = \bigcup_{k \geq 0} E_{n,k} = \{0\} \times \{0, 1\} \times \dots \times \{0, 1, \dots, n-1\}.$$

Then E_n is the set of **Lehmer codes** (backwards); see [24, p. 34]. There is a bijection $\gamma : E_n \rightarrow \mathfrak{S}_n$ from Lehmer codes to permutations: given $w \in E_n$, construct $\gamma(w)$ by starting with the empty word and inserting

$1, 2, \dots, n$ successively, placing i to the left of w_i letters. It is not hard to see that for all $w \in E_n$,

$$(29) \quad \text{Asc}(w) = \text{Des}(\gamma(w)^{-1})$$

and

$$(30) \quad \sum_{i=1}^n w_i = \text{inv}(\gamma(w)).$$

Applying (29) to (28) and observing that $\text{asc}(\sigma) + \text{des}(\sigma) = n - 1$ for all $\sigma \in \mathfrak{S}_n$, we obtain

$$\text{upf}_{n,k}^{\text{inv}} = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{inv}(\sigma)=k}} 2^{n-\text{des}(\sigma^{-1})-1} = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{inv}(\sigma)=k}} 2^{\text{asc}(\sigma)},$$

which recovers the $q = 1$ specialization of Theorem 3.3.

Define the **underlying code** of a cipher $T \in \mathcal{G}_n(k)$ to be the word $\text{code}(T) \in E_n$ obtained from T by removing all bars. Likewise, the underlying code of a UPF α is $\text{code}(\alpha) = \text{code}(\psi(\alpha))$, where ψ is the Avalos–Bly bijection.

Corollary 4.6. *For a unit interval parking function α , the Lehmer code of spot_α is the underlying code of α . In particular, if β is a unit interval parking function, then α and β have the same outcome permutation if and only if $\psi(\alpha)$ and $\psi(\beta)$ have the same underlying code.*

Proof. Let $\pi_1 | \pi_2 | \dots | \pi_m$ be the block structure of $\alpha = (a_1, a_2, \dots, a_n)$. As in the Avalos–Bly construction described above, for each $i \in [m]$, let T_i denote the multiset of all numbers $\text{inv}(\alpha, j)$ (see (22)) such that $a_j \in \pi_i$. Then $w = \text{code}(\psi(\alpha))$ is the concatenation $T_1 T_2 \dots T_m$, where each T_i is sorted in weakly decreasing order. Let $\sigma = \gamma(w)$. Then $(i, j) \in \text{Inv}(\gamma(w))$ if and only if $\sigma_i > \sigma_j$ and $w_{\sigma_i} > w_{\sigma_j} + w_{\sigma_{j+1}} + \dots + w_{\sigma_{i-1}}$; in this case a_i must belong to a later block than a_j , so $(i, j) \in \text{Inv}(\alpha)$. Thus $\text{Inv}(\gamma(w)) \subseteq \text{Inv}(\alpha)$, and since

$$|\text{Inv}(\gamma(w))| = \sum_{i=1}^n w_i = \sum_{i=1}^n \psi(\alpha)_i = |\text{Inv}(\alpha)|$$

(by the constructions of γ , w and ψ , respectively), we conclude that $\text{Inv}(\gamma(w)) = \text{Inv}(\alpha)$.

On the other hand, $\text{Inv}(\text{spot}_\alpha) = \text{Inv}(\alpha)$. A permutation is uniquely determined by its inversion set, so $\text{spot}_\alpha = \sigma = \gamma(w)$. Hence, $\gamma^{-1}(\text{spot}_\alpha)$ must be the underlying code of $\psi(\alpha)$. \square

Example 4.7. Let $w = 00110311 \in E_{8,7}$. The corresponding ciphers are of the form $0 \cdot 0 | 1 \cdot 1 \cdot 0 | 3 \cdot 1 \cdot 1$, where each dot is replaced either with $|$ or with the null symbol. The unit interval parking function corresponding to any of these 32 ciphers has $\sigma = \text{spot}_\alpha = 13642785$, e.g.,

$$\begin{array}{rcl} 00|110|311 & \xrightarrow{\psi^{-1}} & 13631674 \xrightarrow{\mathcal{C}} 13642785, \\ 00|1|10|3|11 & \xrightarrow{\psi^{-1}} & 13641774 \xrightarrow{\mathcal{C}} 13642785. \end{array}$$

Then $\sigma^{-1} = 15248367 = \gamma(w)^{-1}$. Note that $\text{Des}(\sigma^{-1}) = \{2, 5\} = \text{Asc}(w)$.

4.3. Unit Fubini rankings. A **unit Fubini ranking** is a unit interval parking function whose blocks all have size 1 or 2. Unit Fubini rankings were introduced in [10] to study Boolean intervals of \mathfrak{S}_n under the right weak order. We can use ciphers to provide a new proof of the main result of that paper.

Theorem 4.8 ([10, Theorem 1.2]). *Unit Fubini rankings with $n - r$ blocks and k inversions are in bijection with rank- r Boolean intervals $[\sigma, \tau]$ of \mathfrak{S}_n under the right weak order with $\text{inv}(\sigma) = k$.*

This statement is ostensibly a refinement of [10, Theorem 1.2], which does not mention inversions explicitly, but it is not difficult to use the methods of [10] together with elementary facts such as Lemma 2.14 to obtain the more refined version.

Proof. Call a set of integers **sparse** if it contains no two consecutive integers. Let $\text{UFR}_{n,k}^m$ be the set of unit Fubini rankings of length n with m blocks and k inversions. The bijection (24) restricts to a bijection

$$\psi: \text{UFR}_{n,k}^m \rightarrow \hat{\mathcal{T}}_n^m(k) := \{(T_1, T_2, \dots, T_m) \in \mathcal{T}_n^m(k): |T_i| \in \{1, 2\} \text{ for all } i\}.$$

As in Example 4.7, the elements of $\mathcal{T}_n^m(k)$ can be represented as ciphers as follows: choose a word $w \in E_{n,k}$, insert bars at all its ascents, and insert additional bars as needed so as to obtain m blocks, all of size 1 or 2. That means that we need a total of $m - 1$ bars; if B is the set of non-ascents where there are bars, then $|B| = m - 1 - |\text{Asc}(w)|$. Thus we have a bijection

$$\hat{\mathcal{T}}_n^m(k) \rightarrow \left\{ (w, B) : \begin{array}{l} w \in E_{n,k}, \\ B \subseteq [n-1] \setminus \text{Asc}(w), \\ |B| = m - 1 - |\text{Asc}(w)|, \text{ and} \\ [n-1] \setminus (\text{Asc}(w) \cup B) \text{ is sparse} \end{array} \right\}.$$

We may replace (w, B) with the equivalent data (σ, C) , where $\sigma = \gamma(w)^{-1}$ and $C = \text{Asc}(\sigma) \setminus B$. (By (29), we have $[n-1] \setminus \text{Asc}(w) = [n-1] \setminus \text{Des}(\sigma) = \text{Asc}(\sigma)$.) Translating the conditions on (w, B) into conditions on (σ, C) , we obtain a bijection

$$\hat{\mathcal{T}}_n^m(k) \rightarrow \left\{ (\sigma, C) : \begin{array}{l} \sigma \in \mathfrak{S}_n, \text{ inv}(\sigma) = k, \\ C \subseteq \text{Asc}(\sigma), |C| = n - m, \text{ and} \\ C \text{ is sparse} \end{array} \right\}.$$

