Contents

1. Mon 3/24: Motivation for Homology

The fundamental group is great for studying cell complexes of dimension 2. However, it isn't useful for higher-dimensional spaces (as we have proved).

One solution is to consider higher homotopy groups. Just as $\pi_1(X)$ is the group of homotopy classes of maps $\mathbb{S}^1 \to X$, one can define $\pi_k(X)$ to be the group of homotopy classes of maps $\mathbb{S}^k \to X$. The problem is that these groups are very hard to compute. For example, there is no analogue of Van Kampen's theorem or deck transformations. Even the homotopy groups of spheres have surprises (see table on p.339 of Hatcher):

$$\pi_k(\mathbb{S}^n) = \begin{cases} 0 & \text{if } k < n, \\ \mathbb{Z} & \text{if } k = n, \\ \text{very complicated} & \text{if } k > n. \end{cases}$$

The basic idea of homology theory is to abelianize. Instead of regarding the boundary of the triangle below as the composition of paths $a \cdot b \cdot c$, we regard it as a *formal sum* of paths: $a + b + c$. We are going to lose some information this way (for example, we will no longer to be able to distinguish $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = \mathbb{Z} * \mathbb{Z}$ from $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$) but we obtain invariants that can actually be calculated, both by brute-force computation for explicit cell complexes and by theoretical tools like the Mayer-Vietoris sequence (which is to homology what Van Kampen's theorem is for the fundamental group, only better).

Example (taken from Hatcher): Start with the graph X with two vertices x_0, x_1 and four edges a, b, c. The fundamental group is free on two generators. If we write them as ab^{-1} and ac^{-1}, then we see that the fundamental group $\pi_1(X, x_0)$ consists of all words of even length $g_1 \cdots g_{2n}$ with $g_i \in \{a, b, c\}$ for i odd and $g_i \in \{a^{-1}, b^{-1}, c^{-1}\}$ for i even.

It turns out that the homology group $H_1(X)$ is the abelianization of $\pi_1(X)$, which consists of all $\mathbb{Z}$-linear combinations $ka + \ell b + mc$ such that $k + \ell + m = 0$. These things are called **1-cycles**. In general, a formal linear combination of 1-cells in a cell complex is called a **cellular 1-chain**, and the group of all cellular 1-chains is denoted $C_1(X)$. . For now, a 1-cycle is a 1-chain that enters every vertex the same number of times that it leaves it.

Suppose we attach a 2-cell D along the word ab^{-1} to obtain a cell complex X'. Back in π_1-land, we know that this has the effect of killing off ab^{-1}. The same thing happens to H_1. The 1-chain $a-b$ is now a **1-boundary**: it is the boundary of a 2-cell and therefore is zero in $H_1(X')$. Specifically,

$$\pi_1(X') = \langle ab^{-1},\ ac^{-1} \mid ab^{-1}\rangle = \langle ab^{-1}\rangle \cong \mathbb{Z},$$
$$H_1(X') = \mathbb{Z}\langle a-b,\ a-c\rangle/\mathbb{Z}\langle a-b\rangle = \mathbb{Z}\langle a-c\rangle \cong \mathbb{Z}.$$

This is an example of the general definition of homology:

$$H_1(X) \ = \ \text{1-cycles / 1-boundaries.}$$

Note that for this to make sense, we have to check that every boundary is a cycle. But let's go on with the example.

Take X' and attach another 2-cell D' along ab^{-1} to obtain a complex X''. This does not change $\pi_1(X')$ or $H_1(X')$. However, it changes the second homology group. The 2-chain $D-D'$ is now a **2-cycle**. To see this, observe that each of D and D' has the 1-chain $a-b$ as its boundary, so if we regard "boundary" as a linear map ∂_2 then

$$\partial(D-D') = 0.$$

In general, we are going to define $\partial_i(X)$ as a linear map from i-chains to $(i-1)$-chains, and define

$$\begin{aligned}
Z_i(X) &= \ker \partial_i && (i\text{-cycles}),\\
B_i(X) &= \operatorname{im} \partial_{i+1} && (i\text{-boundaries}),\\
&B_i(X) \subseteq Z_i(X) \subseteq C_i(X), &&\\
H_i(X) &= Z_i(X)/B_i(X) && (i\text{-homology}).
\end{aligned}$$

Here $H_2(X'') = \mathbb{Z}$; it is generated by the 2-cycle $D-D'$. Had we attached D' to ac^{-1} instead, then $Z_2(X'')$ and $H_2(X'')$ would both remain trivial.

If we form X''' by attaching a 3-cell to X'' along the sphere $D-D'$, then $B_2(X''') = Z_2(X''')$ and so $H_2(X''') = 0$ again.

With this example in mind, here is a general overview of homology. For every space, we are going to construct a sequence of abelian groups

$$\cdots \ \to \ C_{n+1}(X) \ \xrightarrow{\partial_{n+1}} \ C_n(X) \ \xrightarrow{\partial_n} \ C_{n-1}(X) \ \to \ \cdots$$

in which $C_n(X)$ is the group of n-chains in X and ∂_n maps each n-chain to its boundary, an $(n-1)$-chain. These groups and maps form a **chain complex**, which means that $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$ for every n; equivalently, $\partial_n \circ \partial_{n+1} = 0$. We can then define

$$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

and show that the groups $H_n(X)$ are topological invariants of X.

There are lots of different ways to construct homology groups. We will start by developing a homology theory for **$\boldsymbol{\Delta}$-complexes**, which are particularly nice cell complexes. This will have the advantage that the computations are feasible, but it is not at all clear why the homology groups are topological invariants, or how to use it to compute homology of more general spaces.

In the general case, we need something called **singular homology**. This is great for theory and is easily seen to be a topological invariant; unfortunately, it leads to enormous chain groups that cannot be handled explicitly. So one of our goals will to be show that these two homology theories coincide.

2. Wed 3/26: Homology of simplicial complexes

2.1. **Simplices, orientations and boundary maps.** **Definition:** Let $n \in \mathbb{N}{\geq 0}$ and let $e_0, \dots, e_n$ be points in a real vector space that are *affinely independent*. This means that no space of dimension $n-1$ contains all of them; equivalently, the vectors $\{v_1 - v_0, \dots, v_n - v_0\}$ are linearly independent. The ***n*-simplex** $[v_0, \dots, v_n]$ is the convex hull of these points:

$$[v_0, \dots, v_n] = \{\sum_{i=0}^{n} a_i v_i \mid a_i \in [0,1],\ \sum_i a_i = 1\}.$$

The numbers a_i are the **barycentric coordinates** of the point $\sum_{i=0}^{n} a_i e_i$. The **standard *n*-simplex** Δ^n is $[e_0, \dots, e_n]$, where the e_i are the standard basis of $\mathbb{R}^{n+1}$.

The order of the points matters! For instance, you should think of the 1-simplex $[v_0, v_1]$ as a directed edge from v_0 to v_1, and you should think of the 2-simplex $[a, b, c]$ as a triangle with a curvy arrow pointing from a to b to c. Typically we fix an ordering on the points from the get-go, which means that there is a canonical homeomorphism $[v_0, \dots, v_n] \to [w_0, \dots, w_n]$ mapping $\sum a_i v_i \mapsto \sum a_i w_i$.

A **facet** of Δ^n is the convex hull of any subset of n of the $n+1$ vertices, i.e.,

$$\operatorname{conv}\{v_0, \dots, \widehat{v_i}, \dots, v_n\}.$$

Here and forevermore, the hat denotes removal, so that $\{v_0, \dots, \widehat{v_i}, \dots, v_n\} = \{v_0, \dots, v_n\} \setminus \{v_i\}$.

The boundary of Δ^n as a topological subspace of $\mathbb{R}^{n+1}$ is the union of its facets. However, this definition does not take orientation into account. Instead, we define the boundary of a simplex to be a certain signed sum of its facets whose orientations are chosen appropriately.

Remember that we want to think of a 1-simplex $[v_0, v_1]$ as an oriented edge from v_0 to v_1. Therefore we define

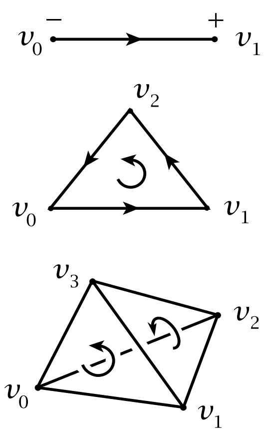
$$\partial[v_0, v_1] = [v_1] - [v_0].$$

(Heuristically, the edge represents a unit of flow that takes one unit of stuff from v_0 and transfers it to v_1.) If we follow the orientation arrow of the triangle, we see that the edges $[v_0, v_1]$ and $[v_1, v_2]$ are traversed in the right direction, but the edge $[v_0, v_2]$ is backwards. Therefore

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$

For the tetrahedron, we again would like a consistent orientation — this means that the arrows along all triangles should look consistent (say counterclockwise) when viewed from the outside. This gives

$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2].$$



$$\partial[v_0, v_1] = [v_1] - [v_0]$$

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

The general pattern is

$$\boxed{\partial[v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \widehat{v_i} \ldots, v_n]} \tag{2.1}$$

where the hat denotes removal.

2.2. **Simplicial complexes.** An **abstract simplicial complex** on vertex set $[n]$ is a subset $X \subseteq 2^{[n]}$ satisfying the following conditions:

- $\emptyset \in X$.
- If $\sigma \in X$ and $\tau \subseteq \sigma$, then $\tau \in X$.

The elements of $[n]$ should be regarded as vertices, and each face σ represents the convex hull of its vertices.

The **standard geometric realization** $|X|$ of an abstract complex X on $[n]$ is defined as follows. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$; then

$$|X| \;=\; \bigcup_{\sigma \in X} \operatorname{conv}(e_i \;:\; i \in \sigma) \;\subset\; \mathbb{R}^n.$$

However, it is often possible to construct a geometric realization of X (i.e., a space homeomorphic to $|X|$) in much smaller ambient dimension. The realization of each $\sigma \in X$ will be a geometric simplex of dimension $|\sigma| - 1$.

More generally, any topological space homeomorphic to $|X|$ could be referred to as a geometric realization of X. It is usually a good idea to ignore the distinction between X and $|X|$.

Some lingo: The elements of X are called its **faces** or its **simplices**. The **dimension** of a face F is $\dim F = |F| - 1$. (In particular, $\dim \emptyset = -1$.) A **facet** is a face that is maximal with respect to inclusion; note that not all facets need have the same size. The **dimension** of X is the largest dimension of a face(t).

The free abelian group generated by the n-dimensional simplices of X is denoted $\Delta_n(X)$. Its elements are called **(simplicial) $\boldsymbol{n}$-chains**. The boundary map defined earlier can be extended linearly to chains, so that we have maps

$$\cdots \to \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \to \cdots \xrightarrow{\partial_1} \Delta_0(X) \to 0. \tag{2.2}$$

(Note: We are ignoring the empty face $\emptyset$ here. More on that later.)

Fact: $\partial_n \circ \partial_{n+1} = 0$ for every n. (For short, $\partial^2 = 0$.) An equivalent statement is that $\ker \partial_n \supseteq \operatorname{im} \partial_{n+1}$.

The proof is left as an exercise — everyone should verify this formula at least once in their life.

This means that the groups and maps and in (2.2) form an algebraic *chain complex*, called the **simplicial chain complex** of X. The **reduced simplicial homology groups of $\boldsymbol{X}$** can now be defined as

$$\tilde{H}_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Note that these are finitely generated $\mathbb{Z}$-modules, and that $H_n(X) = 0$ for $n > \dim X$. Some computations in a moment.

3. **Fri 3/26: Δ-complexes; examples**

Example: Triangulating $\mathbb{S}^2$. The simplest triangulation of $\mathbb{S}^2$ (i.e., simplicial complex X whose geometric realization $|X|$ is homeomorphic to $\mathbb{S}^2$) is the hollow tetrahedron: $X = \langle 123, 124, 134, 234\rangle$. It has 4 vertices, 6 edges, and 4 triangles. Its simplicial chain complex is

$$\Delta_2 = \mathbb{Z}^4 \xrightarrow[\begin{array}{c|cccc} & 123 & 124 & 134 & 234 \\ \hline 12 & 1 & 1 & 0 & 0 \\ 13 & -1 & 0 & 1 & 0 \\ 14 & 0 & -1 & -1 & 0 \\ 23 & 1 & 0 & 0 & 1 \\ 24 & 0 & 1 & 0 & -1 \\ 34 & 0 & 0 & 1 & 1 \end{array}]{\partial_2} \Delta_1 = \mathbb{Z}^6 \xrightarrow[\begin{array}{c|cccccc} & 12 & 13 & 14 & 23 & 24 & 34 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 1 & 0 \\ 3 & 0 & -1 & 0 & -1 & 0 & 1 \\ 4 & 0 & 0 & -1 & 0 & -1 & -1 \end{array}]{\partial_1} \Delta_0 = \mathbb{Z}^4 \to 0$$

Wouldn't it be simpler to take two solid triangles T, S and identify their boundary $\mathbb{S}^1$'s? This would give us a cell complex X with

$$\begin{aligned} X^0 &= \{1, 2, 3\} \\ X^1 &= \{e = 12, f = 13, g = 23\} \\ X^2 &= \{T, S\} \end{aligned}$$

and simplicial chain complex

$$\Delta_2 = \mathbb{Z}^2 \xrightarrow[\begin{array}{c|cc} & T & S \\ \hline e & 1 & 1 \\ f & -1 & -1 \\ g & 1 & 1 \end{array}]{\partial_2} \Delta_1 = \mathbb{Z}^3 \xrightarrow[\begin{array}{c|ccc} & e & f & g \\ \hline 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 3 & 0 & -1 & -1 \end{array}]{\partial_1} \Delta_0 = \mathbb{Z}^3 \to 0$$

Compute the homology of this chain complex:

$$\begin{aligned} H_2^\Delta(X) &= \ker \partial_2 = \mathbb{Z}\langle e - f\rangle \cong \mathbb{Z} \\ H_1^\Delta(X) &= \ker \partial_1 / \operatorname{im} \partial_2 \\ &= \mathbb{Z}\langle [12] - [13] + [23]\rangle / \text{itself} = 0 \\ H_0^\Delta(X) &= \mathbb{Z}^3 / \operatorname{im} \partial_1 \\ &= \mathbb{Z}^3 / \mathbb{Z}\langle [1] - [2], [1] - [3]\rangle \cong \mathbb{Z}. \end{aligned}$$

While this second cell complex is not a simplicial complex, it might as well be (in the sense that the simplicial boundary formula holds). This brings us to the idea of a **Δ-complex**, which is a generalization of a simplicial complex in which repeated faces are allowed, as long as the orientation is determined by a fixed ordering on the vertices. Specifically:

Definition: A **Δ-complex** is a cell complex X that consists of:

(1) A totally ordered vertex set X_0.
(2) For each $n > 0$, a set $X_n = \{e^n_\alpha\}$ of n-dimensional simplices, each with a specified list $V(e_\alpha) = (v_0 \leq \cdots \leq v_n)$ of vertices.
(The vertices need not be unique, and different n-simplices can have the same lists of vertices.)
(3) Boundary maps $\mathbb{Z}X_n \to \mathbb{Z}X_{n-1}$ given as follows: for each $e^n_\alpha \in X_n$ as above, we have

$$\partial[v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i F_i$$

where $F_i \in X_{n-1}$ has vertices $(v_0, \ldots, \widehat{v_i} \ldots, v_n)$.

The chain complex of a Δ-complex is constructed in the same way as that of a simplicial complex, and its homology is defined in the same way. Each n-simplex will have as its boundary a signed sum of $(n-1)$-simplices, which need not all be different.

3.1. **Examples. 1. The circle** $\mathbb{S}^1$ can be realized as a Δ-complex with two vertices v, w and two edges a, b, each oriented from v to w. The chain complex is

$$0 \to \Delta_1 = \mathbb{Z}\langle a, b\rangle \xrightarrow{\partial_1} \Delta_0 = \mathbb{Z}\langle v, w\rangle \to 0$$

where $\partial a = \partial b = w - v$. Thus

$$\ker \partial_1 = \mathbb{Z}\langle a - b\rangle, \qquad \operatorname{im} \partial_1 = \mathbb{Z}\langle w - v\rangle$$

and

$$H_1(\mathbb{S}^1) = \mathbb{Z}\langle a - b\rangle = \mathbb{Z},$$
$$H_0(\mathbb{S}^1) = \mathbb{Z}\langle v, w\rangle / \mathbb{Z}\langle w - v\rangle = \mathbb{Z}.$$

What if we add another vertex x and another edge c from w to x? Then the simplicial chain complex is

$$0 \to \Delta_1 = \mathbb{Z}\langle a,b,c\rangle \xrightarrow[\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} v \\ w \\ x \end{array} & \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}]{\partial_1} \Delta_0 = \mathbb{Z}\langle v,w,x\rangle \to 0.$$

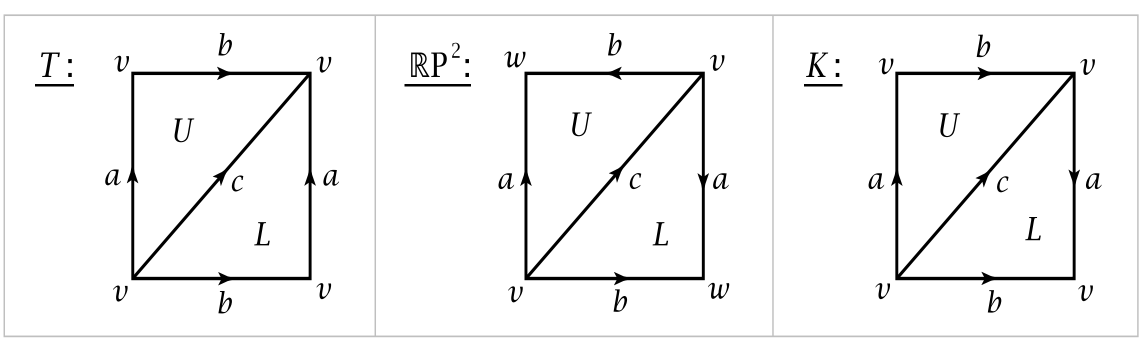
It turns out that this does not change the homology. The kernel is still generated by $a-b$, and the increase in rank of Δ_0 and the increase in rank of ∂_1 cancel each other out.

2. The disk D^2 is homeomorphic to the simplex $[a,b,c]$, whose chain complex is

$$0 \to \underbrace{\mathbb{Z}\langle abc\rangle}_{=\Delta_2} \xrightarrow[\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}]{\partial_2} \underbrace{\mathbb{Z}\langle ab,ac,bc\rangle}_{=\Delta_1} \xrightarrow[\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}]{\partial_1} \underbrace{\mathbb{Z}\langle a,b,c\rangle}_{=\Delta_0} \to 0.$$

Algebraists will recognize this as (more or less) the *Koszul complex*, which is exact (i.e., the homology groups are zero). Well, almost zero; the last bit of the Koszul complex has been cut off, so we get $H_0 = \mathbb{Z}$. The same thing will happen for disks of higher dimensions.

2a. The sphere $\mathbb{S}^2$ can be constructed by identifying two copies N,S of the disk D^2 along their boundary circles. I.e., as a Δ-complex, we could take a triangle and attach two 2-cells N,S with the same oriented boundaries. This has no effect on H_0 or H_1 (since it does not change the image of ∂_2), but on the other hand we will now have $N - S \in \ker \partial_2$, and clearly this element generates the kernel, whence $H_2(\mathbb{S}^2) = \mathbb{Z}$.



3. For the torus T,

$$\partial_2 = \begin{array}{c c} & \begin{array}{cc} U & L \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \qquad \partial_1 = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ v & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{array}.$$

So

$$\begin{aligned} H_2(T) &= \ker \partial_2 / \operatorname{im} \partial_3 = \ker \partial_2 \\ &= \{(p,q) \in \mathbb{Z}^2 \mid p = q\} = \mathbb{Z} \\ H_1(T) &= \ker \partial_1 / \operatorname{im} \partial_2 \\ &= \mathbb{Z}^3 / \mathbb{Z}\langle(1,1,-1)\rangle = \mathbb{Z}^2 \\ H_0(T) &= \ker \partial_0 / \operatorname{im} \partial_1 \\ &= \Delta_0(T)/0 = \mathbb{Z}. \end{aligned}$$

Note that the calculation of H_1 relies on the fact that the vector $(1,1,-1)$ can be extended to a $\mathbb{Z}$-module basis of $\mathbb{Z}^3$.

