Math 824, Fall 2012
Problem Set #4

Instructions: Type up your solutions using LaTeX; there is a header file on the course website with macros that may be useful. E-mail me (`jmartin@math.ku.edu`) the PDF file under the name {*your-name*}*4.pdf*. Deadline: **3:00 PM on Friday, October 26.**

Problem #1 (Stanley, EC1, 2nd ed., 3.119) Prove the *q-binomial theorem*:

$$\prod_{k=0}^{n-1}(x-q^k) \;=\; \sum_{k=0}^{n}\binom{\mathbf{n}}{\mathbf{k}}(-1)^k q^{\binom{k}{2}} x^{n-k}.$$

Here $\binom{\mathbf{n}}{\mathbf{k}}$ denotes the *q-binomial coefficient*:

$$\binom{\mathbf{n}}{\mathbf{k}} \;=\; \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}.$$

You may, with appropriate citation, use identities such as those on p. 55 of Stanley's EC1. (Hint: Let $V=\mathbb{F}_q^n$ and let X be a vector space over $\mathbb{F}_q$ with x elements. Count the number of one-to-one linear transformations $V \to X$ in two ways.) Derive the ordinary binomial theorem as a corollary.

Problem #2 (Stanley, EC1, 3.129) Here is a cute application of combinatorics to elementary number theory. Let P be a finite poset, and let μ be the Möbius function of $\hat{P}=P\cup\{\hat{0},\hat{1}\}$. Suppose that P has a fixed-point-free automorphism $\sigma : P\to P$ of prime order p; that is, $\sigma(x)\neq x$ and $\sigma^p(x)=x$ for all $x\in P$. Prove that $\mu_{\hat{P}}(\hat{0},\hat{1}) \cong -1 \pmod{p}$. What does this say in the case that $\hat{P}=\Pi_p$?

Problem #3 (Stanley, HA, 2.5) Let G be a graph on n vertices, let $\mathcal{A}_G$ be its graphic arrangement in $\mathbb{R}^n$, and let $\mathcal{B}_G=\mathscr{B}_n\cup\mathcal{A}_G$. (That is, $\mathcal{B}$ consists of the coordinate hyperplanes $x_i=0$ in $\mathbb{R}^n$ together with the hyperplanes $x_i=x_j$ for all edges ij of G.) Calculate $\chi_{\mathcal{B}_G}(q)$ in terms of $\chi_{\mathcal{A}_G}(q)$.

Problem #4 (Stanley, EC2, 3.115(d)] Calculate the characteristic polynomial and the number of regions of the *type-B braid arrangement* $\mathcal{O}_n\subseteq\mathbb{R}^n$, with $n(n-1)$ hyperplanes $x_i=x_j$, $x_i=-x_j$ for $1\le i<j\le n$.

Problem #5 Recall that each permutation $w=(w_1,\dots,w_n)\in\mathfrak{S}_n$ corresponds to a region of the braid arrangement Br_n, namely the open cone $C_w=\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_{w_1}<x_{w_2}<\cdots<x_{w_n}\}$. Denote its closure by $\overline{C_w}$. For any set $W\subseteq\mathfrak{S}_n$, consider the closed fan

$$F(W)=\bigcup_{w\in W}\overline{C_w}=\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_{w_1}\le\cdots\le x_{w_n}\ \text{ for some } w\in W\}.$$

Prove that $F(W)$ is a convex set if and only if W is the set of linear extensions of some poset P on $[n]$. (A *linear extension* of P is a total ordering $\prec$ consistent with the ordering of P, i.e., if $x<_P y$ then $x\prec y$.)

Problem #6 The runners in a race are seeded $1,\dots,n$ (stronger runners are assigned higher numbers). To even the playing field, the rules specify that you earn one point for each higher-ranked opponent you beat, and one point for each lower-ranked opponent you beat *by at least one second*. (If a higher-ranked runner beats a lower-ranked runner by less than 1 second, no one gets a the point for that matchup.) Let s_i be the number of points scored by the i^{th} player and let $s=(s_1,\dots,s_n)$ be the *score vector*.

(#6a) Show that the possible score vectors are in bijection with the regions of the Shi arrangement.

(#6b) Work out all possible score vectors in the cases of 2 and 3 players. Conjecture a necessary and sufficient condition for $(s_1,\dots,s_n)$ to be a possible score vector for n players. Prove it if you can.