On the other hand, a Boolean interval in the right weak order with bottom element σ and rank r is given precisely by choosing r pairwise commuting elementary transpositions s_i such that $\text{inv}(\sigma s_i) > \text{inv}(\sigma)$ for every i , which is equivalent to choosing a sparse subset of $\text{Asc}(\sigma)$ of size r . \square

Remark 4.9. Let Fib_i denote the i th Fibonacci number: $(\text{Fib}_0, \text{Fib}_1, \text{Fib}_2, \text{Fib}_3, \dots) = (1, 1, 2, 3, \dots)$. Theorem 1.1 of [10] states that for each $\sigma \in \mathfrak{S}_n$, the number of Boolean intervals in right weak order with bottom element σ is $\prod_{i=1}^k \text{Fib}_{a_i(\sigma)}$, where $a_1(\sigma), \dots, a_k(\sigma)$ are the lengths of the maximal ascending subwords of $\sigma = (\sigma_1, \dots, \sigma_n)$, or, equivalently, of the maximal non-ascending subwords of $w = \gamma^{-1}(\sigma^{-1})$. The proof of Theorem 4.8 provides a generating function for Boolean intervals by rank, as follows. Define the **Fibonacci polynomials** $\text{Fib}_n(q)$ by

$$\text{Fib}_n(q) = \sum_{\substack{S \subseteq [n-1] \\ S \text{ sparse}}} q^{|S|}$$

(see sequence #A011973 in [18]), so that $\text{Fib}_n(1) = \text{Fib}_n$. Then

$$(31) \quad \sum_{\substack{\tau \in \mathfrak{S}_n \\ [\sigma, \tau] \text{ Boolean}}} q^{\text{inv}(\tau) - \text{inv}(\sigma)} = \prod_{i=1}^k \text{Fib}_{a_i(\sigma)}(q).$$

5. FOATA INVARIANCE AND INV-MAJ EQUIDISTRIBUTION

In this section, we prove that a classical bijection of Foata [11] preserves the classes of ℓ -interval parking functions for $\ell \in \{1, 2, n-2, n-1\}$, from which it follows that the inversion and major index statistics are equidistributed on each of these classes.

Theorem 5.1. *Let $1 \leq \ell < n$, and let $\text{IPF}_n(\ell)$ denote the class of ℓ -interval parking functions of length n .*

- *If $\ell \in \{1, 2, n-2, n-1\}$, then the Foata transform restricts to a bijection $\text{IPF}_n(\ell) \rightarrow \text{IPF}_n(\ell)$, and consequently the inversion and major index statistics are equidistributed on $\text{IPF}_n(\ell)$, i.e.,*

$$\sum_{p \in \text{IPF}_n(\ell)} t^{\text{inv}(p)} = \sum_{p \in \text{IPF}_n(\ell)} t^{\text{maj}(p)}.$$

- *If $\ell \notin \{1, 2, n-2, n-1\}$, then inversion and major index are not equidistributed on $\text{IPF}_n(\ell)$, so the Foata transform cannot be a bijection $\text{IPF}_n(\ell) \rightarrow \text{IPF}_n(\ell)$.*

The proof spans several cases, which we order by increasing difficulty.

- $\ell = n-2$: Proposition 5.7
- $3 \leq \ell \leq n-3$: Proposition 5.8
- $\ell = 1$: Proposition 5.9
- $\ell = 2$: Proposition 5.10

5.1. The Foata transform on words. We begin by describing Foata's bijection [11] and its relevant properties.

Definition 5.2 ([11]). The **Foata transform** is the function F defined inductively on words $w = w_1 \cdots w_n$ as follows.

- For $n = 1$, we set $F(w) = w$.
- For $n > 1$, $F(w)$ is defined as follows. Let $w' = F(w_1 \cdots w_{n-1}) = w'_1 \cdots w'_{n-1}$.
 - If $w_n \geq w'_{n-1}$, then place a separator after every w'_i such that $w_n \geq w'_i$.
 - If $w_n < w'_{n-1}$, then place a separator after every w'_i such that $w_n < w'_i$.
- Cycle each **segment** (i.e., each maximal subword between consecutive separators) by moving its rightmost entry to the start.
- Finally, append w_n . The result is $F(w)$.

For an example of the algorithm in action, see [24, Example 1.4.7].

Foata [11] proved that F is a bijection, and that $\text{inv}(F(w)) = \text{maj}(w)$ for all words w , which implies equidistribution of inversion number and major index for permutations (which are 0-interval parking functions), i.e.,

$$(32) \quad \sum_{w \in \mathfrak{S}_n} q^{\text{inv}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\text{maj}(w)}.$$

Equidistribution had been proved previously by MacMahon [20], but the Foata transform gave the first combinatorial proof. (For more detail, see [24, pp. 41–43].)

It is immediate from the definition of F that it preserves the content of a word, i.e., the number of occurrences of each letter. In particular, it preserves the class of parking functions (which are $(n-1)$ -interval parking functions). Moreover, the definition of F also implies that for each i , the “partial Foata transform” F_i defined by $F_i(w) = F(w_1 \cdots w_i)w_{i+1} \cdots w_n$ is also a bijection.

The motivating question of this section is: *When does the Foata transform preserve the class of ℓ -interval parking functions of length n ?* When the answer is positive, equidistribution of inversion number and major index for that class follows immediately.

5.2. Permuting parking functions. Here we compare different parking functions related by permutations (specifically, by partial Foata transforms). It is convenient to assign each car a permanent unique identifier (its “license plate”, so to speak) and denote a parking function $p = (p_1, \dots, p_n)$ by a *two-line array*

$$p = \begin{pmatrix} C_1 & \cdots & C_n \\ p_1 & \cdots & p_n \end{pmatrix}$$

in which C_i means the car with license plate i ; p_i is its preference; and the order of columns indicates the order with which cars enter the parking lot.

Remark 5.3. The objective of using C_i is to be able to change the order in which the cars park but not change their preferences. For example, when $p = 112$, we say car 2 prefers to park in spot 1. Then we can tell those cars to keep their preferences but change their order in the queue to now park enter in the order $1 \rightarrow 3 \rightarrow 2$. This creates the parking function $q = 121$. By using the notation C_i we are able to then reference that the car that was the 2nd car in p is now the 3rd car in q . In other words,

$$p = \begin{pmatrix} C_1 & C_2 & C_3 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{becomes} \quad q = \begin{pmatrix} C_1 & C_3 & C_2 \\ 1 & 2 & 1 \end{pmatrix}.$$

This notation is convenient, as it allows one to say clearly how we compare the displacements of the cars as we make small changes to a parking function.