4. For the projective plane $\mathbb{R}P^2$:

$$\partial_2 = \begin{array}{c c} & \begin{array}{cc} U & L \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}, \qquad \partial_1 = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} v \\ w \end{array} & \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

$$\begin{aligned} H_2(\mathbb{R}P^2) &= \ker \partial_2 / \operatorname{im} \partial_3 = \ker \partial_2 = \{(p,q) \in \mathbb{Z}^2 \mid p = q,\ p = -q\} \\ &= 0 \\ H_1(\mathbb{R}P^2) &= \ker \partial_1 / \operatorname{im} \partial_2 \\ &= \{(p,q,r) \in \mathbb{Z}^3 | p = -q\} / \mathbb{Z}\langle(1,-1,-1),(1,-1,1)\rangle \\ &\quad \text{(hold that thought)} \\ H_0(\mathbb{R}P^2) &= \ker \partial_0 / \operatorname{im} \partial_1 = \mathbb{Z}\langle v, w\rangle / \mathbb{Z}\langle v - w\rangle \\ &= \mathbb{Z}. \end{aligned}$$

What about that H_1 term? Given any $(p,q,r) \in \mathbb{Z}^3$ with $p = -q$, we want to solve the equation

$$s(1,-1,-1) + t(1,-1,1) = (p,q,r)$$

for s and t. There are lots of choices such that $s+t=a=-b=-s-t$, but then $r=s-t$ must have the same parity as p for the equation to have a solution. What we are seeing is that $\operatorname{im}\partial_2$ is a $\mathbb{Z}$-submodule of index 2 in $\ker\partial_1$, and so $\boxed{H_1(\mathbb{R}P^2)=\mathbb{Z}_2.}$

Note: The calculation of H_2 would be different if the ground ring we were working with was $\mathbb{Z}_2$ instead of $\mathbb{Z}$. Then the conditions $p=q$ and $p=-q$ would be identical and we'd end up with $\mathbb{Z}_2$. Often it is useful to do homology over a ring other than $\mathbb{Z}$ — but we'll get to that later.

5. For the Klein bottle K (a problem from HW #5),

$$\partial_2 = \begin{array}{c c} & \begin{array}{cc} U & \;L \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}, \qquad \partial_1 = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ v & \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \end{array}.$$

We now have $H_2(K)=0$ for the same reason that $H_2(\mathbb{R}P^2)=0$, and yet again $H_0(K)=0$.

To calculate $H_1(K)$, observe that

$$\{v_1=(1,1,-1),\ v_2=(1,0,0),\ v_3=(0,1,0)\}$$

generates $\mathbb{Z}^3$ as a $\mathbb{Z}$-module, and that

$\operatorname{im}\partial_2 = \mathbb{Z}\langle(1,1,-1),\ (1,-1,1)\rangle = \mathbb{Z}\langle(1,1,-1),\ (1,-1,1)+(1,1,-1)\rangle = \mathbb{Z}\langle(1,1,-1),\ ($

Therefore

$$H_1(K)=\mathbb{Z}^3/\operatorname{im}\partial_2=\mathbb{Z}\oplus\mathbb{Z}_2.$$

This is an example of a Smith normal form calculation, about which more later.

To summarize:

Space	H_2	H_1	H_0
$\mathbb{S}^1$	0	$\mathbb{Z}$	$\mathbb{Z}$
D^2	0	0	$\mathbb{Z}$
$\mathbb{S}^2$	$\mathbb{Z}$	0	$\mathbb{Z}$
T	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
$\mathbb{R}P^2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$
K	0	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}$

It looks like H_0 is *always* $\mathbb{Z}$ — is that in general true?

The simplest space X for which $H_0(X) \neq \mathbb{Z}$ is $X = \mathbb{S}^0$. Here the simplicial chain complex has only one nontrivial group, namely Δ_0, and all the boundary maps are zero so $H_0(X) = \Delta_0(X) = \mathbb{Z}^2$. More generally, if X is a discrete space with k points then $H_0(X) = \mathbb{Z}^k$.

4. **Mon 3/31: Lab Day!**

Learn how to use Macaulay2. See the handout.

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5. **Wed 4/2: Calculating simplicial homology**

5.1. **Calculating homology of f.g. $\mathbb{Z}$-modules.** Every subgroup $A \subseteq \mathbb{Z}^n$ is free abelian. (But that doesn't mean that the quotient $\mathbb{Z}^n/A$ is free!) Therefore, it is isomorphic to $\mathbb{Z}^p$ for some $p \leq n$. The number p is the *rank* of A, and is an isomorphism invariant. A set of elements $\{a_1, \dots, a_p\}$ that generate A as an abelian group is called a $\mathbb{Z}$-*basis* of A. (Or "$\mathbb{Z}$-module basis," "integral basis," "lattice basis", etc.) If the vectors a_i are made into the columns of a $n \times p$ matrix, then they form an integral basis if and only if the gcd of all $p \times p$ minors is 1. (So this is stronger than being linearly independent over $\mathbb{Q}$, for which we would just need that at least one minor is nonzero.)

Let $B \subseteq A$ be f.g. abelian groups with $\operatorname{rank}(A) = p$ and $\operatorname{rank}(B) = q$. Then $p \geq q$. Moreover, there always exists a $\mathbb{Z}$-basis $\{a_1, \dots, a_p\}$ of A and scalars $\lambda_1, \dots, \lambda_q \in \mathbb{Z}_{>0}$ such that $\{\lambda_1 a_1, \dots, \lambda_q a_q\}$ is a $\mathbb{Z}$-basis for B. Once we have this data, we can see that

$$G = A/B = \mathbb{Z}^{p-q} \oplus \bigoplus_{i=1}^{q} \mathbb{Z}_{\lambda_i}. \tag{5.1}$$

In the case $A = \mathbb{Z}^n$ and $B = \operatorname{im} M$ for some $n \times m$ matrix M of rank $q \leq \min(m, n)$, this is expressed by the matrix equation

$$\underbrace{M}_{n\times m} = \underbrace{X}_{n\times n}\underbrace{S}_{n\times m}\underbrace{Y}_{m\times m}$$

where X represents a change of basis between the standard basis and the basis $\{a_1, \dots, a_n\}$; and Y converts between the generating set of B given by the columns of M, and the basis $\{\lambda_1 a_1, \dots, \lambda_q a_q\}$. Both of these matrices are invertible over $\mathbb{Z}$ — i.e., their determinants are ± 1. Meanwhile, the matrix S will be diagonal:

$$S = \operatorname{diag}(\lambda_1, \dots, \lambda_q).$$

Since X and Y represent isomorphisms, it follows that $A/B \cong \mathbb{Z}^n/\operatorname{im} S = \mathbb{Z}^{n=q} \oplus \bigoplus_i \mathbb{Z}_{\lambda i}$. Note that some of the λ_i's can be zero (if $q < m$), in which case we get additional $\mathbb{Z}$ summands.

Recall that every f.g. abelian group G has a unique representation (up to isomorphism) of the form (5.1) such that λ_i divides λ_j whenever $i|j$. In this case, the numbers λ_i are the **invariant factors** of G and the matrix S is called the **Smith normal form**.

Useful Fact 1: Let $M \in \mathbb{Z}^{n\times m}$. The i^{th} invariant factor of M equals the gcd of all the $i \times i$ minors of M.

Useful Fact 2: For any homomorphism $f : \mathbb{Z}^m \to \mathbb{Z}^n$, there is a basis of $\ker d$ that extends to a basis of $\mathbb{Z}^m$. Equivalently, the group $\mathbb{Z}^m/\ker d$ is free (all its invariant factors are 1).

Now consider the simplicial chain complex associated with a Δ-complex X:

$$\cdots \to \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \to \cdots.$$

By Useful Fact 2, the invariant factors of $H_n^\Delta(X) = \ker \partial_n/\operatorname{im} \partial_{n+1}$ are the same as those of $\Delta_n(X)/\operatorname{im} \partial_{n+1}$. Therefore, to compute $H_n^\Delta(X)$ up to isomorphism, one calculates the invariant factors (either by inspection, or by using Useful Fact 1) and observes that the free rank (the number of copies of $\mathbb{Z}$) is $\operatorname{rank} \ker \partial_n - \operatorname{rank} \operatorname{im} \partial_{n+1} = \operatorname{rank} \Delta_n - \operatorname{rank} \partial_n - \operatorname{rank} \partial_{n+1}$.

This can all be automated: computer algebra systems such as Sage and Macaulay2 can compute the homology of chain complexes of f.g. $\mathbb{Z}$-modules in a jiffy. Here is the Macaulay2 computation of the simplicial homology of the Δ-complex structure on $\mathbb{R}P^2$:

```
D2 = matrix{{1,1},{-1,-1},{-1,1}};
D1 = matrix{{-1,-1,0},{1,1,0}};
RP2 = chainComplex( D1, D2 )
  -- notice the order of the differential maps!
prune HH RP2
```

5.2. **Reduced and unreduced homology.** Consider the simplicial complex $X = \langle v_1v_2v_3, v_2v_4, v_3v_4\rangle$. If you construct the simplicial chain complex the way Hatcher tells you to, it will look like (5.2). However, if you tell `Macaulay2` something like

```
R=ZZ[a,b,c,d]; chainComplex simplicialComplex {a*b*c, b*d, c*d}
```

you will get something bulkier, namely (5.3). What is going on?

$$\tilde{\mathscr{C}}: \quad \underset{=\mathbb{Z}}{\Delta_2} \xrightarrow{\partial_2} \underset{=\mathbb{Z}^5}{\Delta_1} \xrightarrow{\partial_1} \underset{=\mathbb{Z}^4}{\Delta_0} \xrightarrow{\partial_0} \underset{=\mathbb{Z}}{\Delta_{-1}} \longrightarrow 0 \tag{5.2}$$

$$\mathscr{C}: \quad \underset{=\mathbb{Z}}{\Delta_2} \xrightarrow{\partial_2} \underset{=\mathbb{Z}^5}{\Delta_1} \xrightarrow{\partial_1} \underset{=\mathbb{Z}^4}{\Delta_0} \longrightarrow 0 \tag{5.3}$$

Let $H_i^\Delta(X)$ be the i^{th} homology of the chain complex $\mathscr{C}$, and let $\tilde{H}_i^\Delta(X)$ be the i^{th} homology of the chain complex $\tilde{\mathscr{C}}$.

First of all, it is clear that $H_i^\Delta(X) = \tilde{H}_i^\Delta(X)$ for all $i > 0$. The only difference occurs in dimension 0, where

$$H_0^\Delta(X) = \Delta_0 / \operatorname{im} \partial_1, \qquad \tilde{H}_0^\Delta(X) = \ker \partial_0 / \operatorname{im} \partial_1.$$

The boundary of each vertex is the (-1)-cell $[\emptyset]$, which is the generator of Δ_{-1}. Therefore, the map ∂_0 is given by the matrix $[1\ 1\ 1\ 1]$, and

$$\ker \partial_0 = \left\{ \sum_{i=1}^{r} a_i v_i \ + \ a_i \in \mathbb{Z},\ \sum a_i = 0 \right\} \subseteq \Delta_0.$$

This is a corank-1 submodule of Δ_0. In fact it is a *summand* (as is any submodule that is the kernel of some map): we can write $\Delta_0 = \ker \partial_0 \oplus F$ where F is infinite cyclic (e.g., spanned by the class of any vertex). Therefore

$$\begin{aligned} H_0^\Delta(X) &= \Delta_0 / \operatorname{im} \partial_1 = (\ker \partial_0 \oplus F) / \operatorname{im} \partial_1 = (\ker \partial_0 / \operatorname{im} \partial_1) \oplus F \\ &\cong \tilde{H}_0^\Delta(X) \oplus \mathbb{Z}. \end{aligned}$$

This relationship holds for all simplicial complexes (and, once we have defined singular homology, for all spaces). Sometimes reduced homology is more convenient, sometimes unreduced homology, depending on the context.

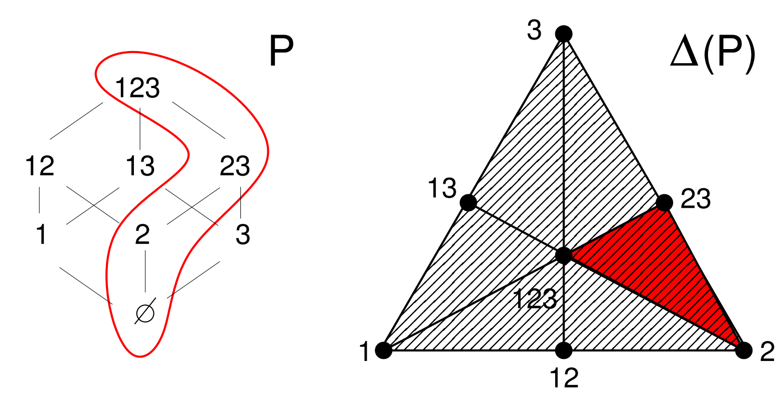
5.3. **A combinatorial motivation for topology.** A **partially ordered set** $\boldsymbol{P}$ (or **poset**) is a set equipped with an relation $\leq$ that behaves the way you think it does:

(1) $x \leq x$ for all $x \in P$;
(2) if $x \leq y$ and $y \leq x$, then $x = y$;
(3) if $x \leq y$ and $y \leq z$, then $x \leq z$;

A **chain** in P is a subset C that is totally ordered: either $x \leq y$ or $y \leq x$ for every $x, y \in C$. Note that every singleton subset of P is a chain, as is every subset of a chain. Therefore, the set of chains form a simplicial complex $\Delta(P)$, the **order complex** of P, whose vertex set is P itself.

Posets are ubiquitous in combinatorics, and the mapping $P \mapsto \Delta(P)$ is one of many connections between combinatorics with algebraic topology. Note that a simplicial complex can itself be regarded as a poset — it is a subposet of the power set of its vertices.

As an example, let $P = 2^{[3]} = \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}$. The complex $\Delta(P)$ is called the **barycentric subdivision of the 2-simplex**: it is obtained from Δ^2 by drawing a new vertex in the center of each simplex, then connecting two vertices with an edge (or three vertices with a triangle, etc.) if they come from simplices that form a chain.



6. **Fri 4/4: Singular homology**

In order to extend homology theory to general spaces we need the notions of **singular** chains, cycles, boundaries and homology groups.

Definition: Let X be a topological space. A **singular $\boldsymbol{n}$-simplex** is a continuous map $\sigma : \Delta^n \to X$.

There's no requirement that σ be one-to-one, so many singular simplices look very unlike "honest" simplices (hence the name).

The **$\boldsymbol{n}$th singular chain group of $\boldsymbol{X}$** is the free abelian group $C_n(X)$ of linear combinations of singular n-simplices. This is an enormous group — where $\Delta_0(X)$ is generated by the (often finite) set of vertices of a complex, by contrast $C_0(X)$ is the free abelian group on the *points* of X.

The **singular boundary map** $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]}. \tag{6.1}$$

Convention: In general, any p-simplex $[w_0, \ldots, w_p]$ can be identified with the standard p-simplex by the map that preserves the ordering of vertices. Such maps are implicit in a formula such as (6.1): each of the summands is to be treated as a map $\Delta^{n-1} \to X$). The notation is going to get crowded enough without specifying these maps explicitly!

This map satisfies $\partial^2 = 0$ for the same reason as does the simplicial boundary map: the terms in $\partial^2\sigma$ are, up to sign, of the form

$$\sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,\widehat{v_j},\ldots,v_n]}$$

for $i < j$. The term that comes from removing j first and then i carries sign $(-1)^j(-1)^i$, while the term that comes from removing i first and then j carries sign $(-1)^i(-1)^{j-1}$, so everything cancels. Therefore, the modules and maps

$$C_\bullet(X) : \qquad 0 \to C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0$$

form a chain complex, the (unreduced) **singular chain complex** of X.

Definition: The $\boldsymbol{n^{th}}$ **(unreduced) singular homology group of $\boldsymbol{X}$** is
$$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

As in the simplicial case, we can define the reduced singular chain complex $\tilde{C}_\bullet(X)$ by appending a map $C_{-1}(X) = \mathbb{Z}$. The resulting reduced singular homology groups are denoted $\tilde{H}_n(X)$. The relation between reduced and unreduced homology is the same as in the simplicial setting:
$$H_n(X) = \begin{cases} \tilde{H}_n(X) & \text{if } n > 0, \\ \tilde{H}_n(X) \oplus \mathbb{Z} & \text{if } n = 0. \end{cases}$$

Everything we have defined so far, including the groups $H_n(X)$, are clearly homeomorphism invariants of X. However, we can't calculate singular homology directly, except in some extreme cases. What is nice about the singular theory is that it allows us to work combinatorially and algebraically with simplices and boundary maps without worrying about the structure of X.

Proposition: Let $\{X_\alpha\}$ be the (path-)connected components of X. Then
$$C_\bullet(X) = \bigoplus_\alpha C_\bullet(X_\alpha)$$
and consequently
$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha) \qquad \forall n.$$

Proof. Simplices are (path-)connected, so the image of every singular simplex is in some (path-)component of X. Therefore $C_\bullet(X)$ breaks up as a direct sum. ☐

Proposition: If X is path-connected, then $H_0(X) = \mathbb{Z}$.

Proof. For every singular 1-simplex σ, we have $\partial_1 \sigma = [x] - [y]$ for some $x, y \in X$ (namely $\sigma(v_0)$ and $\sigma(v_1)$ respectively). Moreover, every such 0-chain is the boundary of a singular 1-simplex, namely a path in X from v to w. Therefore
$$\operatorname{im} \partial_1 = G = \left\{ \sum c_x[x] \in C_0(X) \mid \sum c_x = 0 \right\}$$
and $H_0(X) = C_0(X)/\operatorname{im} \partial_1 = \mathbb{Z}$. (The quotient map $C_0(X) \twoheadrightarrow H_0(X)$ is given by adding up the coordinates in a chain.) ☐

Corollary 6.1. *If X has c connected components then $H_0(X) = \mathbb{Z}^c$ and $\tilde{H}_0(X) = \mathbb{Z}^{c-1}$.*

Proposition: If X consists of a single point, then $\tilde{H}_n(X) = 0$ for all $n > 0$.

Proof. There is only one singular n-simplex, namely the constant map $k_n : \Delta^n \to X$, so $C_n(X) \cong \mathbb{Z}$ for all $n \geq -1$. The boundary formula says that

$$\partial k_n = \begin{cases} k_{n-1} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

so the reduced singular chain complex is:

$$\cdots \to C_3 \xrightarrow{0} C_2 \xrightarrow{\cong} C_1 \xrightarrow{0} C_0 \xrightarrow{\cong} C_{-1} \xrightarrow{0} 0.$$

The homology at the even positions is $0/0 = 0$, and at the odd positions is $C_n/C_n = 0$. □

6.1. **Induced Maps.** Suppose we have two topological spaces X, Y and a map $f : X \to Y$.

If $\sigma : \Delta^n \to X$ is a singular d-simplex in X, then the composition

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$$

is a singular d-simplex in Y. Therefore we have a map

$$f_\sharp : C_n(X) \to C_n(Y)$$

given by $f_\sharp(\sigma) = f \circ \sigma$.

Lemma 6.2. *The map $f_\sharp$ is a* ***homomorphism of chain complexes****; that is, it commutes with the boundary maps.*

Proof. The key fact is that restricting a function's domain commutes with precomposing it with another function.

$$\begin{aligned} f_\sharp(\partial\sigma) &= f_\sharp\left(\sum_{i=0}^{n}(-1)^i \sigma|_{[v_0 \cdots \widehat{v_i} \cdots v_n]}\right) \\ &= \sum_{i=0}^{n}(-1)^i (f \circ \sigma)|_{[v_0 \cdots \widehat{v_i} \cdots v_n]} \\ &= \partial(f \circ \sigma) \;=\; \partial(f_\sharp\sigma). \end{aligned}$$

□

From this it follows that $f_\sharp$ maps cycles to cycles (if $\partial z = 0$ then $\partial(f_\sharp z) = f_\sharp(\partial z) = 0$) and boundaries to boundaries (if $b = \partial a \in \operatorname{im}\partial_X$, then $f_\sharp(b) = f_\sharp(\partial a) = \partial(f_\sharp(a)) \in \operatorname{im}\partial_Y$). Therefore:

Proposition 2.9: A continuous map $f : X \to Y$ induces homomorphisms $f_* : H_k(X) \to H_k(Y)$ for every k.

If $g : Y \to Z$ is another continuous function, then $(fg)_* = f_* g_*$. Also, $(\mathbb{1}_X)_* = \mathbb{1}_{H_k(X)}$ for every k.

