Math 824, Fall 2012
Problem Set #5

Instructions: Type up your solutions using LaTeX; there is a header file on the course website with macros that may be useful. E-mail me (`jmartin@math.ku.edu`) the PDF file under the name {*your-name*}*5.pdf*. Deadline: **5:00 PM on Monday, December 3.**

Problem #1 Consider the permutation action of the symmetric group $\mathfrak{S}_4$ on the vertices of the complete graph K_4, whose corresponding representation is the defining representation ρ_{def} (let's say over $\mathbb{C}$). Let σ be the 3-dimensional representation corresponding to the action of $\mathfrak{S}_4$ on pairs of opposite edges of K_4.

(#1a) Compute the character of σ.

(#1b) Explicitly describe all G-equivariant linear transformations $\phi : \rho_{\text{def}} \to \sigma$. (Hint: Schur's lemma should be useful.)

Problem #2 Recall that the *alternating group* $\mathfrak{A}_n$ consists of the $n!/2$ even permutations in $\mathfrak{S}_n$, that is, those with an even number of even-length cycles.

(#2a) Show that the conjugacy classes in $\mathfrak{A}_4$ are not simply the conjugacy classes in $\mathfrak{S}_4$. (Hint: Consider the possibilities for the dimensions of the irreducible characters of $\mathfrak{A}_4$.)

(#2b) Determine the conjugacy classes in $\mathfrak{A}_4$, and the complete list of irreducible characters.

(#2c) Use this information to determine $[\mathfrak{A}_4, \mathfrak{A}_4]$ without actually computing any commutators.

Problem #3 Supply the proofs for the identities (9.12) on p.108 of the lecture notes:

$$\prod_{i,j\geq 1} (1 + x_i y_j) \;=\; \sum_\lambda e_\lambda(\mathbf{x}) m_\lambda(\mathbf{y}) \;=\; \sum_\lambda \varepsilon_\lambda \frac{p_\lambda(\mathbf{x}) p_\lambda(\mathbf{y})}{z_\lambda}.$$

Problem #4 Prove part (vii) of Theorem 9.25 on p.118 of the lecture notes, namely that the Schur functions $\{s_\lambda \mid \lambda \vdash n\}$ are the images of the irreducible characters of $\mathfrak{S}_n$ under the Frobenius characteristic **ch**. (Hint: Use the fact that **ch** is an isometry. This is one of those proofs that may seem like sleight-of-hand!)

Problem #5 **(#5a)** For $w \in \mathfrak{S}_n$, let $(P(w), Q(w))$ be the pair of tableaux produced by the RSK algorithm from w. Denote by w^* the reversal of w in one-line notation (for instance, if $w = 57214836$ then $w^* = 63841275$). Prove that $P(w^*) = P(w)^T$ (where ^{T} means transpose). Hint: Figure out how to describe the rows and columns of $P(w)$ in terms of subsequences of w.

(#5b) *(Open problem; optional)* For which permutations does $Q(w^*) = Q(w)$? Maple computation indicates that the number of such permutations is

$$\begin{cases} \dfrac{2^{(n-1)/2}(n-1)!}{((n-1)/2)!^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

but I don't know a combinatorial (or even an algebraic) reason.

(#5c) *(Open problem; optional)* For which permutations does $Q(w^*) = Q(w)^T$? I have no idea what the answer is. The sequence $(q_1, q_2, \dots) = (1, 2, 2, 12, 24, 136, 344, 2872, 7108, \dots)$, where $q_n = \#\{w \in \mathfrak{S}_n \mid Q(w^*) = Q(w)^T\}$, does not seem to appear in the Online Encyclopedia of Integer Sequences.

Problem #6 (*Open problem; optional*) Let $n \geq 2$ and for $\sigma \in \mathfrak{S}_n$, let $f(\sigma)$ denote the number of fixed points. As a warmup, prove that $\sum_{\sigma \in \mathfrak{S}_n} f(\sigma)^2 = 2 \cdot n!$. Open problem (to the best of my knowledge): Prove that for any n, k, the number $\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(\sigma)^k$ is an integer. It appears to be A203647 in OEIS. Find a formula and/or a representation-theoretic interpretation.