Accordingly, we write $\text{spot}_p(C_i)$ for the spot in which car C_i parks with respect to p , and define $\text{disp}_p(C_i) = \text{spot}_p(C_i) - p_i$. For example, if p is the parking function

$$p = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ 1 & 1 & 1 & 1 & 4 & 3 \end{pmatrix}$$

then its Foata transform is

$$r = F(p) = \begin{pmatrix} C_5 & C_1 & C_2 & C_3 & C_4 & C_6 \\ 4 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

We can see for example that $\text{spot}_r(C_4) = 5$ and $\text{disp}_r(C_4) = 5 - 1 = 4$. Recall that a parking function p is an ℓ -interval parking function if and only if $\max_i \{\text{disp}_p(C_i)\} \leq \ell$.

Lemma 5.4 (Interval Rearrangement). *Let p be a parking function of length n . Partition $[n]$ into intervals B_1, \dots, B_k , and let $q = p \cdot w$, where $w \in \mathfrak{S}_{B_1} \times \dots \times \mathfrak{S}_{B_k}$. (See Section 3.1 for the action of permutations on parking functions.) Then for each $i \in [k]$ we have*

$$\{\text{spot}_p(j) : j \in B_i\} = \{\text{spot}_q(j) : j \in B_i\}.$$

This fact may be known, but we have not found it in the literature, so we give a short proof.

Proof. The permutation w can be factored as a product of adjacent transpositions, each one belonging to some \mathfrak{S}_{B_i} , so it suffices to show that interchanging the order of two adjacent cars either preserves or swaps their outcomes. Suppose that S is the set of parking spots available at some point in the algorithm, and that the next two cars in line to park prefer spots u and v . Depending on who parks first, the two spots they park in are either

$$a = \min\{s \in S : s \geq u\}, \quad b = \min\{s \in S \setminus a : s \geq v\}$$

or

$$y = \min\{s \in S : s \geq v\}, \quad z = \min\{s \in S \setminus y : s \geq u\}.$$

If $u = v$ then evidently $a = y$ and $b = z$. Without loss of generality, assume $u < v$.

Evidently $u \leq a \leq z$ and $v \leq y \leq b$. If $(a, y) = (z, b)$ then we are done. Otherwise, either $a < z$ or $y < b$. In the first case, the definitions of a and z imply that $y = a$, from which it follows that $u, u+1, \dots, v-1 \notin S$, and these two conditions together imply $z = b$. A similar argument shows that $y < b$ implies $(y, z) = (a, b)$. \square

Lemma 5.4 generalizes easily from parking functions to parking completions (see [1]). One useful consequence is the following.

Proposition 5.5. *Suppose p, q are two parking functions obtained by switching two adjacent cars, as follows:*

$$p = \begin{pmatrix} \dots & C_1 & C_2 & \dots \\ \dots & x_1 & x_2 & \dots \end{pmatrix}, \quad q = \begin{pmatrix} \dots & C_2 & C_1 & \dots \\ \dots & x_2 & x_1 & \dots \end{pmatrix},$$

with $x_1 > x_2$. Then $\text{maxdisp}(p) \geq \text{maxdisp}(q)$. In particular, the increasing rearrangement of every ℓ -interval parking function is an ℓ -interval parking function.

Proof. By interval rearrangement (Lemma 5.4), the sets $\{\text{spot}_p(C_x), \text{spot}_p(C_y)\}$ and $\{\text{spot}_q(C_x), \text{spot}_q(C_y)\}$ are equal, say to $\{a, b\}$, where $a < b$. If $\text{spot}_p(C_x) = \text{spot}_q(C_x)$ and $\text{spot}_p(C_y) = \text{spot}_q(C_y)$, then $\text{maxdisp}(p) = \text{maxdisp}(q)$. Otherwise, we have without loss of generality

$$a = \text{spot}_p(C_1) = \text{spot}_q(C_2) < \text{spot}_p(C_2) = \text{spot}_q(C_1) = b.$$

In particular, $b > a \geq x_1 > x_2$, and

$$\begin{aligned} \max(\text{disp}_p(C_1), \text{disp}_p(C_2)) &= \max(a - x_1, b - x_2) \\ &> \max(\text{disp}_q(C_1), \text{disp}_q(C_2)) = \max(b - x_1, a - x_2). \end{aligned}$$

Thus, replacing p with q can only decrease the maximum displacement, as claimed. Iterating this argument shows that bubble-sorting (taking a ℓ -interval parking function and sorting the smallest element to position 1, then the second smallest to position 2, etc.) produces an ℓ -interval parking function. \square

Lemma 5.6 (Reserved Spot Lemma). *For $1 \leq j < k \leq n$, let*

$$p = \begin{pmatrix} C_1 & \dots & C_n \\ p_1 & \dots & p_n \end{pmatrix} \quad \text{and} \quad q = \left(\begin{array}{c|ccccc} C_1 & \dots & C_{j-1} & \mid & C_k & C_j & \dots & C_{k-1} & \mid & C_{k+1} & \dots & C_n \\ p_1 & \dots & p_{j-1} & \mid & p_k & p_j & \dots & p_{k-1} & \mid & p_{k+1} & \dots & p_n \end{array} \right)$$

be parking functions (where the bars are included merely for clarity), and suppose that $\text{spot}_p(C_j) = \text{spot}_q(C_j)$. Then $\text{spot}_p(C_i) = \text{spot}_q(C_i)$ for all $i \in [n]$.

Proof. Let p and q be parking functions as above with indices $j < k$.

If $i < j$ or $i < k$, then the conclusion follows from interval rearrangement (Lemma 5.4).

If $k = j + 1$, then we know that $\text{spot}_p(C_i) = \text{spot}_q(C_i)$ for all $i \neq k$, so it must be the case that $\text{spot}_p(C_k) = \text{spot}_q(C_k)$ as well.

If $k > j + 1$, the result follows from the $k = j + 1$ case by factoring the cycle $(j \ j + 1 \ \cdots \ k - 1 \ k)$ into the product of adjacent transpositions $(k - 1 \ k)(k - 2 \ k - 1) \cdots (j \ j + 1)$. \square

5.3. Proof of Theorem 5.1. We begin with the easiest cases, showing that the Foata transform is a bijection when $\ell = n - 2$, but not when $3 \leq \ell \leq n - 3$.

Proposition 5.7. *For all $n \geq 3$ the Foata transform restricts to a bijection $\text{IPF}_n(n - 2) \rightarrow \text{IPF}_n(n - 2)$.*

Proof. A parking function p of length n fails to be an $(n - 2)$ -parking function precisely when some car has displacement $n - 1$. That car must be the last to park, must prefer spot 1, and must park in spot n , which implies that $\hat{p} = (p_1, \dots, p_{n-1})$ is a parking function of length $n - 1$. (In particular, $|\text{IPF}_n(n - 2)| = (n + 1)^{n-1} - n^{n-2}$.) Since F fixes the last digit of p and permutes the entries of \hat{p} , it follows that it fixes $\text{PF}_n \setminus \text{IPF}_n(n - 2)$, hence must fix $\text{IPF}_n(n - 2)$. \square

Proposition 5.8. *For $n \geq 6$ and $3 \leq \ell \leq n - 3$, the statistics inv and maj are not equidistributed on $\text{IPF}_n(\ell)$.*