7. **Mon 4/7: Homotopy Invariance of Homology Groups**

Recall that a continuous function $f : X \to Y$ induces maps on singular chain groups

$$f_\sharp : C_n(X) \to C_n(Y), \qquad f_\sharp(\sigma) = f \circ \sigma.$$

This is in fact a chain map (i.e., $\partial f_\sharp = f_\sharp \partial$) hence induces homomorphisms

$$f_* : H_n(X) \to H_n(Y).$$

These induced maps are functorial:

$$(\mathbb{1}_X)_* = \mathbb{1}_{H_n(X)} \quad \text{and} \quad (fg)_* = f_* g_*.$$

The punchline of this section is:

Theorem 2.10: If $f, g : X \to Y$ are homotopic maps, then $f_* = g_*$.

In particular, suppose that $f : X \to Y$ is a homotopy equivalence; recall that this means there is a map $g : Y \to X$ such that $fg \simeq \mathbb{1}_Y$ and $gf \simeq \mathbb{1}_X$. Then $f_* g_* = \mathbb{1}$ and $g_* f_* = \mathbb{1}$, so both f_* and g_* are isomorphisms. So an extremely useful corollary of Theorem 2.10 is:

Corollary 2.11: If two spaces are homotopy equivalent, they have the same singular homology groups in all dimensions.

In particular, if a space is contractible then its homology groups are all zero. (The converse is not true!)

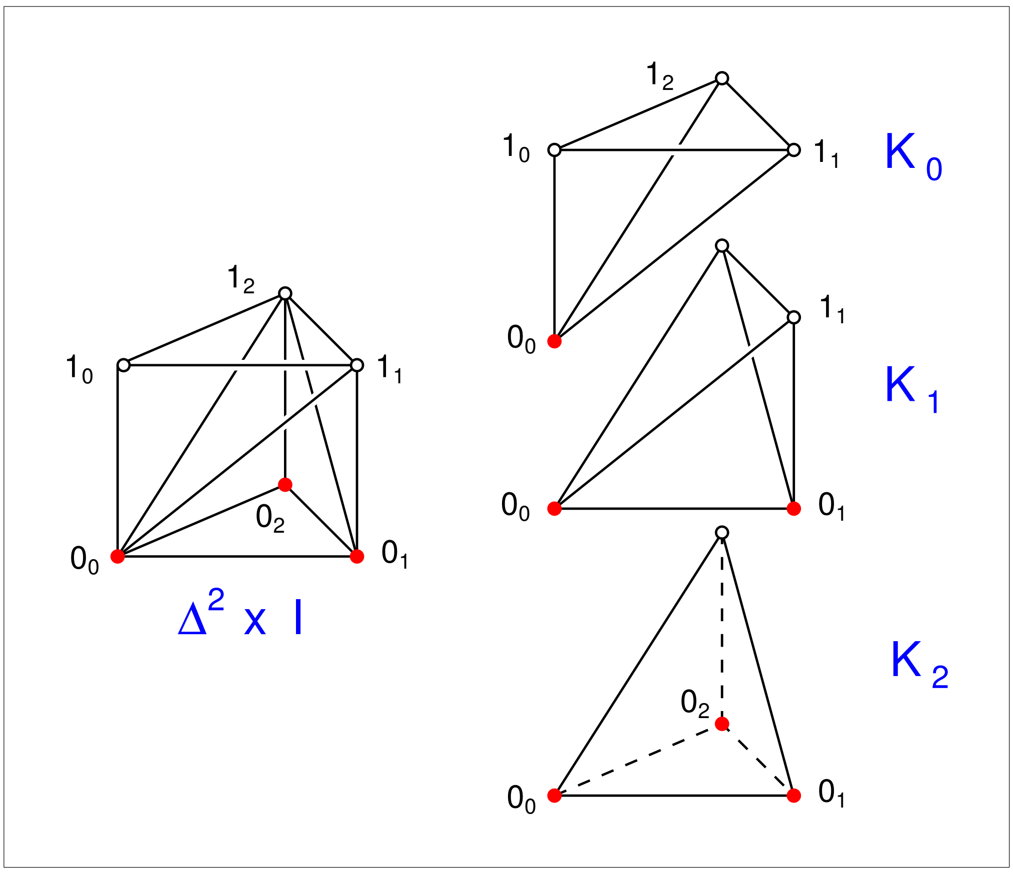
Proof of Theorem 2.10. Consider the prism $\Delta^n \times I = [v_0, \ldots, v_n] \times I$. I will label its vertices in an unorthodox way:

$$0_i = (v_i, 0), \qquad 1_i = (v_i, 1).$$

The prism decomposes into the $(n+1)$-simplices

$$
\begin{aligned}
K_0 &= [0_0, 1_0, 1_1, \ldots, 1_{n-1}, 1_n] \\
K_1 &= [0_0, 0_1, 1_1, \ldots, 1_{n-1}, 1_n] \\
&\cdots \\
K_j &= [0_0, 0_1, \ldots, 0_j, 1_j, \ldots, 1_n] \\
&\cdots \\
K_n &= [0_0, 0_1, \ldots, 0_{n-1}, 0_n, 1_n]
\end{aligned}
$$

These simplices triangulate $\Delta^n \times I$ because each one is the region of the prism between the graphs of two functions of the form $t_i + \cdots + t_n$, $t_{i+1} + \cdots + t_n$, where the t_i's are barycentric coordinates.)



Let $F : X \times I \to Y$ be the homotopy between f and g, i.e., $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

Consider the map $G = F \circ (\sigma \times \mathbb{1}) : \Delta \times I \to Y$. The decomposition of $\Delta \times I$ into $(n+1)$-simplices enables us to regard G as a singular $(n+1)$-chain. That is, we consider the linear *prism operator*

$$\begin{aligned} P_n : \ C_n(X) &\to C_{n+1}(Y) \\ \sigma &\mapsto \sum_{j=0}^{n} (-1)^j G|_{K_j} \end{aligned}$$

As usual, we identify each $(n+1)$-simplex in the sum with Δ^{n+1}, preserving the order of vertices. Also, I'll often drop the subscript on P.

Here is a picture of the maps. Note that the diagonal edges do NOT commute with the others!

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"', "g_\sharp"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"', "g_\sharp"] \arrow[dl, dotted, "P_n"'] & C_{n-1}(X) \cdots \arrow[r] \arrow[d, "f_\sharp"', "g_\sharp"] \arrow[dl, dotted, "P_{n-1}"'] & {} \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"'] & C_n(Y) \arrow[r, "\partial"'] & C_{n-1}(Y) \cdots \arrow[r] & {} \end{tikzcd} \tag{7.1}$$

Claim: $\partial P = g_\sharp - f_\sharp - P\partial$.

Proof: Let $\sigma \in C_n(X)$. We first calculate $\partial P_n \sigma$. We have:

$$\partial P \sigma = \sum_{j=0}^{n} (-1)^j \partial \left(G|_{K_j} \right)$$

$$= \sum_{j=0}^{n} (-1)^j \left(\underbrace{\sum_{i=0}^{j} (-1)^i G|_{[0_0,\ldots,\hat{0}_i,\ldots,0_j,1_j,\ldots,1_n]}}_{A} - \underbrace{\sum_{i=j}^{n} (-1)^i G|_{[0_0,\ldots,0_j,1_j,\ldots,\hat{1}_i,\ldots,1_n]}}_{B} \right)$$

For each k, the $i = j = k$ summand in B cancels the $i = j = k+1$ summand in A. So we can rescue the $i = j = 0$ summand in A and the $i = j = n$ summand in B, giving

$$\partial P(\sigma) \;=\; \underbrace{G|_{[1_0,\ldots,1_n]}}_{g_\sharp(\sigma)} - \underbrace{G|_{[0_0,\ldots,0_n]}}_{f_\sharp(\sigma)} + \sum_{j=0}^{n}(-1)^{j+i}\left(\sum_{i<j} G|_{[0_0,\ldots,\widehat{0_i},\ldots,0_j,1_j,\ldots,1_n]} - \sum_{i>j} G|_{[0_0,\ldots,0_j,1_j,\ldots,\widehat{1_i},\ldots,1_n]}\right). \tag{7.2}$$

Now we calculate $P_{n-1}\partial\sigma$. We start with another abuse of notation:

$$P\partial\sigma \;=\; \sum_{i=0}^{n}(-1)^i P\left(\sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]}\right) \;=\; \sum_{i=0}^{n}(-1)^i P\left(\sigma|_{[w_0,\ldots,w_{n-1}]}\right).$$

The list $[w_0,\ldots,w_{n-1}]$ depends on i — remember this. Set

$$r_i = (w_i, 0), \qquad s_i = (w_i, 1) \qquad \in \Delta^{n-1}\times I.$$

Applying the definition of P, we get

$$\begin{aligned}
P\partial\sigma &= \sum_{i=0}^{n}(-1)^i \sum_{k=0}^{n-1}(-1)^k F\circ(\sigma|_{[w_0,\ldots,w_{n-1}]}\times \mathbb{1})|_{[r_0,\ldots,r_k,s_k,\ldots,s_{n-1}]}\\
&= \sum_{i=0}^{n}(-1)^i \sum_{k=0}^{n-1}(-1)^k G|_{[r_0,\ldots,r_k,s_k,\ldots,s_{n-1}]}\\
&= \sum_{j=0}^{n}(-1)^j\left(-\sum_{i<j}(-1)^i G|_{[0_0,\ldots,\widehat{0_i},\ldots,0_j,1_j,\ldots,1_n]}\right.\\
&\qquad\left. + \sum_{i>j}(-1)^i G|_{[0_0,\ldots,0_j,1_j,\ldots,\widehat{1_i},\ldots,1_n]}\right)
\end{aligned} \tag{7.3}$$

To explain the bookkeeping in this last equation, j indicates the index of the vertex of Δ_{n-1} that appears twice (once as $(v_j,0)$ and once as $(v_j,1)$) in the indicated n-simplex. Meanwhile, $k+1$ is the number of r's (or y's) in that simplex. If $i\le k$ then $j=k+1>i$, while if $i>k$ then $j=k<i$.

A miracle has occurred: the garbage that shows up in (7.3) is the same as the garbage in (7.2) (with a minus sign). This proves the claim, which we repeat here:

$$\partial P = g_\sharp - f_\sharp - P\partial \qquad \text{or equivalently} \qquad g_\sharp - f_\sharp = \partial P - P\partial. \tag{7.4}$$

To convince yourself that this computation is correct, you may want to working out the explicit expressions for small cases such as $n = 2$ and $n = 3$ by hand.

Now, suppose that $\alpha \in C_n(X)$ is a cycle, i.e., $\partial\alpha = 0$. Then

$$g_\sharp(\alpha) - f_\sharp(\alpha) = (\partial P - P\partial)\alpha = \partial P(\alpha).$$

That is, $g_\sharp(\alpha)$ and $f_\sharp(\alpha)$ differ by a boundary, which is precisely the statement that $g_*[\alpha] = f_*[\alpha]$, where $[\alpha]$ denotes the class of α in $H_n(X)$. So we have proved the theorem. $\square$

Note: The maps P are collectively called a *chain homotopy*. Specifically, suppose that A and B are chain complexes with chain maps $f, g : A \to B$. Then a chain homotopy is a collection of maps $P : A_n \to A_{n+1}$ satisfying (7.4). (For the picture, see (7.1).)

8. Wed 4/9: Exact Sequences; Relative Homology

The next step is to understand the relationship between the homology groups of a space X, a subspace A, and the quotient space X/A. This relationship is expressed algebraically by a *long exact sequence.*

Recall that a chain complex is a sequence of abelian groups and maps

$$A: \qquad \cdots \to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots$$

such that $\ker \partial_n \supseteq \operatorname{im} \partial_{n+1}$ for all n. It is called **exact at $\boldsymbol{A_n}$** if $\ker \partial_n = \operatorname{im} \partial_{n+1}$, and **exact** if it is exact at every group between two maps. (So, e.g., to say that $A \to B \to C$ is exact is to say that it is exact at B.)

Observe that for groups P, Q, R,

$$\begin{aligned}
0 \to P \to 0 \text{ exact} \quad &\iff \quad P = 0\\
0 \to P \xrightarrow{f} Q \text{ exact} \quad &\iff \quad f \text{ is injective}\\
P \xrightarrow{f} Q \to 0 \text{ exact} \quad &\iff \quad f \text{ is surjective}\\
0 \to P \xrightarrow{f} Q \to 0 \text{ exact} \quad &\iff \quad f \text{ is an isomorphism}\\
0 \to P \xrightarrow{f} Q \xrightarrow{g} R \to 0 \text{ exact} \quad &\iff \quad R \cong Q/P
\end{aligned}$$

(or more precisely $R \cong Q/f(P)$, but this is frequently used when f is an inclusion map). An exact sequence of this last form is called a **short exact sequence**. (You have seen this before, in one of the special cases of Van Kampen's Theorem.)

One more note: the *cokernel* of a map $f : A \to B$ is by definition coker $f eqdef B/\operatorname{im} f$. So a map is surjective iff its cokernel is 0.

Definition: Let A, B be chain complexes of abelian groups. A **homomorphism** or **chain map** $f : A \to B$ is a family of group homomorphisms $f_n : A_n \to B_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & A_n \arrow[r] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\dots \arrow[r] & B_{n+1} \arrow[r] & B_n \arrow[r] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

That is, $f_{n+1} \circ \partial = \partial \circ f_n$ for every n. More concisely, we can write $f\partial = \partial f$. For example, we have already seen that a continuous function $f : X \to Y$ gives rise to a homomorphism of chain complexes $f_\sharp : C(X) \to C(Y)$.

A chain map $f : A \to B$ has a well-defined kernel and cokernel, which are also chain complexes. The modules comprising $\ker f$ and $\operatorname{coker} f$ are what you wuold expect:

$$(\ker f)_n = \ker(f_n), \qquad (\operatorname{coker} f)_n = \operatorname{coker}(f_n),$$

and the maps between them are given by f as well. If you've never seen this before, confirm it for yourself. Like many proofs in homological algebra, it's hard to go wrong.

To say that chain maps have well-defined kernels and cokernels is to say that the category of chain complexes with chain maps is an **abelian category**.

We can also speak of complexes of complexes, and of exact sequences of complexes. So a **short exact sequence of complexes** $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "i_{n+1}"] & A_n \arrow[r] \arrow[d, "i_n"] & A_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r] \arrow[d, "j_{n+1}"] & B_n \arrow[r] \arrow[d, "j_n"] & B_{n-1} \arrow[r] \arrow[d, "j_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Theorem 8.1. *A short exact sequence of chain complexes as above induces a long exact sequence of their homology groups:*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots \tag{8.1}$$

Sketch of proof: If you have not seen this before, you should verify it on your own. The maps i_* and j_* are induced by the chain maps i and j respectively (as before), while the map $\partial : H_n(C) \to H_{n-1}(A)$ (which is called a **connecting homomorphism**) is defined as follows.

Every element of $H_n(C)$ is the image of a cycle $c \in C_n$. Since j is onto, there exists $b \in B_n$ such that $jb = c$. Thus $\partial b \in B_{n-1}$. We have $j\partial b = \partial jb = \partial c = 0$, so $\partial b \in \ker j = \operatorname{im} i$ and there exists a unique $a \in A_{n-1}$ such that $ia = \partial b$. Note that $i\partial a = \partial ia = \partial\partial b = 0$, so $\partial a = 0$ (since it is mapped to zero by the injective map i). Therefore, a is a cycle, and we can define $\partial[c]$ to be the homology class $[a]$ of a in $H_{n-1}(a)$.

(This construction justifies the name ∂ for the connecting homomorphism: we regard c as equivalent to b and a as equivalent to ia.)

Does the choice of b matter? Suppose we have another b' with $jb' = c$. Then $b - b' \in \ker j = \operatorname{im} i$, say $b - b' = ix$. Choosing a' such that $ia' = \partial b'$, we have

$$i(a - a') = \partial b - \partial b' = \partial(b - b') = \partial ix = i\partial x$$

Since i is injective this says $a - a' = \partial x$. Therefore a, a' represent the same homology class in $H_{n-1}(a)$.

Does the choice of c matter? Suppose we have another cycle c' with $c - c' = \partial y$ for some $y \in C_{n+1}$. Choose $z \in B_{n+1}$ such that $jz = y$; then

$$c' = c - \partial y = jb - \partial jz = jb - j\partial z = j(b - \partial z)$$

so changing c to c' has the effect of changing b to $b - \partial z$. But since $\partial^2 = 0$, this does not change the element ∂b.

This is the kind of "diagram-chasing" argument you have to make to verify that the sequence (8.1) is exact. I omit the rest of the details. $\square$

Now, back to topology. An inclusion of spaces $i : A \hookrightarrow X$ induces monomorphisms $i_* : C_n(A) \hookrightarrow C_n(X)$ for every n, given simply by regarding a chain in A as a chain in X. Therefore we have a monomorphism of chain complexes $i_* : C_\bullet(A) \hookrightarrow C_\bullet(X)$. The **relative chain complex** $C_\bullet(X, A)$ is defined to be the cokernel of this map, and its homology groups are called the **relative homology groups** of the pair (X, A). By the algebra we have just done, there is a **long exact sequence for relative homology**, which is not easy to typeset:

$$
\begin{tikzcd}
\tilde{H}_n(A) \arrow[r, "i_*"] & \tilde{H}_n(X) \arrow[r, "j_*"] & \tilde{H}_n(X,A) \arrow[dll, "\partial"] \\
\tilde{H}_{n-1}(A) \arrow[r, "i_*"] & \tilde{H}_{n-1}(X) \arrow[r, "j_*"] & \tilde{H}_{n-1}(X,A) \arrow[dll, "\partial"] \\
\cdots & &
\end{tikzcd}
\tag{8.2}
$$

The relative complex (X, A) is in fact an algebraic stand-in for the quotient space X/A, as we will prove.

Example: Suppose that $A = \mathbf{pt}$ consists of a single point in X. Then the exact sequence of chain complexes $0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X, A) \to 0$ is

$$
\begin{tikzcd}
 & & \mathbb{Z} \arrow[d] & \\
\cdots & C_1(X) \arrow[r, "\partial"] \arrow[d, equal] & C_0(X) \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots & C_1(X,A) \arrow[r, "\partial"'] & C_0(X,A) \arrow[r] & 0
\end{tikzcd}
$$

We have $\operatorname{im}\alpha \cap \operatorname{im}\partial = 0$ (since, as we know, the 0-boundaries are 0-chains whose sum of coefficients is 0). So both maps marked ∂ have the same kernel. We have proved something important:

$$
\boxed{H_n(X, \mathbf{pt}) = \tilde{H}_n(X).} \tag{8.3}
$$

Example: Let $X = D^2$ and $A = \mathbb{S}^1$. The (unreduced) chain complexes of these spaces are as follows:

$$
\begin{tikzcd}
\text{dimension} & 3 & 2 & 1 & 0 & \\
C_\bullet(A): & & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
C_\bullet(X): & 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\
C_\bullet(X,A): & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 & &
\end{tikzcd}
$$

All the vertical maps $\mathbb{Z} \to \mathbb{Z}$ are isomorphisms. All the maps in $C_\bullet(X,A)$ are zero, so $H_2(X,A) = \mathbb{Z}$ and $H_1(X,A) = H_0(X,A) = 0$. Note that this coincides with the reduced simplicial homology of the 2-sphere, which is the space X/A.

On the other hand, if A were contractible, hence acyclic, then the long exact sequence (8.2) would give isomorphisms $\tilde{H}_n(X) \to H_n(X,A)$ for every n. (Remember that if $0 \to P \to Q \to 0$ is exact, then $P \cong Q$.) On the other hand, contracting a contractible subspace of X gives a homotopy equivalence $X \simeq X/A$.

This is not very strong evidence yet, but we could provide more examples. At all events, the next part of the game is to identify the relative homology groups $H_n(X,A)$ with the absolute reduced homology groups $\tilde{H}_n(X/A)$ of the quotient space. For this we will need the inclusion $A \to X$ to be "nice" (or a **good pair**); fortunately, CW-pairs are good.

These relative objects need some more study. If $f : X \to Y$ is a continuous map with $A \subset X$, $B \subset Y$, and $f(A) \subset B$, then f defines a **relative map** $(X,A) \to (Y,B)$. Relative maps induce homomorphisms on chains and homology: since

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ B \arrow[r, "i"] & Y \end{tikzcd} \qquad \text{gives rise to} \qquad \begin{tikzcd} C_n(A) \arrow[r, "i_\sharp"] \arrow[d, "f_\sharp"'] & C_n(X) \arrow[d, "f_\sharp"] \\ C_n(B) \arrow[r, "i_\sharp"] & C_n(Y) \end{tikzcd}$$

we get an induced map

$$f_\sharp : C_n(X)/C_n(A) = C_n(X,A) \to C_n(Y)/C_n(B) = C_n(Y,B)$$

and since $\partial f_\sharp = f_\sharp \partial$ by the same calculation as for absolute chains (Lemma 6.2), we also get an induced map

$$f_* : H_n(X,A) \to H_n(Y,B).$$

It is also true that a *triple* of spaces $X \supset A \supset B$ gives rise to a long exact sequence of relative homology:

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \to H_{n-1}(X,B) \to \cdots \tag{8.4}$$

In the case that B is a point, this is just the "ordinary" long exact sequence for the pair (X,A).

9. **Fri 4/11: Excision and the LES for Quotient Spaces**

The technical tool that enables us to make the key observation $H_n(X,A) \cong \tilde{H}_n(X/A)$ is called *excision.*

Theorem 9.1 (Excision I)**.** *Let $X \supset A \supset Z$ be spaces such that the closure of Z is contained in the interior of A. Then the inclusion $(X \setminus Z,\ A \setminus Z) \hookrightarrow (X,\ A)$ induces isomorphisms of relative homology*

$$H_n(X \setminus Z,\ A \setminus Z) \to H_n(X,\ A)$$

for every n.

The conclusion is also true for reduced homology. Setting $B = X \setminus Z$ (so $Z = X \setminus B$ and $A \setminus Z = A \cap B$) gives the following equivalent formulation:

Theorem 9.2 (Excision II)**.** *Let $A, B \subset X$ be sets whose interiors cover X. Then the inclusion $(B,\ A \cap B) \hookrightarrow (X,\ A)$ induces isomorphisms of relative homology*

$$H_n(B,\ A \cap B) \to H_n(X,\ A)$$

for every n.

Let's hold off on the proof for the moment — this is one of the most technical parts of Chapter 2. Here is one way excision is useful.

Definition 9.3. A **good pair** is a pair of spaces (X, A) such that A is closed in X and is a deformation retract of some open set in X.

For example, CW-pairs are good — you can enlarge any cell closure $\overline{e^n_\alpha}$ by adding a little bit of each other cell whose closure meets ∂e^n_α. Doing this for each cell in A gives the desired open set in X that deformation-retracts back to A.

Theorem 9.4. *If (X, A) is a good pair then $H_n(X, A) \cong \tilde{H}_n(X/A)$.*

Proof. The quotient map $q : X \to X/A$ gives rise to a relative map $(X, A) \to (X/A, A/A)$ that we will also denote q. Let V be a neighborhood of A that deformation-retracts to A. Then:

(1) In the relative long exact sequence (8.4) for the triple $A \subset V \subset X$, all the terms $H_n(V, A)$ vanish because the deformation-retraction gives a homotopy

equivalence $(V, A) \simeq (A, A)$ and $H_n(A, A) = 0$. So we have an isomorphism

$$\alpha : H_n(X,\ A) \to H_n(X,\ V).$$

(2) Applying the same logic to the triple $A/A \subset V/A \subset X/A$, we get an isomorphism

$$\beta : H_n(X/A,\ A/A) \to H_n(X/A,\ V/A).$$

We therefore have a diagram[1]

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\sim", "\alpha"'] \arrow[d, "q_*"] & H_n(X,V) \arrow[r, "\sim", "\text{exc}"'] \arrow[d, "q_*"] & H_n(X-A, V-A) \arrow[d, "q_*"] \\
H_n(X/A, A/A) \arrow[r, "\sim", "\beta"'] & H_n(X/A, V/A) \arrow[r, "\sim", "\text{exc}"'] & H_n(X/A - A/A, V/A - A/A)
\end{tikzcd}$$

in which the maps marked "exc" are isomorphisms by excision. The right-hand map is an isomorphism because the relative spaces in question are actually identical, and q restricts to the identity map. So the left-hand vertical arrow is also an isomorphism, and $H_n(X/A, A/A) = \tilde{H}_n(X/A)$. $\square$

Excision can be proved from the following proposition:

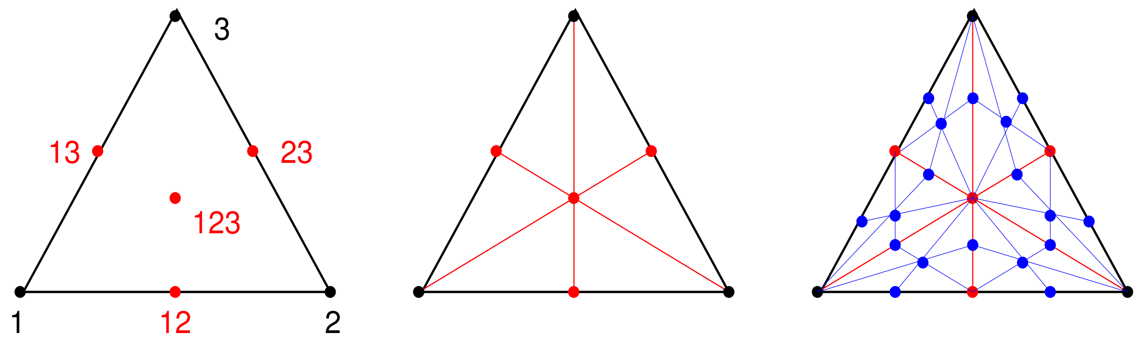
Lemma 9.5 (Subdivision). *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X (or more generally a family of sets whose interiors are a cover) and let $C_n^{\mathcal{U}}(X)$ be the subgroup of X consisting of chains $\sum a_i\sigma_i$ such that each σ_i lies entirely inside some U_α. Then the $C_n^{\mathcal{U}}(X)$ form a chain complex, and the inclusion $C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain homotopy equivalence.*

Sketch of proof: The details are long and technical [Hatcher, pp.119–124]. The key idea is *barycentric subdivision*: take the standard simplex Δ^n, put a point p_σ in the middle of each nonempty face, and chop Δ up into $(n+1)!$ subsimplices whose facets consist of the points

$$\{p_{\sigma_1},\ p_{\sigma_2},\ \dots,\ p_{\sigma_{n+1}}\}$$

for every maximal chain of faces $\{\sigma_1 \subset \cdots \subset \sigma_{n+1}\}$ with $|\sigma_i| = i$. More concretely, each open simplex in the subdivision corresponds to a total ordering on the barycentric coordinates (e.g., the simplex coming from the faces p_2, p_{23}, p_{123} consists of the points in Δ^2 whose barycentric coordinates satisfy $t_2 > t_3 > t_1$.

[1] Ordinarily I prefer $\setminus$ for set difference, but I'm using $-$ in the right-hand column of this diagram, since the bookkeeping is challenging enough without having to parse (or parenthesize) something like $H_n(X/A \setminus A/A, V/A \setminus A/A)$.



One can extend this operation linearly to chains, show that it induces a chain homotopy, and iterate it. Eventually, the simplices get small enough that each one lies inside some $\mathcal{U}$ (using the Lebesgue number lemma), which means that every singular chain in X is equivalent (in a suitable algebraic sense) to the image of a $\mathcal{U}$-chain.

Proof of Excision II. Let $A, B \subset X$ be sets whose interiors cover X and let $\mathcal{U} = \{A, B\}$. The chain homotopy $i : C^{\mathcal{U}}(X) \to C(X)$ gives rise to maps

$$C_n^{\mathcal{U}}(X) \,/\, C_n(A) \;\to\; C_n(X) \,/\, C_n(A) \tag{9.1}$$

that also induces an isomorphism on homology. (This requires the fact that both i and its homotopy inverse send chains in A to chains in A; this comes from the omitted proof of the Subdivision Lemma.)

On the other hand, the natural map

$$C_n(B) \,/\, C_n(A \cap B) \;\to\; C_n^{\mathcal{U}}(X) \,/\, C_n(A) \tag{9.2}$$

is an isomorphism because each one has its basis the singular n-simplices not contained in A. Composing (9.1) and (9.2), we see that the inclusion map

$$C_n(B) \,/\, C_n(A \cap B) \;\to\; C_n(X)/C_n(A) \tag{9.3}$$

induces an isomorphism on homology, as desired. ∎

Review of the logical flow of all this:

$\boxed{\text{Subdivision lemma} \implies \text{Excision} \implies \tilde{H}_n(X/A) = H_n(X, A) \text{ for good pairs.}}$

We can now finally write down the long exact sequence for reduced homology of a good pair (X, A) and the quotient space:

$$\cdots \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \cdots \to \tilde{H}_0(X/A) \to 0. \tag{9.4}$$

Example: Even for nongood pairs, we can use excision to interpret relative homology topologically: it is the homology of the mapping cone of the inclusion $A \hookrightarrow X$. Recall that the cone over A is the space $CA = A \times I/A \times \{1\}$, which is contractible (it deformation-retracts to the cone point p). Identifying $A \times \emptyset$ with $A \subset X$ gives the mapping cone:

$$X \cup CA = X \cup \big(A \times I/A \times \{1\}\big).$$

The long exact sequence for the inclusion $CA \subset X \cup CA$ is

$$\to \cdots H_n(CA) \to H_n(X \cup CA) \to H_n(X \cup CA, CA) \to H_{n-1}(CA) \to \cdots$$

but CA is contractible, hence acyclic, and we get isomorphisms

$$H_n(X \cup CA) \cong H_n(X \cup CA, CA). \tag{9.5}$$

Meanwhile, excision gives isomorphisms

$$H_n(X \cup CA, CA) \cong H_n(X \cup CA \setminus \{p\}, CA \setminus \{p\}). \tag{9.6}$$

But $CA \setminus \{p\}$ deformation-retracts to A, so (9.6) becomes

$$H_n(X \cup CA, CA) \cong H_n(X, A). \tag{9.7}$$

and combining (9.5) with (9.7) gives

$$H_n(X \cup CA) \cong H_n(X, A) \tag{9.8}$$

which is the desired interpretation of relative homology.

10. Mon 4/14: Applications; simplicial/singular equivalence

Homology groups of the sphere. The long exact sequence for reduced homology (9.4) lets us compute $H_k(\mathbb{S}^n)$ for all k and n. Consider the inclusion $\mathbb{S}^n \to \partial D^{n+1} \subset D^{n+1}$. Then $D^{n+1}/\mathbb{S}^n$ is the one-point compactification of $\mathbb{R}^{n+1}$, namely $\mathbb{S}^{n+1}$. The LES for reduced homology is

$$\cdots \to \tilde{H}_k(D^{n+1}) \to \tilde{H}_k(\mathbb{S}^{n+1}) \to \tilde{H}_{k-1}(\mathbb{S}^n) \to \tilde{H}_{k-1}(D^{n+1}) \to \cdots \to \tilde{H}_1(\mathbb{S}^{n+1}) \to \tilde{H}_0(\mathbb{S}^n) -$$

But D^{n+1} is contractible, so in fact we get isomorphisms

$$\tilde{H}_k(\mathbb{S}^{n+1}) \to \tilde{H}_{k-1}(\mathbb{S}^n)$$

for all k and n. Since the only nonvanishing homology of $\mathbb{S}^0 = \{\bullet\bullet\}$ is $\tilde{H}_0(\mathbb{S}^0) = \mathbb{Z}$, it follows by induction that

$$\boxed{\tilde{H}_k(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}}$$

Wedges. Let X, Y be two spaces. There is a natural inclusion $X \hookrightarrow X \wedge Y$, and clearly $(X \wedge Y)/X = Y$. Provided that the points chosen for the wedge sum are good in each summand We get a long exact sequence

$$\cdots \to \tilde{H}_n(X) \to \tilde{H}_n(X \vee Y) \to \tilde{H}_n(Y) \to^{-} * \tilde{H}_{n-1}(X) \to \cdots$$

The maps marked $*$ are all zero, so this in fact splits up into a bunch of short exact sequence

$$0 \to \tilde{H}_n(X) \to \tilde{H}_n(X \vee Y) \to \tilde{H}_n(Y) \to 0.$$

The roles of X and Y are symmetric, so we have in general

$$\tilde{H}_n(X \vee Y) = \tilde{H}_n(X) \oplus \tilde{H}_n(Y).$$

More generally, it is true that

$$\tilde{H}_n(\bigvee_\alpha X_\alpha) = \bigoplus_\alpha \tilde{H}_n(X_\alpha)$$

again provided that the wedge points are good in each summand. We'll see another way to prove this later.

10.1. Equivalence of Singular and Simplicial Homology.

If X is a Δ-complex and $Y \subset X$ is a subcomplex, we can define the **relative simplicial chain complex** $\Delta(X, Y)$ as the cokernel of the inclusion map $\Delta(Y) \to \Delta(X)$ (which by definition maps simplices to simplices). The homology groups of this chain complex are the **relative simplicial homology groups**

$H_n^\Delta(X,Y)$. We then have a long exact sequence for relative simplicial homology (remember, this was just a matter of algebra — all we need is a short exact sequence of chain complexes).

In addition, there are canonical homomorphisms $\Delta_n(X) \to C_n(X)$ that sends every n-simplex in X to its characteristic map. Accordingly, we get induced homorphisms

$$H_n^\Delta(X) \to H_n(X).$$

Theorem 10.1. *Let X be a Δ-complex and $A \subset X$ a subcomplex. The canonical maps just defined induce isomorphisms*

$$H_n^\Delta(X) \xrightarrow{\cong} H_n(X), \qquad H_n^\Delta(X,A) \xrightarrow{\cong} H_n(X,A).$$

Proof. First consider the case that A is empty. The desired isomorphism certainly holds when $n=0$ (since we know that both $H_0(X)$ and $H_0^\Delta(X)$ are free abelian on the path-components of X).

Now assume $k>0$ and consider the pair (X^k, X^{k-1}) (the k- and $(k-1)$-skeletons of X). We get a map of long exact sequences:

$$\begin{tikzcd}
H_{n+1}^\Delta(X^k,X^{k-1}) \arrow[r] \arrow[d, "\alpha"] & H_n^\Delta(X^{k-1}) \arrow[r] \arrow[d, "\beta"] & H_n^\Delta(X^k) \arrow[r] \arrow[d, "\gamma"] & H_n^\Delta(X^k,X^{k-1}) \arrow[r] \arrow[d, "\delta"] & H_{n-1}^\Delta(X^{k-1}) \arrow[d, "\varepsilon"] \\
H_{n+1}(X^k,X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k,X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1})
\end{tikzcd}$$

Claim: The maps α and δ are isomorphisms. The relative chain complex $\Delta(X^k, X^{k-1})$ is nonzero only in the k^{th} position, where it is spanned by the k-simplices of Δ (say there are p of them). Therefore $H_n^\Delta(X^k,X^{k-1}) = \mathbb{Z}^p$ for $n=k$ and 0 otherwise. OTOH, $H_n(X^k,X^{k-1}) = \tilde{H}_n(X^k/X^{k-1}) = \tilde{H}_n((\mathbb{S}^k)^{\vee p}) \cong \mathbb{Z}^p$ as well, and the characteristic maps of the k-faces form a basis. So α and δ take bases of relative simplicial homology to bases of relative singular homology, proving the claim.

By induction on k we may assume that β and ε are isomorphisms as well. (It is certainly true for the base case $k=1$, where the spaces in question are discrete point sets.)

The proof now follows from the Five Lemma of homological algebra, which is as follows (proof omitted; it's in Hatcher; check the diagram chase yourself):

Five Lemma: Consider the diagram with exact rows

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

Then:

$$\begin{array}{lcc}
\beta, \delta \text{ surjective}, \quad \varepsilon \text{ injective} & \Longrightarrow & \gamma \text{ surjective.} \\
\beta, \delta \text{ injective}, \quad \alpha \text{ surjective} & \Longrightarrow & \gamma \text{ injective.} \\
\therefore\ \alpha, \beta, \delta, \varepsilon \text{ isomorphisms} & \Longrightarrow & \gamma \text{ isomorphism.}
\end{array}$$

We have (finally!) shown that $H_n^\Delta(X) = H_n(X)$ for all n and all finite-dimensional Δ-complexes X.

For the relative case, write down the long exact sequences in simplicial and singular homology for the pair (X, A):

$$\begin{tikzcd}
H_n^\Delta(A) \arrow[r] \arrow[d] & H_n^\Delta(X) \arrow[r] \arrow[d] & H_n^\Delta(X,\ A) \arrow[r] \arrow[d] & H_{n-1}^\Delta(A) \arrow[r] \arrow[d] & H_{n-1}^\Delta(X) \arrow[d] \\
H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,\ A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

Now apply the Five Lemma again. The 1st, 2nd, 4th and 5th vertical arrows are isomorphisms by the first part of the proof, so the 3rd map is an isomorphism as well. $\square$

The theorem holds for infinite complexes as well, though one has to do a little more work (I won't).

11. **Wed 4/16: Invariance of dimension; cellular homology**

11.1. **Invariance of Dimension.** Here is an application of all these tools to make a purely topological (as opposed to algebraic-topological) statement:

Theorem: Let $n \geq 0$. The number n is a topological invariant of $\mathbb{R}^n$. That is, if $\mathbb{R}^n \cong \mathbb{R}^m$ then $n = m$. Even more generally, if $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are homeomorphic open sets, then $n = m$.

Proof. Suppose there is a homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^m$. Let $p \in \mathbb{R}^n$ and $q = \phi(p) \in \mathbb{R}^m$, so that ϕ restricts to a homeomorphism $\mathbb{R}^n \setminus \{p\} \to \mathbb{R}^m \setminus \{q\}$. But these spaces are homotopy-equivalent to $\mathbb{S}^{n-1}$ and $\mathbb{S}^{m-1}$ respectively, which do not have the same homology groups. So no such ϕ can exist.

For the stronger statement, the tool we need is excision. Fix $u \in U$ and apply Excision II with $X = \mathbb{R}^n$, $A = \mathbb{R}^n \setminus \{u\}$, $B = U$ to get isomorphisms

$$\tilde{H}_k(U, U \setminus \{u\}) \cong \tilde{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{u\}) \tag{11.1}$$

for all n. Meanwhile, since $\mathbb{R}^n \setminus \{u\} \simeq \mathbb{S}^{n-1}$, the long exact sequence for the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{u\})$ is

$$\cdots \to \underbrace{\tilde{H}_k(\mathbb{R}^n)}_{0} \to \tilde{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{u\}) \to \tilde{H}_{k-1}(\mathbb{S}^{n-1}) \to \underbrace{\tilde{H}_{k-1}(\mathbb{R}^n)}_{0} \to \cdots$$

which breaks up into isomorphisms $\tilde{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{u\}) \cong \tilde{H}_{k-1}(\mathbb{S}^{n-1})$. Combining this with (11.1) gives

$$\tilde{H}_k(U, U \setminus \{u\}) \cong \tilde{H}_{k-1}(\mathbb{S}^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

which says that n is a topological invariant of U. $\square$

Question: Is there a proof of this fact that does not use tools from algebraic topology?

11.2. **Cellular homology.** We finally know that we can compute the homology groups of any space we can triangulate as a simplicial or Δ-complex. But even that turns out to be unnecessary — even a cell complex structure is enough. This is terribly convenient, since lots of interesting spaces can be expressed nicely as cell complexes but are a hassle to triangulate.

Let X be a cell complex, and let $E_n = \{e^n_\alpha\}$ denote the set of n-dimensional cells in X. The n^{th} **cellular chain group** of X is $\mathbb{Z}E_n$. We will show that there is a **cellular chain complex**

$$\cdots \to \mathbb{Z}E_{n+1} \xrightarrow{d_{n+1}} \mathbb{Z}E_n \xrightarrow{d_n} \mathbb{Z}E_{n-1} \to \cdots \to \mathbb{Z}E_0 \to 0 \tag{11.2}$$

that computes the homology of X; that is, $H_n(X) = \ker d_n / \operatorname{im} d_{n+1}$.

Recall that X^n means the n-skeleton of X, that is, the union of all cells of dimension $\leq n$.

Lemma 11.1 (Lemma 2.34)**.** (1) $H_n(X^n, X^{n-1})$ *is free abelian, with basis elements in bijection with* E_n*.*
(2) $H_k(X^n, X^{n-1}) = 0$ *for* $k \neq n$*.*
(3) $H_k(X^n) = 0$ *for* $k > n$*.*
(4) *The inclusion* $i : X^n \hookrightarrow X$ *induces an isomorphism* $i_* : H_k(X^n) \to H_k(X^n) = 0$ *for* $k < n$*.*

Proof. (1,2) We know that $H_n(X^n, X^{n-1}) = \tilde{H}_n(X^n/X^{n-1})$, and this space is a wedge of n-spheres, one for each n-cell in X.