Proof. Let $\mathcal{I}_{n,\ell} = \{p \in \text{IPF}_n(\ell) : \text{inv}(p) = 1\}$ and $\mathcal{M}_{n,\ell} = \{p \in \text{IPF}_n(\ell) : \text{maj}(p) = 1\}$. We show that $|\mathcal{M}_{n,\ell}| < |\mathcal{I}_{n,\ell}|$. To do so, define

$$\mathcal{P}_{n,\ell}^I = \left\{ (\alpha, i) \in \text{IPF}_n^\uparrow(\ell) \times [n-1] : \begin{array}{l} (\mathbf{I1}) \quad a_i < a_{i+1}, \text{ and} \\ (\mathbf{I2}) \quad \text{either } a_{i+1} > i \text{ or } \text{disp}_\alpha(C_i) \leq \ell - 1 \end{array} \right\},$$

$$\mathcal{P}_{n,\ell}^M = \left\{ (\alpha, i) \in \text{IPF}_n^\uparrow(\ell) \times [n-1] : \begin{array}{l} (\mathbf{M1}) \quad a_i < a_{i+1}, \text{ and} \\ (\mathbf{M2}) \quad \text{for all } j \in [n], \text{ if } a_{i+1} \leq j \leq i, \text{ then } \text{disp}_\alpha(C_j) \leq \ell - 1 \end{array} \right\}.$$

Our plan is to define bijections $\kappa : \mathcal{P}_{n,\ell}^I \rightarrow \mathcal{I}_{n,\ell}$ and $\lambda : \mathcal{P}_{n,\ell}^M \rightarrow \mathcal{M}_{n,\ell}$, then to show that $\mathcal{P}_{n,\ell}^I \supsetneq \mathcal{P}_{n,\ell}^M$.

First, given a pair $(\alpha, i) \in \mathcal{P}_{n,\ell}^I$, let $\beta = \kappa(\alpha, i)$ be the parking function obtained by swapping a_i and a_{i+1} . Since $\alpha = \alpha^\uparrow$, it follows that β is a parking function with one inversion, and α can be recovered from β as its increasing rearrangement.

By interval rearrangement (Lemma 5.4), every car has the same outcome in β as does in α , except that C_i and C_{i+1} may switch places. Since $\alpha = \alpha^\uparrow$, the two parking spots in question are i and $i + 1$. There are two possibilities:

- If $a_{i+1} = i + 1$, then $\text{spot}_\beta(C_j) = \text{spot}_\alpha(C_j)$ for every $j \in [n]$, and it follows that $\beta \in \text{IPF}_n(\ell)$.
- If $a_{i+1} \leq i$, then $(\text{spot}_\beta(C_i), \text{spot}_\beta(C_{i+1})) = (\text{spot}_\alpha(C_{i+1}), \text{spot}_\alpha(C_i)) = (i + 1, i)$. By condition **(I2)**, $\text{disp}_\beta(C_{i+1}) = \text{disp}_\alpha(C_{i+1}) - 1$ and $\text{disp}_\beta(C_i) = \text{disp}_\alpha(C_i) + 1$. Hence $\beta \in \text{IPF}_n(\ell)$ if and only if $\text{disp}_\alpha(C_i) \leq \ell - 1$.

Thus we have a injective map $\kappa : \mathcal{P}_{n,\ell}^I \rightarrow \mathcal{I}_{n,\ell}$. For surjectivity, observe that if $\beta \in \mathcal{I}_{n,\ell}$ has a unique inversion $b_i > b_{i+1}$, then $\alpha = \beta^\uparrow \in \text{IPF}_n^\uparrow(\ell)$ by Proposition 5.5, and the same case argument shows that $(\alpha, i) \in \mathcal{P}_{n,\ell}^I$.

Second, given a pair $(\alpha, i) \in \mathcal{P}_{n,\ell}^M$, let $\beta = \lambda(\alpha, i)$ be the parking function obtained by moving a_{i+1} to the front. Similarly to the first case, β is a parking function with major index 1, and $\alpha = \beta^\uparrow$.

By interval rearrangement (Lemma 5.4), the spots and displacements of cars C_{i+2}, \dots, C_n are unchanged from α to β . By construction, we have for all $j \in [n]$,

$$\text{spot}_\beta(C_j) = \begin{cases} a_{i+1} & \text{if } j = i + 1, \\ \text{spot}_\alpha(C_j) & \text{if } j \leq i \text{ and } \text{spot}_\alpha(C_j) < a_{i+1}, \\ \text{spot}_\alpha(C_j) + 1 & \text{if } j \leq i \text{ and } \text{spot}_\alpha(C_j) \geq a_{i+1}, \end{cases}$$

and

$$\text{disp}_\beta(C_j) = \begin{cases} 0 & \text{if } j = i + 1, \\ \text{disp}_\alpha(C_j) & \text{if } j \leq i \text{ and } \text{spot}_\alpha(C_j) < a_{i+1}, \\ \text{disp}_\alpha(C_j) + 1 & \text{if } j \leq i \text{ and } \text{spot}_\alpha(C_j) \geq a_{i+1}, \end{cases} = \begin{cases} 0 & \text{if } j = i + 1, \\ \text{disp}_\alpha(C_j) & \text{if } j < a_{i+1}, \\ \text{disp}_\alpha(C_j) + 1 & \text{if } a_{i+1} \leq j \leq i, \end{cases}$$

and now condition **(M2)** in the definition of $\mathcal{P}_{n,\ell}^M$ implies that β has maximum displacement ℓ , hence belongs to $\mathcal{M}_{n,\ell}$.

Thus we have an injective map $\lambda : \mathcal{P}_{n,\ell}^M \rightarrow \mathcal{M}_{n,\ell}$. To see that it is a surjection, let $\beta \in \mathcal{M}_{n,\ell}$; then $\alpha = \beta^\uparrow$ has the form

$$\alpha = \begin{pmatrix} C_2 & C_3 & \cdots & C_{i+1} & C_1 & C_{i+2} & \cdots & C_n \\ b_2 & b_3 & \cdots & b_{i+1} & b_1 & b_{i+2} & \cdots & C_n \end{pmatrix},$$

where i is the greatest index such that $b_1 > b_{i+1}$. Then $\alpha \in \text{IPF}_n^\uparrow(\ell)$ by Proposition 5.5, and again the calculation above shows that $(\alpha, i) \in \mathcal{P}_{n,\ell}^M$.

Third, it is immediate from their definitions that $\mathcal{P}_{n,\ell}^I \supseteq \mathcal{P}_{n,\ell}^M$. To prove that containment is strict, consider the parking function

$$p = (\underbrace{1, \dots, 1}_{\ell+1 \text{ copies}}, \ell+1, \ell, \ell+4, \ell+5, \dots, n) \in \mathcal{I}_{n,\ell}$$

(recall that $n \geq 6$ and $3 \leq \ell \leq n-3$). Then $\kappa^{-1}(p) = (\alpha, \ell+2)$, where

$$\alpha = (\underbrace{1, \dots, 1}_{\ell+1 \text{ copies}}, \ell, \ell+1, \ell+4, \ell+5, \dots, n).$$

Thus $(\alpha, \ell+2) \in \mathcal{P}_{n,\ell}^I$. However, setting $i = \ell+2$ and $j = \ell+1$, we have

$$a_{i+1} = \ell+1 \leq j \leq i, \quad \text{but} \quad \text{disp}_\alpha(C_j) = (\ell+1) - a_{\ell+1} = (\ell+1) - 1 = \ell.$$

So condition **(M2)** fails and $(\alpha, \ell+2) \notin \mathcal{P}_{n,\ell}^M$, proving strict containment as desired. \square

We next consider the case $\ell = 1$.