(3) The LES of this pair is

$$\cdots \to H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \xrightarrow{i_*} H_k(X^n) \to H_k(X^n, X^{n-1}) \to \cdots$$

If $k \notin \{n-1, n\}$ then the two outer groups are zero and so i_* is an isomorphism. In particular, if $k > n$ then the result of (1,2) gives

$$H_k(X^n) \cong H_k(X^{n-1}) \cong \cdots \cong H_k(X^0) \cong 0$$

(4) I will just prove this in the case that $\dim X = d < \infty$. The LES says that In this case the result of (1,2)) gives isomorphisms

$$H_k(X^n) \cong H_k(X^{n+1}) \cong \cdots \cong H_k(X^d) = H_k(X)$$

where each map is i_* for some inclusion i of successive skeletons.

For the proof of (4) when $\dim X = \infty$, see Hatcher, pp. 138–139. $\square$

This is a strong hint that the cellular chain groups $\mathbb{Z}E_n$ of (11.2) should be identified with the relative homology groups of successive skeletons. What should the maps between them be? The answer is to splice together pieces of the long exact sequences in relative homology for consecutive skeletons:

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & & \\
 & & & H_{n-1}(X^{n-1}) \arrow[dr, "j_{n-1}"] & \\
\cdots \to H_{n+1}(X^{n+1}, X^n) \arrow[rr, "d_{n+1}"] \arrow[dr, "\partial_{n+1}"'] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[ur, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots \\
 & H_n(X^n) \arrow[ur, "j_n"'] \arrow[dr, "i_n"] & & & \\
0 \arrow[ur] & & H_n(X^{n+1}) = H_n(X) \arrow[dr] & & \\
 & & & H_n(X^{n+1}, X^n) = 0 &
\end{tikzcd}
$$

Definition 11.2. The **cellular chain complex** of a CW-complex X is

$$\cdots \to H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$$

where $d_n = j_{n-1} \circ \partial_n$ is the map as indicated above. The **cellular homology groups** are

$$H_n^{CW}(X) = \ker d_n / \operatorname{im} d_{n+1}.$$

Proposition 11.3. *$H_n^{CW}(X) \cong H_n(X)$.*

Proof. First note that

$$H_n(X) = H_n(X^{n+1}) \cong H_n(X^n)/\ker i_n = H_n(X^n)/\operatorname{im}\partial_{n+1}.$$

The map j_n is injective, so it maps $\operatorname{im}\partial_{n+1}$ isomorphically onto its image, namely $\operatorname{im} d_{n+1}$, and it maps $H_n(X^n)/\operatorname{im}\partial_{n+1}$ onto $\operatorname{im} j_n/\operatorname{im} d_{n+1}$. On the other hand

$$
\begin{aligned}
\operatorname{im} j_n/\operatorname{im} d_{n+1} &= \ker \partial_n/\operatorname{im} d_{n+1} \\
&= \ker(j_{n-1}\partial_n)/\operatorname{im} d_{n+1} \qquad \text{(since } j_{n-1} \text{ is injective)} \\
&= \ker d_n/\operatorname{im} d_{n+1} \\
&= H_n^{CW}(X).
\end{aligned}
$$

□

12. Fri 4/18: Degrees of endomorphisms on spheres

Recall from last time:

X = CW-complex
X^n = n-skeleton of X
$E_n = \{n\text{-cells } e^n_\alpha\}$

***n*th cellular chain group:** $\mathbb{Z}E_n$, identified with $H_n(X^n, X^{n-1})$

Cellular chain complex:

$$\begin{tikzcd}
 & & H_{n-1}(X^{n-1}) \arrow[dr, "j_{n-1}"] & & \\
\cdots \arrow[r] & \mathbb{Z}E_n \arrow[ur, "\partial_n"] \arrow[rr, "d_n"] & & \mathbb{Z}E_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where the maps j_{n-1} and ∂_n come from the appropriate long exact sequences for relative homology.

Proposition: $\ker \partial_n \supseteq \operatorname{im} \partial_{n+1}$ and $H_n(X) \cong \ker \partial_n / \operatorname{im} \partial_{n+1}$.

Corollary 12.1. *For any cell complex X:*

- *If X has a cell structure with k n-cells, then $H_n(X)$ is generated by at most k elements.*
- *In particular, if X has no n-cells then $H_n(X) = 0$.*
- *If X has no two cells in adjacent dimensions, then $H_n(X)$ is free abelian of rank equal to the number of n-cells. (The cellular boundary maps must all be zero.)*

Example 12.2. Complex projective space $\mathbb{C}P^n$ has a cell structure with one cell in each even dimension $0, 2, \ldots, 2n$. Therefore,

$$H_k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{if } k \in \{0, 2, \ldots, 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

More generally, the complex Grassmannian (remember the complex Grassmannian?) has a cell decomposition into Schubert cells, all of which are complex vector spaces, hence even-dimensional cells. So the cell structure instantly tells you the homology groups. Note that things are more complicated over $\mathbb{R}$.

This is an excellent start. But if we have an explicit CW-complex, how do we calculate the cellular boundary maps? This brings us to the notion of *degree*.

A continuous function $f : \mathbb{S}^n \to \mathbb{S}^n$ induces a map $f_* : H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n)$. Since this group is $\mathbb{Z}$, the map f_* sends $x \mapsto dx$ for some integer d, which is called the **degree** of the map, written $\deg(f)$. (Note that we are identifying the domain and range of both f and f_*, so we can distinguish positive-degree from negative-degree maps.)

For a map $\mathbb{S}^1 \to \mathbb{S}^1$, degree equals winding number. In general, $\deg(f)$ counts the points in $f^{-1}(y)$ for any $y \in \mathbb{S}^1$, taking into account orientations. We will make this explicit in the local degree formula below.

(1) If f is a homeomorphism then $\deg(f) = 1$.

(2) If f is not onto, then $\deg(f) = 0$. Say $p \notin \operatorname{im} f$; then f can be factored as $\mathbb{S}^n \xrightarrow{f} \mathbb{S}^n \setminus \{p\} \hookrightarrow \mathbb{S}^n$, so f_* factors as $\mathbb{Z} \to 0 \to \mathbb{Z}$.

(3) If $f \simeq g$ then $f_* = g_*$ so $\deg(f) = \deg(g)$.

(4) $\deg(fg) = \deg(f)\deg(g)$ since $(fg)_* = f_* g_*$.

(5) $\deg(f) = -1$ if f is a reflection. Consider the cell structure on $\mathbb{S}^n$ with two cells of each dimension $0, \dots, n$. Let P, Q be the n-dimensional cells and let r, s be the $(n-1)$-cells, and let f be the reflection across the "equator" $\overline{r \cup s}$, so that f interchanges p and q. Orient everything so that $\partial P = \partial Q = r + s$. Then the induced map of f is

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}E_n \arrow[r, "\begin{bmatrix}1&1\\1&1\end{bmatrix}", "\partial"'] \arrow[d, "f_\sharp", "\begin{bmatrix}0&1\\1&0\end{bmatrix}"'] & \mathbb{Z}E_{n-1} \arrow[r] \arrow[d, "f_\sharp"', "\begin{bmatrix}1&0\\0&1\end{bmatrix}"] & \cdots \\
0 \arrow[r] & \mathbb{Z}E_n \arrow[r, "\partial", "\begin{bmatrix}1&1\\1&1\end{bmatrix}"'] & \mathbb{Z}E_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

We therefore have $H_n(\mathbb{S}^n) = H_n^{CW}(\mathbb{S}^n) = \ker \partial = \mathbb{Z}\langle(1,-1)\rangle$. On the other hand $f_\sharp$ switches the entries of its argument, hence maps the generator of $H_n(\mathbb{S}^n)$ to its negative.

(6) Consider the antipodal map $A : \mathbb{S}^n \to \mathbb{S}^n$ on the sphere. If we embed $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ as usual, then the antipodal map is a composition of

$n+1$ reflections (namely, across each coordinate hyperplane). Therefore $\deg(A) = (-1)^{n+1}$. This is one of those times that the parity of dimension makes a difference.

Just for fun, here is a quick application of degree.

Theorem 12.3 (Hairy Bowling Ball Theorem)**.** *If n is even, then $\mathbb{S}^n$ has no nonzero continuous tangent vector field.*

Sketch of proof: A tangent vector field can be used to construct a homotopy between the antipodal map and the identity. (Supply the details yourself, or consult Hatcher, p.135.) Since homotopic maps have the same degree, we must have $(-1)^{n+1} = 1$, so n is odd.

13. **Mon 4/21: Degrees and Cellular Homology**

Recall from last time that a map $f : \mathbb{S}^n \to \mathbb{S}^n$ ($n \geq 1$) induces a homomorphism

$$f_* : H_n(\mathbb{S}^n) = \mathbb{Z} \to H_n(\mathbb{S}^n), \qquad f_*(1) = \deg(f).$$

We want to know how to calculate degree in practice.

If we are lucky, some point $y \in \mathbb{S}^n$ has finite preimage $Z = \{z_1, \dots, z_m\}$. (Many maps that arise in real life will have this property.) Let V be a neighborhood of Y such that $f^{-1}(V)$ is a disjoint union $U_1 \cup \dots \cup U_m$ with $z_i \in U_i$. Then we have a big diagram (p.136 of Hatcher):

$$\begin{tikzcd}
 & H_n(U_i, U_i - z_i) \arrow[r, "f_{*i}"] \arrow[dl, "\text{exc}"', "\cong"] \arrow[d, "k_i"] & H_n(V, V - y) \arrow[d, "\cong"', "\text{exc}"] \\
H_n(\mathbb{S}^n, \mathbb{S}^n - z_i) & H_n(\mathbb{S}^n, \mathbb{S}^n - Z) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(\mathbb{S}^n, \mathbb{S}^n - y) \\
 & H_n(\mathbb{S}^n) \arrow[ul, "\cong", "\text{rel}"'] \arrow[u, "j"'] \arrow[r, "f_*"] & H_n(\mathbb{S}^n) \arrow[u, "\cong", "\text{rel}"']
\end{tikzcd}$$

where:

- The isomorphisms marked "exc" come from excision (version I).

- The isomorphisms marked "rel" come from LES's in relative homology. If x is either z_i or y, then part of the LES is

$$H_n(\mathbb{S}^n - x) \to H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n, \mathbb{S}^n - x) \to H_{n-1}(\mathbb{S}^n - x)$$

but the two outer terms vanish since $\mathbb{S}^n - x \cong \mathbb{R}^n$ is contractible.

- The map j comes from an SES that is part of an LES:

$$\underbrace{H_n(\mathbb{S}^n - Z)}_{0} \to H_n(\mathbb{S}^n) \xrightarrow{j} H_n(\mathbb{S}^n, \mathbb{S}^n - Z) \to H_{n-1}(Z) \to \underbrace{H_{n-1}(\mathbb{S}^n)}_{0}.$$

- The maps k_i and p_i are induced homomorphisms of inclusions. (We're dropping the subscript stars.)

Since $H_n(\mathbb{S}^n) \cong \mathbb{Z}$, every group but the one in the middle is a copy of $\mathbb{Z}$. The **local degree of $\boldsymbol{f}$ at $\boldsymbol{z_i}$** is defined as

$$\deg(f|z_i) eqdef \deg(f_{*i}).$$

Proposition 13.1. $\deg(f) = \sum_i \deg(f|z_i)$.

Proof. Let $U = \cup_{i=1}^m U_i$. The pair $(U, \mathbb{S}^n - Z)$ covers $\mathbb{S}^n$, so Version II of excision gives an isomorphism

$$H_n(\mathbb{S}^n, \mathbb{S}^n - Z) \cong H_n(U, U \cap (\mathbb{S}^n - Z)) = \bigoplus_{i=1}^m H_n(U_i, U_i - z_i) \cong \mathbb{Z}^m.$$

(The second equality is just because homology is additive on components.) We can rewrite the previous diagram as

$$\begin{tikzcd}
 & \mathbb{Z}e_i \arrow[r, "f_{*i}"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & \mathbb{Z} \arrow[d, "\cong"] \\
\mathbb{Z} & \mathbb{Z}\langle e_1, \dots, e_m \rangle \arrow[l, "p_i"'] \arrow[r, "g"] & \mathbb{Z} \\
 & \mathbb{Z} \arrow[ul, "\cong"] \arrow[u, "j"] \arrow[r, "f_*"] & \mathbb{Z} \arrow[u, "\cong"']
\end{tikzcd}$$

The maps k_i send $e_i \mapsto e_i$, and the map j sends $e \mapsto \sum e_i$ because $p_i(j(1)) = 1$. By commutativity of the diagram, we have therefore

$$\begin{aligned}
\deg(f) &= f_*(1) = g(j(1)) && \text{(bottom square)} \\
&= g\left(\sum_i e_i\right) = g\left(\sum_i k_i e_i\right) \\
&= \sum_i f_{*i} e_i && \text{(top square)} \\
&= \sum_i \deg(f|z_i). && \square
\end{aligned}$$

Again, this formula says that you can figure out the degree of a map by finding a point with finite preimage (any point will do) and counting the preimage points, keeping track of orientations. The nice thing about degrees is that they give the coefficients of the boundary maps in cellular homology. Specifically:

Theorem 13.2 (Cellular Boundary Formula). *Let X be a cell complex with cellular chain complex*

$$\cdots \to \mathbb{Z}E_n \xrightarrow{d_n} \mathbb{Z}E_{n-1} \to \cdots$$

where E_n denotes the set of n-cells. Then for every pair of cells e^n_α and e^{n-1}_β, the corresponding coefficient $d_{\beta\alpha}$ of d_n is the degree of the map $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ given by the composition

$$\mathbb{S}^{n-1} = \partial e^n_\alpha \xrightarrow{\phi_\alpha} X_{n-1} \xrightarrow{q} X_{n-1}/(X_{n-1} \setminus e^{n-1}_\beta) = \mathbb{S}^{n-1}$$

where ϕ_α is the attaching map of e^n_α and q is the quotient map that collapses everything outside e^{n-1}_β to a point (so that we wind up with a one-point compactification of e^{n-1}_β).

The number $d_{\beta\alpha}$ is called the **degree with which $\boldsymbol{e^n_\alpha}$ is attached to $\boldsymbol{e^{n-1}_\beta}$.** I omit the proof (more diagram-chasing; see p.141 of Hatcher).

Example: Real projective space. Recall that $\mathbb{R}P^n$ has a cell structure $e^0 \cup \cdots \cup e^n$. Every attaching map is 2-1 (it identifies antipodal points on ∂e^n). To find its degree, we can use the local degree formula.

Let A be the antipode on $\partial e^n \cong \mathbb{S}^{n-1}$ and let z_1, z_2 be two antipodal points, with disjoint antipodal neighborhoods U_1, U_2, that map to a point y in a neighborhood $V \subseteq e^{n-1}$. The attaching map of ∂e^n is a local homeomorphism, so the maps f_1, f_2 in the diagram

$$\begin{tikzcd}
H_{n-1}(U_1, U_1 - z_1) \arrow[r, "f_1"] \arrow[d, "A_*"'] & H_{n-1}(V, V - y) \arrow[d, equal] \\
H_{n-1}(U_2, U_2 - z_2) \arrow[r, "f_2"'] & H_{n-1}(V, V - y)
\end{tikzcd}$$

each have degree ± 1. But recall that the degree of the antipode alternates: we have

$$\deg(f_1) = \deg(f_2)\deg(A^*) = (-1)^n.$$

Therefore

$$d : E^n \to E^{n-1} = \begin{cases} \pm 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Note that this sign is just a conventional choice and doesn't affect the kernel or image of the map. Therefore, the cellular chain complex $E_\bullet(\mathbb{R}P^n)$ is

$$\begin{cases} 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \cdots \to \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 & \text{if } n \text{ is odd,} \\ 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \cdots \to \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 & \text{if } n \text{ is even} \end{cases} \tag{13.1}$$

from which it follows that

$$\tilde{H}_k(\mathbb{R}P^n) = \begin{cases} 0 & \text{if } k \text{ is even or } k > n, \\ \mathbb{Z}_2 & \text{if } k \text{ is odd and } k < n, \\ \mathbb{Z} & \text{if } k = n \text{ is odd.} \end{cases}$$

14. Wed 4/23: More cellular homology; homology with coefficients; Euler characteristic

Example: 2-Manifolds. Consider the orientable surface M_g of genus g, with the cell structure from p.5 of Hatcher. The cellular chain complex is

$$E_2 = \mathbb{Z} \xrightarrow{d_2} E_1 = \mathbb{Z}^{2g} \xrightarrow{d_1} E_0 = \mathbb{Z}$$

but the attaching maps are both zero (d_1 because each 1-cell is a loop, d_2 because each pair of sides of the polygon with the same labels point in opposite directions). Therefore

$$H_2(M_g) = \mathbb{Z}, \qquad H_1(M_g) = \mathbb{Z}^{2g}, \qquad H_0(M_g) = \mathbb{Z}. \tag{14.1}$$

That was easy!

Similarly, the nonorientable surface N_g has a cell structure consisting of a 2-cell attached to the wedge of g circles by the wod $a_1^2 \cdots a_g^2$ (Hatcher, p.51). Therefore its cellular chain complex is

$$E_2 = \mathbb{Z} \xrightarrow{d_2} E_1 = \mathbb{Z}^g \xrightarrow{d_1} E_0 = \mathbb{Z}$$

where d_2 is a $g \times 1$ column vector with all entries 2, and $d_1 = 0$. Therefore

$$H_2(N_g) = 0, \qquad H_1(N_g) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2, \qquad H_0(N_g) = \mathbb{Z}. \tag{14.2}$$

In fact, orientable surfaces have top homology class $\mathbb{Z}$ and nonorientable ones have top homology class zero. This is characteristic of all manifolds (not just surfaces).

14.1. Homology with Coefficients.

At the beginning of all this, we defined a (singular, simplicial, cellular) chain to be an integer linear combination of simplices. All the machinery would work the same way if we considered linear combinations of simplices over an arbitrary abelian group G (called the *coefficient group* in this context). For example, the n^{th} singular chain group of a space X opver G consists of formal G-linear combinations of singular n-simplices, i.e.,

$$C_n(X;G) = C_n(X) \otimes_{\mathbb{Z}} G = \{\sum_{i=1}^k g_i \sigma_i \ : \ \sigma_i : \Delta^n \to X, \ g_i \in G\}.$$

The singular boundary map $\partial_n : C_n(X;G) \to C_{n-1}(X;G)$ is defined by the same formula. giving a chain complex $C_\bullet(X;G)$. Again, the homology groups $H_n(X;G)$ of this complex are topological invariants of X.

One important case is that G is a field of characteristic 0, say $\mathbb{R}$ ($\mathbb{Q}$, $\mathbb{C}$, etc. behave the same way). The rank of a matrix over $\mathbb{R}$ is the same as it is over $\mathbb{Z}$, but modules over $\mathbb{R}$ are just vector spaces — there is no possibility of torsion. So passing from $\mathbb{Z}$ to $\mathbb{R}$ has the effect of keeping the free part of homology and throwing away torsion:

$$H_n(X;\mathbb{R}) = H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = H_n(X;\mathbb{Z})/\mathbf{T}(H_n(X;\mathbb{Z})). \tag{14.3}$$

In principle this loses information, but there are many spaces for which the integer homology is torsion-free, so one may as well work over $\mathbb{Q}$, which can simplify computation (no more Smith normal forms!) For finite cell complexes, homology over $\mathbb{R}$ is easily encoded by the *Poincaré polynomial*

$$\mathrm{Poin}(X,q) = \sum_{n\geq 0} \dim H_n(X;\mathbb{R})q^n$$

which has nice properties; for example, it can be interpreted combinatorially for certain nice spaces (like Grassmannians and flag manifolds).

Another important case is $G = \mathbb{Z}_2$. Many attaching maps of cell complexes have degree 2, which become 0 over $\mathbb{Z}_2$. For example, in the cellular chain complex (13.1) for $\mathbb{R}P^n$, all attaching maps become zero, from which it follows immediately that

$$H_k(\mathbb{R}P^n;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } 0 \leq k \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

a much simpler answer than over $\mathbb{Z}$. This is an illustration of the principle that (14.3) **fails in positive characteristic** — more on this in a moment. In general, $\mathbb{Z}_2$-homology is useful for studying manifolds that might or might not be orientable.

It is not hard to come up with spaces that are acyclic with respect to one coefficient group but not another. (On the other hand, statements like "If X is contractible, then it is acyclic" holds over any coefficient group.)

In general, the relationship between $H_n(X;\mathbb{Z})$ and $H_n(X;G)$ is expressed by the **universal coefficient theorem for homology** (Hatcher, Theorem 3A.3, p.264), whose proof requires much more machinery from both topology (cohomology) and algebra (Tor). Here is the statement: for any space X and any coefficient group G, there is a split short exact sequence

$$0 \;\to\; H_n(X)\otimes G \;\to\; H_n(X;G)\otimes \mathrm{Tor}(H_{n-1}(X);G) \;\to\; 0. \tag{14.4}$$

Here "split" means that the middle term is the direct sum of the other two (this is the nicest kind of short exact sequence). What Tor means is a longer story, but it can be calculated explicitly for particular cases (see Hatcher, Prop. 3A.5, p.265); for instance, if A has no torsion then $\mathrm{Tor}(A, B) = 0$, so that (14.3) is a special case of (14.4).

14.2. **Euler Characteristic.**

Definition 14.1. Let X be a topological space such that $\tilde{H}_n(X; \mathbb{R}) = 0$ for $n \gg 0$. The **Euler characteristic** of X is

$$\chi(X) = \sum_{n \geq 0} (-1)^n \dim H_n(X; \mathbb{R}) = \mathrm{Poin}(X, -1).$$

Why work with this rather coarse invariant, which throws away a lot of information? Here's why.

Proposition 14.2 (Euler-Poincaré Formula)**.** *Let X be a finite cell complex and let E_n be the set of n-cells. Then*

$$\chi(X) = \sum_{n \geq 0} (-1)^n |E_n|.$$

Proof. Let $d_n : \mathbb{R}E_n \to \mathbb{R}E_{n-1}$ be the cellular boundary maps over $\mathbb{R}$. We have

$$\begin{aligned}
\chi(X) &= \sum_n (-1)^n \dim H_n(X; \mathbb{R}) \\
&= \sum_n (-1)^n (\dim \ker d_n - \dim \operatorname{im} d_{n-1}) \\
&= \sum_n (-1)^n (\dim \ker d_n + \dim \operatorname{im} d_n) \qquad \text{(by rearranging terms)} \\
&= \sum_n (-1)^n \dim \mathbb{R}E_n \\
&= \sum_n (-1)^n |E_n|. \qquad \square
\end{aligned}$$

Thus the Euler characteristic is a combinatorial constraint on the possible cell structures on a given topological space. For example, from the cell structure on $\mathbb{S}^n$ as the one-point compactification of $\mathbb{R}^n$, it follows that

$$\chi(\mathbb{S}^n) = 1 + (-1)^n = \begin{cases} 2 & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

which restricts the possible face posets of polytopes (i.e., polyhedral cell complexes homeomorphic to spheres). The case $n = 2$ is Euler's famous polyhedral formula: for a polyhedron in 3-space with v vertices, e edges and f faces, we must have $v - e + f = 2$.

The calculations of homology for orientable surfaces M_g (14.1) and nonorientable surfaces N_g (14.2) imply that

$$\chi(M_g) = 2 - 2g, \qquad \chi(N_g) = 2 - g.$$

So Euler characteristic is powerful enough to distinguish between orientable (resp., nonorientable) surfaces of different genera.

15. Fri 4/25: Retractions and the Mayer-Vietoris Sequence

15.1. **Retractions.** Suppose $A \subseteq X$ is a subspace and $r : X \to A$ is a retraction; recall that this means that $r|_A = \mathbb{1}_A$. In other words, the composition $A \xrightarrow{i} X \xrightarrow{r} A$ is the identity map. From this it follows that $(r \circ i)_* = r_* i_*$ is the identity on $H_n(A)$ for all n. In particular, i_* is 1-1 and r_* is onto. This gives rise to a really easy proof of the Brouwer Fixed Point Theorem:

Theorem 15.1 (Brouwer Fixed Point Theorem)**.** *For $n \geq 0$, every continuous map $f : B^n \to B^n$ has a fixed point.*

Proof. The theorem is trivially true for $n = 0$. If $n > 0$, observe that a fixed-point-free function $f : B^n \to B^n$ gives rise to a retraction $r : B^n \to \mathbb{S}^{n-1}$: for each $x \in B^n$, draw the unique ray from $f(x)$ to x and let $r(x)$ be the point where the ray hits $\mathbb{S}^{n-1}$. However, such a retraction cannot exist, since it would induce a surjection $H_{n-1}(B^n) = 0 \to H_{n-1}(\mathbb{S}^n) = \mathbb{Z}$. $\square$

Going back to the general case of a retraction $r : X \to A$, consider the long exact sequence in homology for the pair (X, A). Since $i_* : H_n(A) \to H_n(X)$ is injective, we have $\operatorname{im} \partial = \ker i_* = 0$ — but this says that all of the connecting homomorphisms are zero, and so the long exact sequence actually breaks up into short exact sequences

$$0 \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \to 0.$$

In fact these are actually *split* short exact sequences. That is,

$$H_n(X) = H_n(A) \oplus H_n(X, A).$$

This is an easy piece of algebra (p.147).

If $X = A \times B$ and $b_0 \in B$, then the map $f(a, b) = f(a, b_0)$ is a retraction $X \to A$, so

$$H_n(A \times B) = H_n(A) \oplus H_n(A \times B, A \times \{b_0\}). \tag{15.1}$$

Note, however, that the second thing is certainly *not* $H_n(B)$. For instance, if $B = \mathbb{S}^0$ then $A \times B/A \times \{b_0\} = A \cup \{\bullet\}$, and if $B = \mathbb{S}^1$ then you are going to have to use your imagination. In fact much more work is needed to compute homology of products; the end result is the *Künneth formula*, which says that for every k, there is a split short exact sequence

$$0 \to \bigoplus_k H_k(X) \otimes H_{n-k}(Y) \to H_n(X \times Y) \to \bigoplus_k \mathrm{Tor}_1(H_k(X), H_{n-k-1}(Y)) \to 0.$$

15.2. **Mayer-Vietoris Sequences.** Suppose we have a decomposition $X = A \cup B$. We will assume that A, B are good subspaces of X — i.e., we can if necessary enlarge A to an open set that deformation-retracts onto it. We would like to describe the homology of X in terms of the homologies of A, B, and $A \cap B$.

Let $C_n(A+B) = C_n(A) + C_n(B) \subset C_n(X)$ (i.e., the group of X-chains that are sums of A-chains and B-chains).

The boundary map $\partial : C_n(X) \to C_{n-1}(X)$ certainly maps $C_n(A+B) \to C_{n-1}(A+B)$, so $C_\bullet(A+B)$ is a chain complex.

The Subdivision Lemma (Lemma 9.5) says that $C_\bullet(A+B)$ is chain-homotopic to $C_n(X)$, so we can use it to compute homology of X.

For each n there is a short exact sequence

$$0 \to C_n(A \cap B) \to C_n(A) \oplus C_n(B) \xrightarrow{g} C_n(A+B) \to 0$$

because g is clearly a surjection, and its kernel consists of ordered pairs $(x, -x)$, where $x \in C_n(A \cap B)$. We therefore have a short exact sequence of chain complexes

$$0 \to C_\bullet(A \cap B) \to C_\bullet(A) \oplus C_\bullet(B) \to C_\bullet(A+B) \to 0$$

which gives rise to a long exact sequence in homology called the **Mayer-Vietoris sequence:**

$$\begin{tikzcd}
H_n(A\cap B) \arrow[r] & H_n(A)\oplus H_n(B) \arrow[r] & H_n(X) \arrow[dll, rounded corners] \\
H_{n-1}(A\cap B) \arrow[r] & H_{n-1}(A)\oplus H_{n-1}(B) \arrow[r] & H_{n-1}(X) \arrow[dll, rounded corners] \\
H_{n-2}(A\cap B) \to \dots & &
\end{tikzcd} \tag{15.2}$$

The homology groups may be taken to be either unreduced or reduced — the unreduced version shown above is simply the direct sum of the reduced Mayer-Vietoris sequence with the split short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$.

Example: Recall that the **suspension** SX of a space X is obtained by taking two cones A, B over X and attaching them along their bases. Equivalently, take a prism over X and squash the top and bottom faces to points:

$$SX = X \times [-1,1]/X \times \{-1\} \cup X \times \{1\}.$$

For example, the suspension of $\mathbb{S}^n$ is $\mathbb{S}^{n+1}$. Last week we showed that $H_n(SX) \cong H_{n-1}(X)$ for all $n > 0$, using the LES for relative homology. This is even easier using Mayer-Vietoris. Observe that $A \cap B = X$ and $A \cup B = SX$. Meanwhile, A, B are both contractible, so the Mayer-Vietoris sequence for the pair breaks up into short exact sequences $0 \to H_n(SX) \xrightarrow{\partial} H_{n-1}(X) \to 0$.

Example: Consider the wedge sum $X \vee Y$. The intersection is a single point, so the Mayer-Vietoris sequence again breaks up into isomorphisms $\tilde{H}_n(X \vee Y) = \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$. Again, we already knew this.

Remark 15.2. If $A \cap B$ is path-connected, so that $\tilde{H}_0(A \cap B) = 0$, then the Mayer-Vietoris sequence implies that $\tilde{H}_1(X) = \tilde{H}_1(A) \oplus \tilde{H}_1(B)/\operatorname{im}\Phi$, where Φ is the map $\tilde{H}_1(A \cap B) \to \tilde{H}_1(A) \oplus \tilde{H}_1(B)$. This looks an awful lot like Van Kampen's Theorem, doesn't it?

There is also a relative Mayer-Vietoris sequence. Given a good pair (X, Y) with open covers $X = A \cup B$ and $Y = C \cup D$ with $C \subseteq A$ and $D \subseteq B$, the relative Mayer-Vietoris sequence has the form

$$\begin{tikzcd}
H_n(A\cap B, C\cap D) \arrow[r] & H_n(A,C)\oplus H_n(B,D) \arrow[r] & H_n(X,Y) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]$(\tikztostart.south)!0.5!(\tikztotarget.north)$) -| (\tikztotarget.north)}] \\
H_{n-1}(A\cap B, C\cap D) \arrow[r] & H_{n-1}(A,C)\oplus H_{n-1}(B,D) \arrow[r] & H_{n-1}(X,Y) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]$(\tikztostart.south)!0.5!(\tikztotarget.north)$) -| (\tikztotarget.north)}] \\
H_{n-2}(A\cap B, C\cap D) \to \dots & &
\end{tikzcd} \tag{15.3}$$

Example: [Hatcher, p.158, #36] We want to calculate the homology of $X \times \mathbb{S}^n$ in terms of that of X. From (15.1) we have

$$H_k(X \times \mathbb{S}^n) = H_k(X) \oplus H_k(X \times \mathbb{S}^n, X \times \{x_0\}) \tag{15.4}$$

so we need to calculate the second direct summand. For this, we can use a relative Mayer-Vietoris sequence. Replace X with $X \times \mathbb{S}^n$ in (15.3) and set

$$\begin{aligned}
A &= \text{ northern hemisphere } \cong \mathbb{R}^n, \qquad C = D = X \times \{x_0\} \\
B &= \text{ southern hemisphere } \cong \mathbb{R}^n, \qquad (x_0 = \text{some point on equator})
\end{aligned}$$

Then $H_k(A, C) = H_k(X \times \mathbb{R}^n, X \times \{x_0\}) = H_k(X, X) = 0$, so the relative MVS gives

$$\begin{aligned}
H_k(X \times \mathbb{S}^n, X \times \{x_0\}) &\cong H_{k-1}(X \times \mathbb{S}^{n-1}, X \times \{x_0\}) \\
&\cong \cdots \cong H_{k-n}(X \times \mathbb{S}^0, X \times \{x_0\}) \\
&\cong \tilde{H}_{k-n}(X \times \mathbb{S}^0 \;/\; X \times \{x_0\}) \\
&= \tilde{H}_{k-n}(X \cup \{\bullet\}) \qquad\qquad = H_{k-n}(X).
\end{aligned}$$

Combining this with (15.4) gives an interesting formula:

$$\boxed{H_k(X \times \mathbb{S}^n) = H_k(X) \oplus H_{k-n}(X).}$$

It takes more work to derive the formula for the homology of a product of spaces $X \times Y$ in terms of the homologies of X and Y. The answer, which is called the **Künneth formula**, turns out to be the following: For every n, there is a split short exact sequence

$$0 \to \bigoplus_k H_k(X) \otimes H_{n-k}(Y) \to H_n(X \times Y) \to \bigoplus_k \operatorname{Tor}^1(H_k(X), H_{n-k-1}(Y)) \to 0.$$

Example: Connected sums. Let X, Y be compact real manifolds of dimension n. The **connected sum** $X\#Y$ is obtained by choosing open discs $U \subset X$ and $V \subset Y$, deleting them, and then identifying their boundary $\mathbb{S}^{n-1}$'s. That is,

$$X\#Y = (X \setminus U) \cup (Y \setminus V) \ / \ \partial U \sim \partial V.$$

Note that $X\#Y$ is also a compact manifold. The two spaces $X \setminus U$, $Y \setminus V$ cover $X\#Y$. Now, I *think* that every compact manifold has a cell structure with only one top-dimensional cell; if this is true, then deleting an open disc (which can be taken to be a subset of the big cell) is homotopy-equivalent to removing the entire cell (shrink the punctured cell onto its boundary). That is,

$$X \setminus U \simeq X^{n-1}, \quad Y \setminus V \simeq Y^{n-1}.$$

Therefore, the Mayer-Vietoris sequence has the form

$$\cdots \to \tilde{H}_k(\mathbb{S}^{n-1}) \to \tilde{H}_k(X^{n-1}) \oplus \tilde{H}_k(Y^{n-1}) \to \tilde{H}_k(X\#Y) \to \tilde{H}_{k-1}(\mathbb{S}^{n-1}) \to \cdots$$

In particular, we get isomorphisms $\tilde{H}_k(X\#Y) \cong \tilde{H}_k(X^{n-1}) \oplus \tilde{H}_k(Y^{n-1}) = \tilde{H}_k(X) \oplus \tilde{H}_k(Y)$ for all $k < n-1$. The more exciting part of the LES is

$$0 \to \tilde{H}_n(X\#Y) \xrightarrow{\alpha} \underbrace{\tilde{H}_{n-1}(\mathbb{S}^{n-1})}_{\mathbb{Z}} \xrightarrow{\beta} \tilde{H}_{n-1}(X^{n-1}) \oplus \tilde{H}_{n-1}(Y^{n-1}) \xrightarrow{\gamma} \tilde{H}_{n-1}(X\#Y) \to 0.$$

This is where we have to think geometrically (or use the algebra to tell us about the geometry). For instance, suppose $X = M_g$ and $Y = M_h$ are orientable surfaces ($n = 2$). Then $X \sharp Y = M_{g+h}$. In fact $H_1(M_{g+h}) = \mathbb{Z}^{2(g+h)} = \mathbb{Z}^{2g} \oplus \mathbb{Z}^{2h} = H_1(X^1) \oplus H_1(Y^1)$, which is consistent with the statement that α, γ are isomorphisms and $\beta = 0$. But I don't think this is true if, say, $Z = N_g$ and $Y = N_h$.

For more on this, see this discussion on math.stackexchange.com.

16. Mon 4/28: Combinatorics and Topology

Shifted complexes. A simplicial complex X on vertex set $[n] = \{1, \ldots, n\}$ is **shifted** if, for every face. $\sigma \in X$, replacing some or all of the vertices in σ with numerically smaller vertices gets you another face of X. For example

Proposition: Every shifted simplicial complex is homotopy-equivalent to a wedge of spheres, in which the number of copies of $\mathbb{S}^k$ in the wedge is the number of k-simplices in X not containing vertex 1.

Consequently, calculating the homology of a shifted complex is a purely combinatorial (i.e., fun) problem.

Proof. The **star** and **link** of vertex 1 are defined as

$$S = \text{star}_X(1) = \{\sigma \in X \mid \sigma \cup \{1\} \in X\},$$
$$L = \text{link}_X(1) = \{\sigma \in X \mid 1 \notin \sigma,\ \sigma \cup \{1\} \in X\}.$$

The star is the cone over the link. In particular the star is contractible, so $X/S(1) \simeq X$. (This is true for the star and link of any vertex in any complex.)

Now, suppose X is shifted and that $\sigma \in X \setminus S$. In particular σ is a facet, because if $\sigma \cup\!\!\!\cdot\, \{v\} \in X$ for any v then shiftedness implies $\sigma \cup\!\!\!\cdot\, \{1\} \in X$. Moreover, shiftedness implies that the sets

$$\{\sigma \setminus \{v\} \cup \{1\} \mid v \in \sigma\} \tag{$*$}$$

are all faces as well. Together with σ, these faces form a subcomplex Y_σ of X homeomorphic to $\mathbb{S}^n$. Contracting the link squashes all facets of Y_σ other than σ itself to a point. We conclude that X/S is a wedge of spheres, with an n-dimensional sphere for each n-face not containing 1, and the conclusion follows. □

This result generalizes problem #6 on HW #6. It is easy to see that the complex $\Delta^{n,d}$ (the d-skeleton of the n-simplex) is shifted. Every non-maximal face belongs to the star, so $\Delta^{n,d} \simeq (\mathbb{S}^d)^{\wedge q}$ where $q = \binom{n}{d+1}$ — the number of ways to construct a d-simplex from the n vertices $\{2, 3, \ldots, n+1\}$.

Stanley-Reisner rings. Let Δ be a simplicial complex on vertex set $[n]$, and let $S = \Bbbk[x_1, \ldots, x_n]$ where $\Bbbk$ is some field (it doesn't matter much what). We can regard faces of Δ as squarefree monomials in S (i.e., as products of distinct variables). The **Stanley-Reisner ideal** of Δ is

$$I_\Delta eqdef \left\langle \prod_{i\in\tau} x_i \mid \tau \notin \Delta \right\rangle.$$

The monomials corresponding to minimal nonfaces τ give a minimal generating set for I_Δ. The **Stanley-Reisner ring** is

$$\Bbbk[\Delta] eqdef S/I_\Delta.$$

The natural basis for $\Bbbk[\Delta]$ consists of monomials whose support is a face of Δ. (The *support* of a monomial is the set of variables that appear in it with nonzero power; again, we are associating this set of variables with the simplex on the corresponding vertices of Δ.)

A ring R is **graded** if $R = \bigoplus_{d\geq 0} R_d$ and $R_d \cdot R_e \subseteq R_{d+e}$. The summand R_d is called the ***d^th^* graded piece** and its elements are **homogeneous of degree *d***. The **Hilbert series** of R is the formal power series

$$H(R,q) eqdef \sum_{d\geq 0} \dim_\Bbbk (R_d) q^d.$$

The Stanley-Reisner ring $\Bbbk[\Delta]$ is graded, with $\Bbbk[\Delta]_k$ equal to the $\Bbbk$-linear span of the monomials of degree k supported on a face of Δ. Meanwhile, for each face σ with i vertices, we have

$$\sum_{\substack{\text{monomials } \mu: \\ \text{support}(\mu)=\sigma}} q^{\deg(\mu)} = q^d(1+q+q^2+\cdots)^d = \left(\frac{q}{1-q}\right)^d$$

(this is a basic example of a generating function). Therefore,

$$\begin{aligned} H(\Bbbk[\Delta];q) &= \sum_{n\geq 0} \dim(\Bbbk[\Delta]_d) q^d \\ &= \sum_{\sigma\in\Delta} \left(\frac{q}{1-q}\right)^{1+\dim\sigma} \\ &= \sum_d f_d(\Delta) \left(\frac{q}{1-q}\right)^{1+d} \end{aligned}$$

where $f_d(\Delta)$ means the number of d-dimensional faces. Putting the summands over a common denominator gives a formula

$$H(\Bbbk[\Delta];q) = \frac{h_\Delta(q)}{(1-q)^D}.$$

It turns out that $D = \dim\Delta + 1 = \dim\Bbbk[\Delta]$ (where the second dim means Krull dimension of a ring), and that $h(q)$ is a polynomial with interesting combinatorial properties.

A lot of headway can be made by working back and forth between the combinatoria/topological object Δ and the algebraic object $\Bbbk[\Delta]$. For example, Hochster's formula says that local cohomology of $\Bbbk[\Delta]$ (whatever that means) corresponds to simplicial homology of certain subcomplexes of Δ. Algebraic information about the ring can give combinatorial information about the complex (e.g., if $\Bbbk[\Delta]$ is Cohen-Macaulay, then the coefficients of h are all positive; if in addition $\Bbbk[\Delta]$ is Gorenstein, then the coefficients are palindromic) and vice versa (if Δ is shellable, which means that it can be assembled by attaching facets one at a time in a "nice" manner, then $\Bbbk[\Delta]$ is Cohen-Macaulay).

17. **Wed 4/30: Homology and the Fundamental Group**

Here is the Cliffs Notes version: "H_1 is the abelianization of π_1."