Proposition 5.9. *The Foata transform restricts to a bijection $\text{UPF}_n \rightarrow \text{UPF}_n$.*

Proof. Let $q \in \text{UPF}_n$, and let r be the parking function obtained by cycling a single interval $[a, b]$:

$$q = \left(\begin{array}{c|ccc} \cdots & C_a & \cdots & C_b & \cdots \\ \cdots & q_a & \cdots & q_b & \cdots \end{array} \right) \quad \rightarrow \quad r = \left(\begin{array}{c|ccccc} \cdots & C_b & C_a & \cdots & C_{b-1} & \cdots \\ \cdots & q_b & q_a & \cdots & q_{b-1} & \cdots \end{array} \right).$$

Here the bars are included for clarity. Note that $\text{spot}_r(C_b) \leq \text{spot}_q(C_b)$ because passing from q to r moves C_b toward the front of the queue.

If $\text{spot}_r(C_b) = \text{spot}_q(C_b)$, then the Reserved Spot Lemma (Lemma 5.6) implies that all outcomes are preserved, so $r \in \text{UPF}_n$. Accordingly, we assume henceforth that $\text{spot}_r(C_b) < \text{spot}_q(C_b)$. Since q is a unit interval parking function, we have

$$0 \leq \text{disp}_r(C_b) = \text{spot}_r(C_b) - q_b < \text{spot}_q(C_b) - q_b = \text{disp}_q(C_b) \leq 1$$

so the two inequalities are both actually equalities. In particular,

$$(33) \quad \text{spot}_r(C_b) = q_b \quad \text{and} \quad \text{spot}_q(C_b) = q_b + 1.$$

By interval rearrangement (Lemma 5.4),

$$\sum_{i=a}^b \text{spot}_q(C_i) = \sum_{i=a}^b \text{spot}_r(C_i)$$

so

$$\sum_{i=a}^b \text{disp}_q(C_i) = \sum_{i=a}^b \text{disp}_r(C_i)$$

and

$$1 + \sum_{i=a}^{b-1} \text{disp}_q(C_i) = \sum_{i=a}^{b-1} \text{disp}_r(C_i)$$

and $\text{disp}_r(C_i) \geq \text{disp}_q(C_i)$ for all $i \in [a, b-1]$. From this we may conclude that there is a unique $j \in [a, b-1]$ such that

$$(34) \quad \text{disp}_r(C_j) = \text{disp}_q(C_j) + 1 \text{ and}$$

$$(35) \quad \text{disp}_r(C_i) = \text{disp}_q(C_i) \quad \text{for all } i \in [a, b-1] \setminus \{j\}.$$

If $\text{disp}_r(C_j) = 1$ and $\text{disp}_q(C_j) = 0$, then r is a UPF and we are done. Accordingly, assume for the sake of contradiction that $\text{disp}_r(C_j) = 2$ and $\text{disp}_q(C_j) = 1$. By the Reserved Spot Lemma (Lemma 5.6), the parking function

$$s = \left(\begin{array}{cccc|cccc|c} \cdots & C_b & C_j & C_a & \cdots & C_{j-1} & C_{j+1} & \cdots & C_{b-1} & \cdots \\ \cdots & q_b & q_j & q_a & \cdots & q_{j-1} & C_{j+1} & \cdots & q_{b-1} & \cdots \end{array} \right)$$

satisfies $\text{spot}_s(i) = \text{spot}_r(i)$ for all $i \in [a, b]$. On the other hand, certainly

$$\{\text{spot}_q(C_i) : i \in [a, b]\} = \{\text{spot}_r(C_i) : i \in [a, b]\} = \{\text{spot}_s(C_i) : i \in [a, b]\}$$

and we have seen that for $i \notin \{b, j\}$,

$$\text{spot}_s(C_i) = \text{spot}_r(C_i) = \text{spot}_q(C_i).$$

We conclude that

$$\{\text{spot}_r(C_b), \text{spot}_r(C_j)\} = \{\text{spot}_q(C_b), \text{spot}_q(C_j)\},$$

which forces

$$\begin{aligned} \text{spot}_q(C_j) &= \text{spot}_r(C_b) = q_b, \text{ and} \\ \text{spot}_r(C_j) &= \text{spot}_q(C_b) = q_b + 1. \end{aligned}$$

Combining with (34), we have

$$\text{disp}_q(C_j) = 1 = \text{spot}_q(C_j) - q_j = q_b - q_j$$

i.e., $q_j = q_b - 1$. So it must be the case that spot q_j was occupied by some car in a previous segment, and with respect to q , cars C_j and C_b park in spots $q_j + 1$ and $q_j + 2$ respectively. On the other hand, with respect to r , car C_b parks first in spot $q_j + 1$, and car C_j must park in spot $q_b + 2$, giving it displacement 2.

Now, suppose that the cycling operation that produced r from q was a step in the Foata transform (Definition 5.2). In particular $q_j < p_n \leq q_b$, and in fact $p_n = q_b$ (by the earlier observation $q_j = q_b - 1$). But since q is a UPF, we have

$$\text{spot}_q(C_n) \in \{q_b, q_b + 1\}$$

but this is impossible because

$$\text{spot}_q(C_j) = q_j + 1 = q_b \quad \text{and} \quad \text{spot}_q(C_b) = q_j + 2 = q_b + 1$$

giving the desired contradiction. \square

Finally, we consider the most difficult case, when $\ell = 2$.

Proposition 5.10. *For all $n \geq 1$, if $p \in \text{IPF}_n(2)$, then $F(p) \in \text{IPF}_n(2)$.*

Proof. We define a **Boojum**² to be a parking function p such that (i) $\text{maxdisp}(p) = 2$; (ii) $\text{maxdisp}(F(p)) > 2$; and (iii) p is of minimal length among all parking functions satisfying (i) and (ii). We show that Boojums do not exist.

Suppose for contradiction that $p = (p_1, \dots, p_n)$ is a Boojum. We mostly work with $q = F_{n-1}(p) = (q_1, \dots, q_n)$, where as before F_{n-1} is a partial Foata transform. Accordingly, we assume that the order of cars with respect to q is C_1, \dots, C_n . Note that q is a parking function (because it is a rearrangement of the parking function p). Let $r = F(p)$.

Observe that q itself is a 2-parking function, because

- $\hat{p} = (p_1, \dots, p_{n-1})$ is a 2-parking function, and by minimality of Boojums $\hat{q} = (q_1, \dots, q_{n-1}) = F(\hat{p})$ is as well; and
- $q_n = p_n$, so by interval rearrangement (Lemma 5.4) and the fact that \hat{q} is a permutation of \hat{p} , we have $\text{disp}_q(C_n) = \text{disp}_p(C_n) \leq 2$.

²A creature that does not exist: see [6]

Consider what happens in the last step of calculating the Foata transform of p . We place some set of separators in \hat{q} , cycle each segment, and then append p_n . Suppose there are $k \geq 1$ separators after positions i_1, \dots, i_k where $i_k = n - 1$. That is,

$$q = \left(\begin{array}{ccc|ccc|ccc|c} C_1 & \cdots & C_{i_1} & C_{i_1+1} & \cdots & C_{i_2} & \cdots & C_{i_{k-1}+1} & \cdots & C_{i_k} & C_n \\ q_1 & \cdots & q_{i_1} & q_{i_1+1} & \cdots & q_{i_2} & \cdots & q_{i_{k-1}+1} & \cdots & q_{i_k} & p_n \end{array} \right)$$

and

$$r = \left(\begin{array}{cccc|cccc|ccc|c} C_{i_1} & C_1 & \cdots & C_{i_1-1} & C_{i_2} & C_{i_1+1} & \cdots & C_{i_2-1} & \cdots & C_{i_k} & C_{i_{k-1}+1} & \cdots & C_{i_k-1} & C_n \\ q_{i_1} & q_1 & \cdots & q_{i_1-1} & q_{i_2} & q_{i_1+1} & \cdots & q_{i_2-1} & \cdots & q_{i_k} & q_{i_{k-1}+1} & \cdots & q_{i_k-1} & p_n \end{array} \right).$$

By the definition of a Boojum, q is a 2-interval parking function and r is not. That is, for some j , we have

$$(36) \quad \text{disp}_q(C_j) \leq 2 < \text{disp}_r(C_j).$$

Note that $j \neq n$, because $\text{spot}_q(C_n) = \text{spot}_r(C_n)$ (so, $\text{disp}_r(C_n) = \text{disp}_q(C_n)$). Moreover, if $j \leq i_{k-1}$, then $q_1 \cdots q_{i_{k-1}} p_n$ is the i_{k-1} th partial Foata transform of a smaller Boojum, which contradicts the minimality assumption on p . Therefore, $i_{k-1} + 1 \leq q_j \leq i_k$. Thus, we are concerned only with the behavior of the cars with labels in the interval $B = [u, \dots, n - 1]$, where for convenience we set $u = i_{k-1} + 1$.

We claim that $\text{spot}_q(C_{n-1}) > \text{spot}_r(C_{n-1})$. Indeed, since C_{n-1} parks earlier with respect to r than with respect to q , we have $\text{spot}_q(C_{n-1}) \geq \text{spot}_r(C_{n-1})$. If equality holds, then by the Reserved Spot Lemma (Lemma 5.6), r would in fact be a 2-interval parking function, a contradiction. Consequently,

$$(37) \quad 2 \geq \text{disp}_q(C_{n-1}) > \text{disp}_r(C_{n-1}) \geq 0.$$

Also, $p_n \neq q_j$. To see this, observe that because $\text{disp}_r(C_j) > \text{disp}_r(C_n)$ even though C_j parks before C_n , so their preferred spots cannot possibly be equal.

We now consider two cases of the Foata bijection.

Case 1: Assume $q_u, \dots, q_{n-2} \leq p_n < q_{n-1}$. Observe that no car that originally parked in a spot less than q_{n-1} is affected by the cycling. In particular,

$$(38) \quad \text{spot}_q(C_j) \geq q_{n-1}.$$

The assumptions of Case 1 imply that

$$(39) \quad q_{n-1} > p_n > q_j.$$

Now from (36) and (38) and (39) we get

$$(40) \quad q_j < p_n < q_{n-1} \leq \text{spot}_q(C_j) \leq 2 + q_j.$$

Therefore $p_n = q_j + 1$, and thus $q_{n-1} = \text{spot}_q(C_j) = q_j + 2$. Since $\text{disp}_r(C_j) \geq 3$, we know that spots $q_j + 1, q_j + 2$ were already occupied before C_j parks, and $q_j + 3$ (at least) is occupied after C_j parks. Therefore, C_n , who wanted to park in spot $p_n = q_j + 1$, cannot park in any of the spots in the set $\{q_j + 1, q_j + 2, q_j + 3\}$. So $\text{disp}_r(C_n) \geq 3$. But $\text{disp}_r(C_n) = \text{disp}_q(C_n)$, which contradicts the assumption that p is a 2-interval parking function and eliminates Case 1.

Case 2: Assume $q_u, \dots, q_{n-2} > p_n \geq q_{n-1}$.

Suppose that $\text{spot}_q(C_i) \neq q_{n-1}$ for all $i < u$. That is, spot q_{n-1} is available before the cars in B start to park. Since q_{n-1} is the strictly smallest preference in B , it follows that $\text{spot}_r(C_{n-1}) = \text{spot}_q(C_{n-1}) = q_{n-1}$, so the Reserved Spot Lemma (Lemma 5.6) implies $\text{spot}_r(C_h) = \text{spot}_q(h)$ for all $h \in [u, n - 2]$, contradicting the assumption that p is a Boojum. Hence $\text{spot}_r(C_{n-1}) > q_{n-1}$, i.e., $\text{disp}_r(C_{n-1}) > 0$, so (37) implies that

$$\text{spot}_r(C_{n-1}) = q_{n-1} + 1 \quad \text{and thus} \quad \text{spot}_q(C_{n-1}) = q_{n-1} + 2.$$

This implies in turn that $\text{spot}_q(C_i) \notin \{q_{n-1} + 1, q_{n-1} + 2\}$ for all $i < u$. Therefore, there must be some $h \geq u$ such that $q_h = q_{n-1} + 1$ (which is the reason C_{n-1} is unable to park there with respect to q), but no other car with preference equal to $q_{n-1} + 1$ or $q_{n-1} + 2$ (because if there were then C_{n-1} would not be able to park in $q_{n-1} + 2$ with respect to q). But this says that the effect of cycling C_{n-1} to the front of the segment is limited to changing

$$(\text{spot}_q(C_n), \text{spot}_q(C_h)) = (q_{n-1} + 2, q_{n-1} + 1)$$

to

$$(\text{spot}_r(C_n), \text{spot}_r(C_h)) = (q_{n-1} + 1, q_{n-1} + 2)$$

and all the other cars, in particular C_j , are unaffected. This rules out Case 2, which completes the proof. \square

5.4. Enumeration by major index. The Foata transform F preserves the content of a parking function and therefore its displacement. Therefore, Theorem 5.1 has the following immediate consequence.

Corollary 5.11. *Let $n \geq 1$ and $\ell \in \{0, 1, 2, n-2, n-1\}$. Then*

$$\sum_{\alpha \in \text{IPF}_n(\ell)} q^{\text{disp}(\alpha)} t^{\text{inv}(\alpha)} = \sum_{\alpha \in \text{IPF}_n(\ell)} q^{\text{disp}(\alpha)} t^{\text{maj}(\alpha)}.$$

Combining this observation with Corollary 3.2 and Theorem 3.3 (together with the fact that $\text{asc}(F(\sigma)^{-1}) = \text{asc}(\sigma^{-1})$ for each $\sigma \in \mathfrak{S}_n$ [13, Thm. 1]) yields major-index versions of those results.