Theorem 17.1. *Let X be a space and $x_0 \in X$. Abbreviate $\pi_1(X, x_0)$ by $\pi_1(X)$.*

Then there is a canonical homomorphism $\psi : \pi_1(X) \to H_1(X)$ sending every loop to itself (regarded as a 1-simplex). If X is path-connected, then ψ is surjective and $\ker\psi$ is the commutator subgroup $\operatorname{Comm}(\pi_1(X))$, so that

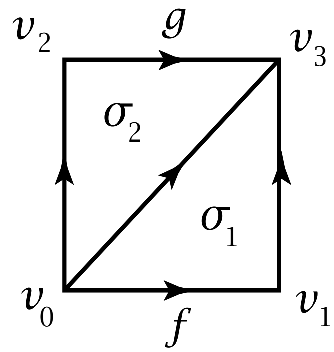
$$H_1(X) \cong \pi_1(X)/\operatorname{Comm}(\pi_1(X)) =: \operatorname{Ab}(\pi_1(X)).$$

This is §2.A of Hatcher, pp. 166–168.

Proof. Every loop $f : I \to X$ can certainly be regarded as a singular 1-simplex. In fact, f is a singular 1-cycle, since $\partial f = f(1) - f(0) = [x_0] - [x_0] = 0 \in C_0(X)$.

Part 1: Show that path-homotopic loops are equal in homology.
In other words, show that $\psi : \pi_1(X) \to H_1(X)$ is a well-defined map.

Suppose that $f \simeq g$ via a path homotopy $F : I \times I \to X$. It's been a while, so recall this means that $F(0, t) = F(1, t) = x_0$ for all t, and we regard F as a homotopy between the loops $f(s) = F(s, 0)$ and $g(s) = F(s, 1)$. Draw a line segment D from $(0, 0)$ to $(1, 1)$ and let σ_1, σ_2 be the restrictions of F to the indicated triangles, so that $\sigma_1 - \sigma_2$ is a 2-chain. (Figure taken from Hatcher, p.166.)



Abbreviating $F|_Y$ by (Y) for an oriented 1-simplex Y, we have

$$\begin{aligned}
\partial(\sigma_1+\sigma_2) &= (\overrightarrow{v_0v_3}) + (\overrightarrow{v_0v_1}) + (\overrightarrow{v_1v_3}) - (\overrightarrow{v_0v_3}) - (\overrightarrow{v_0v_2}) - (\overrightarrow{v_2v_3}) \\
&= (\overrightarrow{v_0v_1}) + (\overrightarrow{v_1v_3}) - (\overrightarrow{v_0v_2}) - (\overrightarrow{v_2v_3}) \\
&= [f_0] + [k_{x_0}] - [f_1] - [k_{x_0}] \\
&= [f_0] - [f_1].
\end{aligned}$$

We have just shown that the 1-simplices f_0, f_1 differ by a boundary and thus are equal in $H_1(X)$. So ψ is a well-defined function.

Part 2: Show that ψ is a group homomorphism.

In other words, we want to show that

$$\psi(f\cdot g) = \psi(f) + \psi(g) \qquad \text{or equivalently} \qquad \psi(f\cdot g) - \psi(f) - \psi(g) = 0.$$

Hatcher has a little trick for this. Define $\sigma : \Delta^2 \to X$ by

$$\Delta^2 = [v_0, v_1, v_2] \xrightarrow{\text{proj}} [v_0, v_2] \xrightarrow{f\cdot g} X$$

where "proj" denotes orthogonal projection. Then

$$\partial\sigma = \sigma|_{[v_0,v_1]} + \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} = [f] + [g] - [f\cdot g] \tag{17.1}$$

and since $\partial\sigma$ is a boundary, this is exactly what we want. Note that this relation in $C_1(X)$ holds for all paths f and g, not just loops.

Part 3: Surjectivity.

It is enough to show that any singular 1-cycle $\sigma : \Delta^1 \to X$ is in the image of ψ. Being a 1-cycle means that $\sigma(0) = \sigma(1) = x_1$ for some point $x_1 \in X$.

Let h be a path from x_0 to x_1 — here we are using the hypothesis that X is path-connected. Then $[h \cdot \sigma \cdot \bar{h}] \in \pi_1(X, x_0)$. By (17.1), both

$$[h] + [\sigma] - [h \cdot \sigma] \qquad \text{and} \qquad [h \cdot \sigma] + [\bar{h}] - [h \cdot \sigma \cdot \bar{h}]$$

are boundaries. Therefore, so is their sum, which boils down to $[\sigma] - [h \cdot \sigma \cdot \bar{h}]$. It follows that

$$[\sigma] = \psi[h \cdot \sigma \cdot \bar{h}] \in \operatorname{im} \psi.$$

Part 4: Figure out $\ker \psi$.

Since $H_1(X)$ is an abelian group, it is immediate that $\ker \psi \supseteq \operatorname{Comm}(\pi_1(X))$. So we need to show that the commutator subgroup is in fact the entire kernel. This is the hardest part.

Suppose that $[f] \in \ker \psi$, so that $[f]$ is the boundary of some 2-chain:

$$[f] = \partial \sum_{i=1}^{n} \varepsilon_i \sigma_i$$

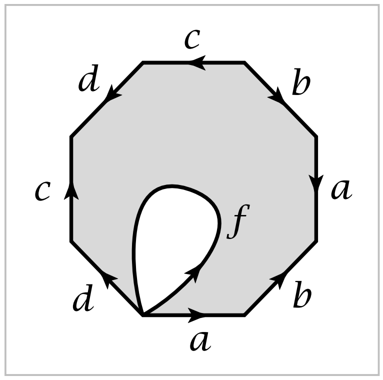
where each ε_i is ± 1 and $(\sigma_1, \ldots, \sigma_n)$ are singular 2-simplices (with possible repeats). By the usual boundary formula, we can write

$$[f] = \sum_{i=1}^{n} \varepsilon_i \partial \sigma_i = \sum_{i=1}^{n} (\varepsilon_i \tau_{i0} - \tau_{i1} + \tau_{i2}).$$

The stuff on the right has to boil down to $[f]$. The only way this can happen (remember, we are working in a free module!) is if one of the τ_{ij}'s equals $[f]$, and all the other ones can be matched into pairs that occur with opposite signs.

Now, build a 2-dimensional Δ-complex Z by starting with n 2-simplices $\sigma_1, \ldots, \sigma_n$ and using this matching to glue them together. I.e., if τ_{ij} and $\tau_{k\ell}$ are matched, then identify the j^{th} edge in σ_i with the ℓ^{th} edge in σ_k. We might as well assume that Z is connected (otherwise the expression for $[f]$ could be simplified). Note that Z is an orientable surface with boundary ∂Z consisting of a single loop. The singular simplices σ_i fit together to define a continuous function $\sigma : Z \to X$ that restricts to f on ∂Z.

Attaching a 2-disk along σ would produce an orientable surface. I am now going to appeal to a theorem we haven't proved, namely that **every** orientable surface is either a 2-sphere or one of the surfaces M_g. If it is a 2-sphere then $[f]$ itself is a boundary (just regard f as the map from the equator and the equator as a triangle) and we're done. Otherwise, the standard construction of M_g means that Z can be given the following cell structure (picture taken from Hatcher, p.168):



Here Z is the grey region. The picture shows the case $g = 2$; in general we know that the polygon has $4g$ sides labeled in cyclic order

$$a_1,\ a_2,\ \overline{a_1},\ \overline{a_2},\ \ldots,\ a_{2g-1},\ a_{2g},\ \overline{a_{2g-1}},\ \overline{a_{2g}}.$$

But the path around the loop labeled f is path-homotopic to $[a_1, a_2] \cdots [a_{2g-1}, a_{2g}]$ in Z, and pushing forward to X (i.e., applying σ_*) gives a path-homotopy of $[f]$ with something in $\mathrm{Comm}(\pi_1(X))$. $\square$

One can also finish the proof without appealing to the classification of surfaces; see p.167 of Hatcher.

17.1. **A Taste of Higher Homotopy Theory.** The fundamental group $\pi_1(X, x_0)$ measures the different ways to map a circle into X. More generally, the higher homotopy groups $\pi_k(X, x_0)$ measure the different ways to map a k-sphere into X.

Let $I = [0, 1]$, so that I^n is the standard n-dimensional cube. Its boundary ∂I^n is homeomorphic to $\mathbb{S}^{n-1}$. The $\boldsymbol{n^{th}}$ **homotopy group** $\pi_n(X, x_0)$ is defined to be the set of homotopy classes of maps

$$(I^n, \partial I^n) \to (X, x_0)$$

i.e., maps $f : I^n \to X$ with $f(s) = x_0$ for every $s \in \partial I^n$. Here "homotopy" means "homotopy fixing the boundary." I.e., a homotopy $f_0 \simeq f_1$ comes from a map

$$F : I^n \times I \to X, \qquad f_t(s) = f(s, t)$$

with $F(s,t) = x_0$ for $(s,t) \in \partial I^n \times I$.

For $f, g \in \pi_n(X, x_0)$, we can put two copies of I^n side by side and map their union to X by hitting the left-hand one with f and the right-hand one with g. This turns out to be a well-defined group operation. In fact it is abelian for all $n \geq 2$.

The bad news is that π_n is very hard to calculate. While there are some exact sequences around, there is no Mayer-Vietoris sequence or Van Kampen theorem.

One interesting fact is the **Hurewicz Theorem** (hoo-RAY-vitch). A space (X, x_0) is ***n*-connected** if $\pi_i(X, x_0) = 0$ for all $i \leq n$. Thus 0-connected means connected and 1-connected means simply-connected. The Hurewicz Theorem states that if a space is $(n-1)$-connected and $n \geq 2$, then $\pi_k(X) \cong H_k(X)$ for $1 \leq k \leq n$

18. **Fri 5/2: Cohomology**

The short story: To define cohomology, you define reverse all the arrows.

Let's take that slower. Let X be a space. Recall that we started our study of homology by defining a singular n-simplex to be a function $\Delta^n \to X$, and defining a singular chain to be a $\mathbb{Z}$-linear combination of singular simplices.

Definition: A **singular n-cochain** is an integer-valued linear function on singular chains. That is, it is an element of the dual group

$$C^n(X) = C_n(X)^* = \mathrm{Hom}_{\mathbb{Z}}(C_n(X), \mathbb{Z}).$$

Duality is a **contravariant functor**. That is, a group homomorphism $\phi : A \to B$ gives rise to a map $B^* \to A^*$, because if $f : B \to \mathbb{Z}$ is a homomorphism then so is $f \circ \phi : A \to \mathbb{Z}$.

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "f\circ\phi"'] & B \arrow[d, "f"] \\ & \mathbb{Z} \end{tikzcd}$$

In this context, the map $B^* \to A^*$ sending $f \mapsto f \circ \phi$ is denoted by ϕ^*. If we represent linear functions by matrices, then the matrix representing ϕ^* is the *transpose* of the matrix representing $\phi : A \to B$.

Note that taking duals is really functorial, in the sense that $\mathbb{1}^* = \mathbb{1}$ and that $(\phi\psi)^* = \psi^*\phi^*$. Note also that a finitely generated free abelian group (or vector space) is isomorphic to its dual: if $\{e_1, \ldots, e_n\}$ is a basis for A, then the linear functions $e_1^*, \ldots, e_n^*$ defined by $e_i^*(e_j) = \delta_{ij}$ (Kronecker delta) form a basis for A^*. However, *this isomorphism is not canonical*, in that changing the basis gives a different isomorphism.

It follows that the singular chain complex $C_\bullet(X)$ gives rise to the **singular cochain complex**

$$C^\bullet(X) : \qquad \cdots \to C^{n-1}(X) \xrightarrow{\partial_n^*} C^n(X) \xrightarrow{\partial_{n+1}^*} C^{n+1}(X) \to \cdots$$

Functoriality implies that this is an algebraic chain complex, because $\partial_{n+1}^*\partial_n^* = (\partial_n\partial_{n+1})^* = 0^* = 0$. Hatcher writes δ_n for ∂_n^*; I have seen this notation elsewhere but don't think it is universal.

The maps δ_n are the **coboundary operators** of X. Their kernels are **cocycles** and their images are **coboundaries**. The ***n^th^* singular cohomology group of *X*** is

$$H^n(X) \overset{\text{def}}{=} \ker \delta_{n+1} \ / \ \operatorname{im} \delta_n.$$

Like the homology groups of X, the cohomology groups are topological invariants. Much of the basic machinery for cohomology works the way you would expect:

- Induced homomorphisms (if $f : X \to Y$ is continuous then there are induced maps $f^* : H^n(Y) \to H^n(X)$)
- Reduced cohomology
- Homotopy invariance
- Simplicial and cellular cohomology
- Cohomology with coefficients in an arbitrary abelian group G (replace $\operatorname{Hom}(C_n, \mathbb{Z})$ with $\operatorname{Hom}(C_n, G)$)
- Excision and the long exact sequence for good pairs
- Mayer-Vietoris sequences

What do cohomology groups look like? A lot like homology groups, it turns out. Say X can be given a cell structure with finitely many cells, say f_n, in each dimension n, so that each ∂_n can be represented by a $f_{n-1} \times f_n$ matrix D_n and δ_n is represented by the transpose D_n^T. If we look at cohomology with coefficients in a field $\Bbbk$, then each cohomology group is a vector space over $\Bbbk$, hence determined by its dimension, and

$$\begin{aligned}
\dim H^n(X) &= \dim \ker \delta_{n+1} - \dim \operatorname{im} \delta_n \\
&= (f_n - \operatorname{rank} D_n^T) - \operatorname{rank} D_{n+1}^T \\
&= (f_n - \operatorname{rank} D^{n+1}) - \operatorname{rank} D_n \\
&= \dim \ker \partial_n - \dim \operatorname{im} \partial_n = \dim H_n(X).
\end{aligned}$$

On the other hand, funnier things can happen over a coefficient group that is not a field.

Example: The real projective plane $\mathbb{R}P^2$, with the standard cell structure, has $C_2 \cong C_1 \cong C_0 \cong \mathbb{Z}$ and cellular chain complex

$$C_\bullet(\mathbb{R}P^2) : \qquad 0 \to C_2 \xrightarrow{[2]} C_1 \xrightarrow{[0]} C_0 \to 0$$

and homology groups

$$H_2(\mathbb{R}P^2) = 0, \quad H_1(\mathbb{R}P^2) = \mathbb{Z}_2, \quad H_0(\mathbb{R}P^2) = \mathbb{Z}.$$

To obtain the cochain complex, just reverse all the arrows (and transpose the 1×1 matrices):

$$C^\bullet(\mathbb{R}P^2): \qquad 0 \leftarrow C^2 \xleftarrow{[2]} C^1 \xleftarrow{[0]} C^0 \leftarrow 0.$$

Again, the cochain groups are all copies of $\mathbb{Z}$. The cohomology groups are

$$\begin{aligned} H^2(\mathbb{R}P^2) &= \ker[0]/\operatorname{im}[2] = \mathbb{Z}_2 && \text{(compare } H_2(\mathbb{R}P^2) = 0\text{)}, \\ H^1(\mathbb{R}P^2) &= \ker[2]/\operatorname{im}[0] = 0 && \text{(compare } H_1(\mathbb{R}P^2) = \mathbb{Z}_2\text{)}, \\ H^0(\mathbb{R}P^2) &= \ker[0]/\operatorname{im}[0] = \mathbb{Z} && \text{(compare } H_0(\mathbb{R}P^2) = 0\text{)}. \end{aligned}$$

In general, the relation between homology and cohomology groups is given by the **Universal Coefficient Theorem**, which states the following (Hatcher, p.195):

Theorem 18.1. *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex* $\operatorname{Hom}(C,G)$ *are determined by split short exact sequences*

$$0 \to \operatorname{Ext}(H_{n-1}(C), G) \to H^n(C) \to \operatorname{Hom}(H_n(C), G) \to 0.$$

Note this is a purely algebraic statement. What Ext means is a long story, but in the case $G = \mathbb{Z}$ it turns out that $\operatorname{Ext}(A,\mathbb{Z})$ is isomorphic to the torsion summand $\mathbf{T}(A)$ of A, while $\operatorname{Hom}(A,\mathbb{Z})$ is isomorphic to the free part of A (since any homomorphism $A \to \mathbb{Z}$ must kill every torsion element). Therefore, the Universal Coefficient Theorem has the following special case:

Corollary: For any topological space X, one has If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $\operatorname{Hom}(C,G)$ are determined by split short exact sequences

$$\boxed{H^n(X) \cong (H_n(X)/\mathbf{T}(H_n(X))) \oplus \mathbf{T}(H_{n-1}(X)).}$$

19. Mon 5/4: Why Bother With Cohomology?

19.1. **Cup product.** If cohomology gives no new information about a space, then why study it? Actually, cohomology has additional structure that homology does not. For any commutative ring R, the cohomology groups $H^n(X;R)$ can be glued together into a graded ring

$$H^*(X;R) = \bigoplus_{n\geq 0} H^n(X;R)$$

which *does* contain additional topological information. It is possible for spaces to have isomorphic homology (hence cohomology) groups in all dimensions, yet have non isomorphic cohomology rings. Also, for many spaces X, the product can be interpreted geometrically — for example, if X is a suitably nice (e.g., smooth) algebraic variety, then every subvariety gives rise to a cohomology class, and intersection of subvarieties corresponds to multiplication of their cohomology classes, so that one can obtain information about intersections by doing computations in the cohomology ring.

The product on $H^*(X;R)$ is called **cup product**, and it is defined as follows. Let $\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$. Given a singular simplex $\sigma : \Delta^{k+\ell} = [v_0,\dots,v_{k+\ell}] \to X$, we define

$$\boxed{(\phi \smile \psi)(\sigma) = \phi\left(\sigma|_{[v_0,\dots,v_k]}\right) \cdot \psi\left(\sigma|_{[v_k,\dots,v_{k+\ell}]}\right)}$$

where the dot denotes multiplication in R.

(LaTeX note: Despite the name, use `\smile` for cup product, not `\cup` which gives the union symbol $\cup$.)

This looks like a silly operation — what is special about the k^{th} vertex that it gets repeated? What about all the other ways of partitioning the vertex set of $\Delta^{k+\ell}$ into a k-set and and an ℓ-set that overlap in one vertex?

Fortunately, a miracle occurs. Cup product plays well with the coboundary operator δ: one can prove that

$$\delta(\phi \smile \psi) \;=\; \delta\phi \smile \psi \;+\; (-1)^k \phi \smile \delta\psi \tag{19.1}$$

(Hatcher, Lemma 3.6, p.206). In particular, it is easy to check that

- If ϕ and ψ are cocycles ($\delta\phi = -\delta\psi = 0$) then so is $\phi \smile \psi$.
- If ϕ and ψ are coboundaries ($\phi = \delta\alpha$, $\psi = \delta\beta$) then so is $\phi \smile \psi$.

This means that cup product is well-defined on cohomology classes, and extending it R-linearly makes the group

$$H^*(X;R) = \bigoplus_{n\geq 0} H^n(X;R)$$

into a ring, the **cohomology ring of $\boldsymbol{X}$ with coefficients in $\boldsymbol{R}$**. The ring is not quite commutative, but rather *skew-commutative* or *graded-commutative* — if $\phi \in H^k(X;R)$ and $\psi \in H^\ell(X;R)$ are homogeneous elements, then (19.1) can be used to show that

$$\phi \smile \psi = (-1)^{k\ell}\psi \smile \phi.$$

[Thm. 3.11, p.210 of Hatcher]. On the other hand, if all the odd cohomology groups of X vanish — for instance, if X has a cell complex structure with no cells of odd dimension — then in fact $H^*(X;R)$ is a commutative ring.

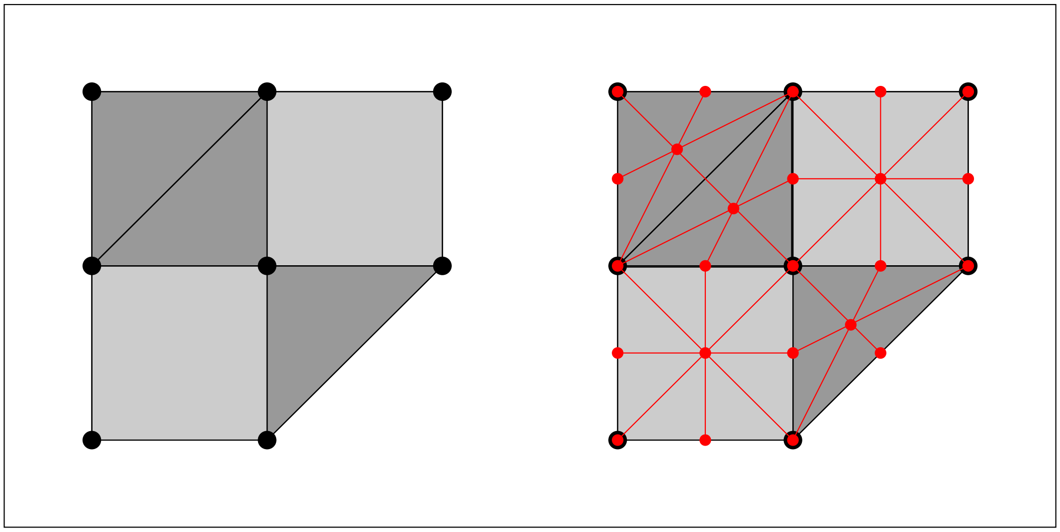
What is the interpretation of this operation? Note that the k- and ℓ-dimensional subsimplices of $\Delta^{k+\ell}$ that show up in the cup product formula intersect as little as possible — namely, in a single vertex. In general, one should think of the cup product $\alpha \smile \beta$ as the cohomology class of the smallest possible intersection of two subspaces cohomologous to α and β.

19.2. **Poincaré duality.** Let X be a cell complex. Let $\mathrm{sd}(X)$ be the abstract simplicial complex whose vertices v_α correspond to the cells σ_α of X, and whose simplices are chains in X. That is,

$$\mathrm{sd}(X) = \{v_1 v_2 \cdots v_k \mid \sigma_1, \ldots, \sigma_k \text{ distinct },\ \overline{\sigma_1} \supset \cdots \supset \overline{\sigma_k}\}.$$

Combinatorially, if $P(X)$ is the poset of cells of X, ordered by containment of closures, then $\mathrm{sd}(X)$ is the order complex of that poset. It turns out that $\mathrm{sd}(X)$ (or, if you perform its geometric realization) is in fact homeomorphic to X.

Example: Let X be the union of a solid square and a solid hexagon, identified along an edge. Here are X and $\mathrm{sd}(X)$.

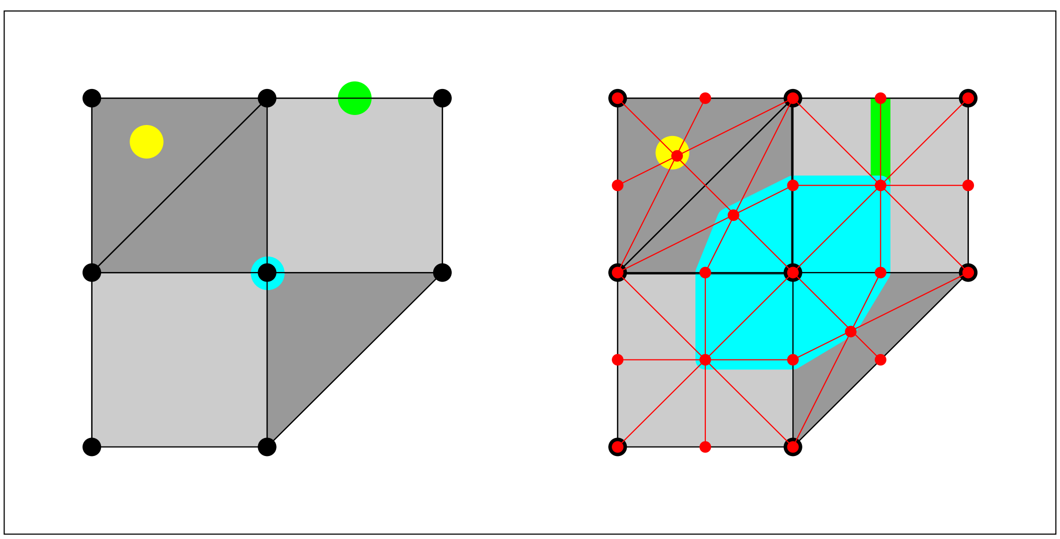


For a simplex $\tau \in \mathrm{sd}(X)$, let $\min(\tau)$ denote the smallest vertex of τ (i.e., the vertex corresponding to the smallest cell of X).

Definition: The **dual block** $D(\sigma)$ of a cell $\sigma \in X$ is the union of all simplices $\tau \in \mathrm{sd}(X)$ such that $\min(\tau) = \sigma$. The **dual block complex** of X is

$$D(X) = \bigcup_{\sigma \in X} D(\sigma).$$

Example: With X as above, here are dual blocks of 0-, 1- and 2-cells:



Geometrically, the dual block $D(\sigma)$ ca be thought of as the "orthogonal complement" of σ — i.e., the set of all directions in which one can leave σ.

Proposition: Let X be a finite regular cell complex that is pure of dimension n. (Recall that *regular* means that the attaching maps $\phi_\alpha : \partial e^n\alpha \to X_{n-1}$ are homeomorphisms for all cells e^n_α. In particular, the matrix entries occurring in the cellular chain complex are all in $\{0, 1, -1\}$.) Then:

(1) The dual blocks are disjoint and their union is $\mathrm{sd}(X)$.
(2) $\overline{D(\sigma)}$ is a $(n-k)$-dimensional subcomplex of $\mathrm{sd}(X)$.
(3) $\overline{D(\sigma)} \supset \overline{D(\tau)}$ if and only if $\overline{\sigma} \subset \overline{e_\tau}$.

The dual blocks don't have to be cells. (E.g., in the example above, the edge marked in green has a dual block that is homeomorphic to a half-open interval.) However:

(4) If X is a manifold then the dual blocks are in fact cells, so that the dual block complex $D(X)$ is a cell complex.

In this case we have two different cell structures on X, and condition #2 says that $E_k(D(X)) \cong E_{n-k}(X)$ for all k. So we have two ways of computing (co)homology on X; how do they compare?

(5) If X is orientable then condition #3 says that the coboundary operators of $D(X)$ are just the boundary operators of X (this does take some proof).

This proves the following theorem:

Theorem 19.1 (Poincaré Duality). *If X is a compact orientable n-dimensional manifold, then $H^k(X) \cong H_{n-k}(X)$ for all k.*

Corollary 19.2. *For any field $\mathbb{F}$ of characteristic 0, we have $H_k(X) \cong H_{n-k}(X)$. I.e., the Poincaré polynomial*

$$\mathrm{Poin}(X, q) = \sum_{k=0}^{n} \dim H_k(X; \mathbb{F}) q^k$$

is a palindrome.

20. Wed 5/6: (Co)homology of Graph Picture Spaces

I can't resist finishing up the course with a theorem I proved ten years ago. The details are in my paper "On the topology of graph picture spaces," `arxiv.org/abs/math.CO/0307405` (hyperlink).

20.1. A generalized Mayer-Vietoris sequence.

Suppose we have a space X and maps $f : A \to X$ and $g : B \to X$ such that $f(A) \cup g(B) = X$. For short, write $A' = f(A)$ and $B' = g(B)$. We want to relate the homology of X to that of A and B (rather than A' and B'). The question is what will take the place of the intersection.

Define $h : A \cupdot B \to X$ by

$$h(p) = \begin{cases} f(p) & \text{if } p \in A, \\ g(p) & \text{if } p \in B. \end{cases}$$

Now consider the induced map on chains:

$$h_\sharp : C_n(A \cupdot B) \to C_n(X).$$

Of course, $C_n(A \cupdot B) = C_n(A) \oplus C_n(B)$. Meanwhile, the image of $h_\sharp$ is $C_n(A') + C_n(B')$. Recall that since $A' \cup B' = X$, the complex $C_\bullet(A + B)$ computes the homology of X by the Subdivision Lemma (Lemma 9.5). Therefore, we can regard $h_\sharp$ as a map

$$h_\sharp : C_n(A) \oplus C_n(B) \to C_n(X), \qquad (\alpha, \beta) \mapsto f_\sharp(\alpha) - g_\sharp(\beta)$$

that induces a map on homology

$$h_* : H_n(A) \oplus H_n(B) \to H_n(X).$$

(Interlude: Why is there a minus sign in the formula for $h_\sharp$? So that there isn't one in the kernel, that's why. We're free to make the sign either $+$ or $-$ by changing bases on $C_n(B)$.)

What is the kernel of $h_\sharp$? The key observation is that the inclusion $\ker(h_\sharp) \hookrightarrow C_n(A) \oplus C_n(B)$ is the homomorphism induced by an honest-to-goodness map of topological spaces. We want to find some space F, together with maps

$i : F \to A$ and $j : F \to B$, such that $f_\sharp(i_\sharp(\gamma)) = g_\sharp(j_\sharp(\gamma))$ for all $\gamma \in C_n(F)$. The answer to our prayers is the **fiber product** $F = A \underset{X}{\times} B$, which is defined by

$$A \underset{X}{\times} B = \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

The maps i, j are just (restrictions of) Cartesian projection to the first and second factors in $A \times B$. We have a commutative diagram of spaces

$$\begin{tikzcd} & A \underset{X}{\times} B \arrow[dl, "i"'] \arrow[dr, "j"] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

which induces a commutative diagram of chain groups

$$\begin{tikzcd} & C_n(A \underset{X}{\times} B) \arrow[dl, "i_\sharp"'] \arrow[dr, "j_\sharp"] & \\ C_n(A) \arrow[dr, "f_\sharp"'] & & C_n(B) \arrow[dl, "g_\sharp"] \\ & C_n(X) & \end{tikzcd}$$

from which we can build a short exact sequence

$$0 \to C_n(A \underset{X}{\times} B) \xrightarrow{i_\sharp \oplus j_\sharp} C_n(A) \oplus C_n(B) \xrightarrow{h_\sharp} C_n(X) \to 0$$

which gives rise to a short exact sequence of chain complexes (replace C_n with $C_\bullet$) which in turn yields a long exact sequence of homology groups:

$$\boxed{\cdots \to H_n(A \underset{X}{\times} B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \underset{X}{\times} B) \to \cdots}$$

which we can regard as a generalization of the Mayer-Vietoris sequence. (The "vanilla" Mayer-Vietoris sequence arises when the maps f and g are simply inclusion of subspaces.)

20.2. **Picture spaces of graphs.** Let G be a graph (i.e., a 1-dimensional cell complex) with vertex set V and edge set E. (Note for combinatorialists: loops and parallel edges are OK.) A **picture** of G consists of a set of points $\{p_v : \ v \in V\}$ and lines $\{\ell_e : \ e \in E\}$ such that

$$v \in e \quad \Longrightarrow \quad p_v \in \ell_e. \tag{20.1}$$

The points and lines are taken to lie in some fixed ambient space, typically $\mathbb{C}P^d$. (Thus "line" means "complex line," i.e., a copy of $\mathbb{C}P^1 \cong \mathbb{S}^2$.) Note that points and lines in $\mathbb{C}P^d$ are identical with 1- and 2-dimensional vector subspaces of $\mathbb{C}^{d+1}$, respectively, so that a picture can be viewed as a point in the product of Grassmannians

$$\operatorname{Gr}(G) \stackrel{\text{def}}{=} \prod_{v \in V} \operatorname{Gr}(1, \mathbb{C}^{d+1}) \ \times \ \prod_{e \in E} \operatorname{Gr}(2, \mathbb{C}^{d+1}).$$

The set of pictures is called the **picture space** of G, denoted $X(G)$. (Note for algebraists and algebraic geometers: The equations (20.1) defining a picture can be written in terms of Plücker coordinates, so $X(G)$ is in fact an algebraic subset of $\operatorname{Gr}(G)$.)

Question: How does the combinatorics of G control the topology of $X(G)$?

Quite a bit, as it turns out. The key is going to be to construct maps between picture spaces from which we can extract a modified Mayer-Vietoris sequence. First let's look at some examples.

1. If G has connected components $G_1, \dots, G_n$, then $X(G) = \prod_{i=1}^n X(G_i)$.

2. In particular, if G has no edges then $X(G)$ is the product of $\mathbb{C}P^d$'s, one for each vertex.

3. Suppose $G = \bullet\!\!\bigcirc$ consists of one vertex and a loop. Then

$$X(\bullet\!\!\bigcirc) = \{(p, \ell) \in \operatorname{Gr}(1, \mathbb{C}^{d+1}) \times \operatorname{Gr}(2, \mathbb{C}^{d+1}) \mid p \subset \ell\}.$$

This is an instance of a *partial flag manifold*; if $d = 2$ it is the complete flag manifold. It comes with a natural map

$$\begin{array}{ccc} X(\bullet\!\!\bigcirc) & \xrightarrow{\pi} & X(\bullet) \\ (p, \ell) & \mapsto & p \end{array}$$

that just forgets ℓ. Note that $\pi^{-1}(p)$ is the space of lines through the point $p \in \mathbb{C}P^d$, so it is naturally homeomorphic to $\mathbb{C}P^{d-1}$.

That is, $\boldsymbol{X(\bullet\!\!\bigcirc)}$ **is a** $\boldsymbol{\mathbb{C}P^{d-1}}$**-bundle over** $\boldsymbol{X(\bullet) = \mathbb{C}P^d}$**.** Bundles are a generalization of products; for example, the cylinder and Möbius band are both I-bundles over $\mathbb{S}^1$. A bundle that is actually a product is called a **trivial bundle** in this context. (By the way, a covering space is more or less a bundle whose fibers are discrete.)

4. Suppose $G = \bullet\!\!-\!\!\bullet$ consists of two vertices joined by an edge. There is again a projection map

$$\begin{array}{ccc} X(\bullet\!\!-\!\!\bullet) & \xrightarrow{\pi} & X(\bullet \quad \bullet) = \mathbb{C}P^d \times \mathbb{C}P^d \\ (p_1, p_2, \ell) & \mapsto & (p_1, p_2) \end{array}$$

Unlike the previous case, the fibers are not all the same. If $p_1 \neq p_2$ then the fiber consists of a single point, since there is only one line through two distinct points. However, if $p_1 = p_2$ (i.e., (p_1, p_2) lies on the **diagonal** Δ of $\mathbb{C}P^d \times \mathbb{C}P^d$) then the fiber is a copy of $\mathbb{C}P^{d-1}$ for the same reason as in the previous case.

Here's a way to think about this: $X(\bullet\!-\!\bullet)$ is the space of pairs of points approaching each other along a line. If the points are different then the line is the line connecting them; if the points have already merged then $X(\bullet\!-\!\bullet)$ remembers how they got there. In algebraic geometry lingo, $X(\bullet\!-\!\bullet)$ is **the blowup of** $\boldsymbol{\mathbb{C}P^d \times \mathbb{C}P^d}$ **along** $\boldsymbol{\Delta}$**.**

The diagonal Δ is called the **center** of the blowup, and the space $\{\pi^{-1}(p, p)\} \subset X(\bullet\!-\!\bullet)$ is called the **exceptional divisor** of the blowup (again, this is algebraic-geometry jargon). We can think of it as a copy of the space $X(\bullet\!\!\bigcirc)$.

Now let's make this more general. Let G be any graph and let $e \in E(G)$. Then there are maps

$$\begin{tikzcd} Y \arrow[r, "i"] & X(G) \arrow[d, "\pi"] \\ & X(G-e) \end{tikzcd}$$

where π is the map that forgets the edge e, and i is inclusion. We can complete the diagram to a square by adding $\pi(Y)$ in the bottom left. Note that $\pi(Y)$ is the center of the blowup, and that π makes Y into a $\mathbb{C}P^{d-1}$-bundle over $\pi(Y)$.

What are these spaces Y and $\pi(Y)$? They're also picture spaces. The center of the blowup can be regarded as $X(G-e)$ — merging the two endpoints of e and deleting e is exactly the operation of contraction. Meanwhile, Y is the picture space of $G-e$ with a loop added at the merged vertex; let's call this graph $G \cdot e$. We can redraw the diagram:

$$\begin{tikzcd} X(G \cdot e) \arrow[r, "i"] \arrow[d, "\pi"'] & X(G) \arrow[d, "\pi"] \\ X(G/e) \arrow[r, "i"'] & X(G-e) \end{tikzcd}$$

Lemma: This diagram is a fiber product square.

Well, sure. If I have a picture of G and a picture of G/e that map to the same picture of $G-e$, then in the picture of G, the two endpoints of e must be represented by the same point, which means that I have a picture of $G \cdot e$.

Therefore, we have a Mayer-Vietoris sequence and can calculate the homology of $X(G)$ inductively. For example, let's figure out the homology of $X(\bullet\!\!-\!\!\bullet)$. Here the diagram is

$$\begin{tikzcd} X(\bullet\!\!\bigcirc) \arrow[r] \arrow[d] & X(\bullet\!\!-\!\!\bullet) \arrow[d] \\ X(\bullet) \arrow[r] & X(\bullet \quad \bullet) \end{tikzcd} \quad = \quad \begin{tikzcd} X(\bullet\!\!\bigcirc) \arrow[r] \arrow[d, "\pi"'] & X(\bullet\!\!-\!\!\bullet) \arrow[d] \\ \mathbb{C}P^d \arrow[r] & \mathbb{C}P^d \times \mathbb{C}P^d \end{tikzcd}$$

where the map marked π makes $X(\bullet\!\!\bigcirc)$ into a $\mathbb{C}P^{d-1}$-bundle over $\mathbb{C}P^d$. The bundle has a cell structure with the same number of cells as a direct product

— and in this case everything in sight has only even cells, so the Mayer-Vietoris sequence breaks up into short exact sequences

$$0 \to H_n(\bullet\!\!\!\bigcirc) \to H_n(\bullet\!\!-\!\!\bullet) \oplus H_n(\bullet) \to H_n(\bullet\quad\bullet) \to 0$$

for n even; when n is odd everything is zero.

We already know that everything but $H_n(\bullet\!\!-\!\!\bullet)$ is torsion-free, and exactness of the sequence implies that $H_n(\bullet\!\!-\!\!\bullet)$ is torsion-free as well (since a torsion element has to be in the kernel of the second map but can't be in the image of the first map). So we might as well sum over all n and work with Poincaré polynomials. Since $\mathbb{C}P^k$ has a cell structure in each dimension $0, 2, \ldots, 2k$, its Poincaré polynomial $P(\mathbb{C}P^k)$ is $q^{2k} + q^{2k-2} + \cdots + q^2 + 1$. Abbreviating this polynomial by Q_k, the Mayer-Vietoris sequence gives

$$\begin{aligned}
P(\bullet\!\!-\!\!\bullet) &\stackrel{\text{def}}{=} \sum_n q^n \operatorname{rank} H_n(\bullet\!\!-\!\!\bullet) \\
&= P(\bullet\!\!\!\bigcirc) + P(\bullet\quad\bullet) - P(\bullet) \\
&= P(\mathbb{C}P^{d-1})P(\mathbb{C}P^d) + P(\mathbb{C}P^d)^2 - P(\mathbb{C}P^d) \\
&= Q_{d-1}Q_d + Q_d^2 - Q^d \\
&= (q^2+1)(q^{2k} + \cdots + q^2 + 1)(q^{2k-2} + \cdots + q^2 + 1).
\end{aligned}$$

Induction leads to the following theorem:

Theorem 20.1. *For every graph G, the homology groups $H_n(X(G))$ are free abelian for n even and zero for n odd. Moreover, the Poincaré polynomial $P(G)$ of $X(G)$ satisfies the following recurrence:*

$$P(G) = \begin{cases} Q_d^n & \text{if } G \text{ has } n \text{ vertices} \\ Q_{d-1}P(G-e) & \text{if } e \text{ is a loop,} \\ Q_2 Q_{d-1} P(G/e) & \text{if } e \text{ is a cut-edge,} \\ P(G-e) + (Q_{d-1}-1)P(G/e) & \text{if } e \text{ is an ordinary edge.} \end{cases}$$

A well-known and extremely important invariant of a graph is its **Tutte polynomial**. This is a two-variable polynomial $T(G) = T_G(x, y)$ which is defined by the deletion-contraction recurrence

$$T(G) = \begin{cases} 1 & \text{if } E(G) = \emptyset, \\ yT(G-e) \text{ if } e \text{ is a loop}, \\ xT(G/e) \text{ if } e \text{ is a cut-edge}, \\ T(G-e) + T(G/e) \text{ if } e \text{ is an ordinary edge}. \end{cases}$$

This is general enough to include all graph invariants satisfying a deletion-contraction recurrence, including the number of spanning trees, the number of acyclic orientations, and the chromatic polynomial, among others. (For more on this, take Math 824!) Comparing these two recurrences says that the homology groups of $X(G)$ are determined entirely by its Tutte polynomial.

It turns out that $X(G)$ is a manifold exactly when G is an **orchard** — every edge is either a loop or a cut-edge. In this case, it is possible to use the theory of line bundles and Chern classes to write down a presentation for the cohomology ring, and to do some intersection theory.