Corollary 5.12. *Let $n \geq 2$. Then*

$$\sum_{\alpha \in \text{IPF}_n(n-2)} q^{\text{disp}(\alpha)} t^{\text{maj}(\alpha)} = \sum_{\alpha \in \text{PF}_n} q^{\text{disp}(\alpha)} t^{\text{maj}(\alpha)} - (qt)^{n-1} \sum_{\beta \in \text{PF}_{n-1}} q^{\text{disp}(\beta)} t^{\text{maj}(\beta) - \text{ones}(\beta)},$$

where $\text{ones}(\beta) = |\{i \in [n-1] : b_i = 1\}|$ and

$$\sum_{\alpha \in \text{UPF}_n} q^{\text{disp}(\alpha)} t^{\text{maj}(\alpha)} = \sum_{\sigma \in \mathfrak{S}_n} (1+q)^{\text{asc}(\sigma)} t^{\text{maj}(\sigma^{-1})}.$$

For 2-interval parking functions (Theorem 3.11), the appearance of the statistics $|\mathcal{R}(\beta)|$ and $|\mathcal{S}(\beta)|$ makes the application of the Foata transform less straightforward. It turns out that F preserves these two statistics, thanks to the following observation.

Lemma 5.13. *Let $(a_1, a_2, \dots, a_n) \in \mathbb{N}^n$. Let $j, k \in [n]$ such that $a_j \leq a_k$, and let $i \geq \max(j, k)$.*

- (1) *If $a_{i+1} \notin \{a_j, \dots, a_k - 1\}$ then a_j and a_k appear in the same order in $F(a_1 a_2 \cdots a_{i+1})$ as they do in $F(a_1 a_2 \cdots a_i)$.*
- (2) *If $\{a_j, \dots, a_k - 1\} \cap \{a_{i+1}, \dots, a_n\} = \emptyset$, then a_j and a_k appear in the same order in $F(a_1 \cdots a_n)$ as they do in $F(a_1 a_2 \cdots a_i)$.*

Proof. (1) The condition $a_{i+1} \notin \{a_j, \dots, a_k - 1\}$ implies that, at the $(i+1)$ st stage of the algorithm, separators are placed either after both a_j and a_k , or after neither. If both, then a_j and a_k belong to different segments. If neither, then if they are in the same segment, then neither one is the last entry in that segment. Therefore, in all cases, the cycling step maintains the relative position of a_j and a_k , so they appear in the same order in $F(a_1 \cdots a_{i+1})$ as in $F(a_1 \cdots a_i)$.

(2) This claim follows from applying (1) repeatedly. \square

Lemma 5.14. *If $\alpha \in \text{UPF}_n$, then $|\mathcal{R}(\alpha)| = |\mathcal{R}(F(\alpha))|$ and $|\mathcal{S}(\alpha)| = |\mathcal{S}(F(\alpha))|$.*

Proof. Since F preserves content, it preserves block structure, hence $|\mathcal{S}(\alpha)| = |\mathcal{S}(F(\alpha))|$ by (14).

For $\alpha = (a_1, a_2, \dots, a_n) \in \text{UPF}_n$, let $\mathcal{P}(\alpha)$ be the set of positions of the second entries of blocks. Equivalently, by block structure,

$$\mathcal{P}(\alpha) = \{i \in [n] : a_j = a_i \text{ for some } j < i\}.$$

Note that $\mathcal{R}(\alpha) \subseteq \mathcal{P}(\alpha)$. Moreover, $|\mathcal{P}(\alpha)| = |\mathcal{P}(F(\alpha))|$, because F preserves content. For convenience, define $\mathcal{R}'(\alpha) = \{a_i : i \in \mathcal{R}(\alpha)\}$, so that $|\mathcal{R}'(\alpha)| = |\mathcal{R}(\alpha)|$.

We claim that $\mathcal{R}'(\alpha) = \mathcal{R}'(F(\alpha))$. Indeed, let $k \in \mathcal{P}(\alpha)$, and let π_s be the block containing a_k . Taking a_j to be either the first entry of π_s or the last entry of π_{s-1} , and taking $i = \max(j, k)$, block structure implies that the elements a_j, a_k satisfy the conditions of Lemma 5.13, hence appear in the same order in $F(\alpha)$ as they do in $F(a_1 a_2 \cdots a_i)$, hence the same order as in $a_1 \cdots a_i$ (due to our choice of i). In particular, $a_k \in \mathcal{R}'(\alpha)$ if and only if $a_k \in \mathcal{R}'(F(\alpha))$. Therefore,

$$|\mathcal{R}(\alpha)| = |\mathcal{R}'(\alpha)| = |\mathcal{R}'(F(\alpha))| = |\mathcal{R}(F(\alpha))|.$$

Example 5.15. Let $\alpha = 34411 \in \text{UPF}_5$, with block structure $11 \mid 3 \mid 44$. Then $F(\alpha) = 13441$. Note that

$$\begin{aligned}\mathcal{P}(\alpha) &= \{3, 5\}, & \mathcal{R}(\alpha) &= \{3\}, \\ \mathcal{P}(F(\alpha)) &= \{4, 5\}, & \mathcal{R}(F(\alpha)) &= \{4\},\end{aligned}$$

and $\mathcal{R}'(\alpha) = \mathcal{R}'(F(\alpha)) = \{4\}$.

Applying Lemma 5.14 and Theorem 5.1 to Theorem 3.11, we obtain a major-index enumeration for 2-interval parking functions.

Corollary 5.16. *For all $n \geq 1$,*

$$\sum_{\alpha \in \text{IPF}_n(2)} q^{\text{disp}(\alpha)} t^{\text{maj}(\alpha)} = \sum_{\beta \in \text{UPF}_n} q^{\text{disp}(\beta)} (1+q)^{|\mathcal{S}(\beta)|} (1+qt)^{|\mathcal{R}(\beta)|} t^{\text{maj}(\beta)}.$$

It would be interesting to provide direct proofs of Corollaries 5.12 and 5.16 that do not rely on the Foata bijection.

6. FURTHER PROBLEMS

We provide some open problems for further study.

Problem 6.1. *Describe a block structure for ℓ -interval parking functions that generalizes the $\ell = 1$ case (Theorem 2.10).*

As we noted in Section 2.2, the block structure of unit interval parking functions appears to be unique to the case $\ell = 1$. For $\ell > 1$, the problem seems to be much harder.

Problem 6.2. *What can be said about the effect of the Foata transform on the outcome of a parking function?*

One obvious thing is that if $\alpha \in \mathfrak{S}_n$, then $\text{spot}_{F(\alpha)} = F(\text{spot}_\alpha)$ (because $\alpha = \text{spot}_\alpha$). However, this does not hold in general. For instance, let $\alpha = 121$. Then $F(\alpha) = 211$, but $\text{spot}_{F(\alpha)} = 213$ whereas $F(\text{spot}_\alpha) = F(123) = 123$. For unit interval parking functions (which the previous α is not), it seems that F acts on the spots permutations as it did for permutations:

Conjecture 6.3. *For any unit interval parking function α , $\text{spot}_{F(\alpha)} = F(\text{spot}_\alpha)$.*

We have verified Conjecture 6.3 for all unit interval parking functions of length at most 8.

Problem 6.4. *Count ℓ -interval parking functions in terms of $(\ell \pm 1)$ -interval parking functions.*

Theorems 3.3 and 3.11 as well as Corollary 3.2 are solutions to the above problem for $\ell = 1, 2, n-2$, respectively. Let $\text{IPF}_n(\ell)^{\max}$ be the set of parking functions α of length n such that $\text{maxdisp}(\alpha) = \ell$. It follows from our enumerative results that for all $n \geq 1$

$$n! = |\text{IPF}_n(0)^{\max}| \leq |\text{IPF}_n(1)^{\max}| \leq |\text{IPF}_n(2)^{\max}| \quad \text{and} \quad |\text{IPF}_n(n-2)^{\max}| \geq |\text{IPF}_n(n-1)^{\max}| = n^{n-2}$$

Experimentally, it seems this pattern continues. See Table 1.

Conjecture 6.5. *For each $n \geq 1$, the sequence $(|\text{IPF}_n(\ell)^{\max}|)_{\ell=0}^{n-1}$ is unimodal i.e. if $a_\ell = |\text{IPF}_n(\ell)^{\max}|$, there exists an integer $c \in \{0, 1, \dots, n-1\}$ such that*

$$n! = a_0 \leq a_1 \leq a_2 \leq \dots \leq a_c \geq \dots \geq a_{n-2} \geq a_{n-1} = n^{n-2}.$$

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$n \setminus \ell$	0	1	2	3	4	5	6	7	8
1	1								
2	2	1							
3	6	7	3						
4	24	51	34	16					
5	120	421	377	253	125				
6	720	3963	4594	3688	2546	1296			
7	5040	42253	62145	57398	46142	32359	16807		
8	40320	505515	929856	979430	865970	702292	497442	262144	
9	362880	6724381	15298809	18289811	17520519	15455851	12587507	8977273	4782969

TABLE 1. $|\text{IPF}_n(\ell)^{\max}|$ up to $n = 9$. OEIS sequence numbers forthcoming.

REFERENCES

- [1] Ayomikun Adeniran, Steve Butler, Galen Dorpalen-Barry, Pamela E. Harris, Cyrus Hettle, Qingzhong Liang, Jeremy L. Martin, and Hayan Nam. Enumerating parking completions using Join and Split. *Electron. J. Combin.*, 27(2):Paper No. 2.44, 19, 2020.
- [2] Tomás Aguilar-Fraga, Jennifer Elder, Rebecca E. Garcia, Kimberly P. Hadaway, Pamela E. Harris, Kimberly J. Harry, Imhotep B. Hogan, Jakeyl Johnson, Jan Kretschmann, Kobe Lawson-Chavarnu, J. Carlos Martínez Mori, Casandra D. Monroe, Daniel Quiñonez, Dirk Tolson, III, and Dwight Anderson Williams, II. Interval and ℓ -interval rational parking functions. *Discrete Math. Theor. Comput. Sci.*, 26(1):Paper No. 10, 29, [2024–2025].
- [3] Yasmin Aguillon, Dylan Alvarenga, Pamela E. Harris, Surya Kotapati, J. Carlos Martínez Mori, Casandra D. Monroe, Zia Saylor, Camelle Tieu, and Dwight Anderson Williams, II. On parking functions and the tower of Hanoi. *Amer. Math. Monthly*, 130(7):618–624, 2023.
- [4] Adrian Avalos and Mark Bly. Sequences, q -multinomial identities, integer partitions with kinds, and generalized Galois numbers. *PUMP J. Undergrad. Res.*, 3:72–94, 2020.
- [5] S. Alex Bradt, Jennifer Elder, Pamela E. Harris, Gordon Rojas Kirby, Eva Reutercrona, Yuxuan Wang, and Juliet Whidden. Unit interval parking functions and the r-Fubini numbers. *Matematica*, 3(1):370–384, 2024.
- [6] Lewis Carroll. The hunting of the snark: An agony in eight fits. Originally published 1876; full text at poetryfoundation.org/poems/43909/the-hunting-of-the-snark.
- [7] Lucas Chaves Meyles, Pamela E. Harris, Richter Jordaan, Gordon Rojas Kirby, Sam Sehayek, and Ethan Spingarn. Unit-interval parking functions and the permutohedron. [arXiv:2305.15554](https://arxiv.org/abs/2305.15554), to appear in *J. Comb.*, 2023.
- [8] Emma Colaric, Ryan DeMuse, Jeremy L. Martin, and Mei Yin. Interval parking functions. *Adv. in Appl. Math.*, 123:Paper No. 102129, 17, 2021. [arXiv:2006.09321](https://arxiv.org/abs/2006.09321).
- [9] Laura Colmenarejo, Pamela E. Harris, Zakiya Jones, Christo Keller, Andrés Ramos Rodríguez, Eunice Sukarto, and Andrés R. Vindas-Meléndez. Counting k -Naples parking functions through permutations and the k -Naples area statistic. *Enumer. Comb. Appl.*, 1(2):Paper No. S2R11, 16, 2021.
- [10] Jennifer Elder, Pamela E. Harris, Jan Kretschmann, and J. Carlos Martínez Mori. Parking functions, Fubini rankings, and Boolean intervals in the weak order of \mathfrak{S}_n . *J. Comb.*, 16(1):65–89, 2025.
- [11] Dominique Foata. On the Netto inversion number of a sequence. *Proc. Amer. Math. Soc.*, 19:236–240, 1968.
- [12] Dominique Foata and John Riordan. Mappings of acyclic and parking functions. *Aequationes Math.*, 10:10–22, 1974.
- [13] Dominique Foata and Marcel-Paul Schützenberger. Major index and inversion number of permutations. *Math. Nachr.*, 83:143–159, 1978.
- [14] Robin Forman. Morse theory for cell complexes. *Adv. Math.*, 134(1):90–145, 1998.
- [15] Robin Forman. A user’s guide to discrete Morse theory. *Sém. Lothar. Combin.*, 48:Art. B48c, 35, 2002.
- [16] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete mathematics*. Addison-Wesley Publishing Company, Reading, MA, second edition, 1994. A foundation for computer science.
- [17] Kimberly P. Hadaway. On Combinatorial Problems of Generalized Parking Functions, 2021. Williams College Honors Thesis.
- [18] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 2025. Published electronically at <https://oeis.org>.
- [19] A. Konheim and B. Weiss. An occupancy discipline and applications. *SIAM Journal on Applied Mathematics*, 14(6):1266–1274, 1966.
- [20] P. A. MacMahon. Two Applications of General Theorems in Combinatory Analysis. *Proc. London Math. Soc. (2)*, 15:314–321, 1916.
- [21] Victor Reiner, Dennis W. Stanton, and Dennis E. White. The cyclic sieving phenomenon. *J. Combin. Theory Ser. A*, 108(1):17–50, 2004.
- [22] SageMath Inc. *CoCalc Collaborative Computation Online*, 2022. <https://cocalc.com/>.

- [23] Richard P. Stanley. *Enumerative Combinatorics. Volume 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
- [24] Richard P. Stanley. *Enumerative Combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.
- [25] William A. Stein et al. *Sage Mathematics Software (Version 9.4)*. The Sage Development Team, 2022. <http://www.sagemath.org>.
- [26] Catherine H. Yan. Parking functions. In *Handbook of Enumerative Combinatorics*, Discrete Math. Appl. (Boca Raton), pages 835–893. CRC Press, Boca Raton, FL, 2015.

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