Contents

1. Mon 2/10: Paths, Path Homotopies, Fundamental Group

Definition: A **path** in X is a continuous function $f : I \to X$. If $f(0) = f(1) = p \in X$, we say that f is a **closed path** or **loop**, or a **closed path with basepoint p.**

1.1. Paths and Path Homotopies.

When we talk about a homotopy of paths, we want to require that the starting and ending points are independent of time.

Definition: Let X be a space and $p, q \in X$. A **path homotopy from p to q** is a family of paths $\{f_t : t \in I\}$ such that

(1) $f_t(0) = p$ and $f_t(1) = q$ for all $t \in I$.
(2) The function $F : I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous.

We might refer to F, rather than the family $\{f_t\}$, as the path homotopy — they contain equivalent data.

Remark 1.1. Path-homotopy is a stronger condition than simply a homotopy of functions $I \to X$. For example, if $X = S^1$, the function $F : I \times I \to X$ defined by $F(t,s) = f_t(s)e^{2\pi its}$ is continuous, and f_0 and f_1 are both closed paths (f_0 is constant and f_1 wraps once around the circle). But F is not a path homotopy, because $f_t(1)$ depends on t.

The condition that the endpoints stay fixed is very important. In fact, any two paths whose images lie in the same path-connected space are homotopic as maps, so that old notion of homotopy is not very useful for paths. Accordingly, whenever we are talking about paths, you can safely assume that "homotopy" means "path-homotopy". We adopt the notation $f \simeq g$ for path homotopy.

Path homotopy is an equivalence relation (the proof is straightforward). Accordingly we denote by $[f]$ the equivalence class of a path f up to path homotopy.

Lemma 1.2 (Reparameterization)**.** *Let $\phi : I \to I$ is any continuous function with $\phi(0) = 0$ and $\phi(1) = 1$. Then $[f \circ \phi] = [f]$ for any path $f \in \pi_1(X,p)$.*

Proof. The path-homotopy between the paths f and $f \circ \phi$ is

$$g_t(s) = f((1-t)s + t\phi(s))$$

which satisfies

$$\begin{aligned} g_0(s) &= f(s), & g_t(0) &= f(0) \\ g_t(1) &= f(1), & g_1(s) &= f(\phi(s)). \end{aligned}$$

□

1.2. **The Fundamental Group.** Let $f : I \to X$ be a path from p to q and let $g : I \to X$ be a path from q to r. The **concatenation** $f \cdot g$ is the path from p to r obtained by following first f and then g:

$$f \cdot g(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ g(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases} \tag{1.1}$$

This operation is well-defined on homotopy equivalence classes: if $f \simeq f'$ are p, q-paths and $g \simeq g'$ are q, r-paths, then $f \cdot g \simeq f' \cdot g'$ (this is straightforward to check). Therefore the symbol $[f] \cdot [g]$ is well-defined.

Theorem 1.3. *Let $p \in X$ and let $\pi_1(X, p)$ denote the set of all homotopy equivalence classes of closed paths with basepoint p. Then the operation of concatenation makes $\pi_1(X, p)$ into a group. The identity element is $[k_p]$, where $k_p(t) = p$ is the stationary path at p. The inverse is $[f]^{-1} = [\overline{f}]$, where $\overline{f}(t) = f(1-t)$.*

Proof of Thm. 1.3. Let $f \in \pi_1(X, p)$. Then $f \cdot k_p(t) = f(\phi(t))$ and $k_p \cdot f(t) = f(\psi(t))$, where

$$\phi(t) = \begin{cases} 2t \text{ if } 0 \le t \le \frac{1}{2}, \\ 1 \text{ if } \frac{1}{2} \le t \le 1, \end{cases} \quad \psi(t) = \begin{cases} 0 \text{ if } 0 \le t \le \frac{1}{2}, \\ 2t-1 \text{ if } \frac{1}{2} \le t \le 1. \end{cases}$$

By the reparameterization lemma, it follows that $[f] = [f \cdot k_p][k_p \cdot f]$.

Let $g(t) = f(1-t)$. Then $f \cdot g$ is given by

$$f \cdot g(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases} = \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ f(2-2t) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

This map is homotopic to k_p via the homotopy

$$h_s(t) = \begin{cases} f(\min(2t, s)) & \text{if } 0 \le t \le \frac{1}{2}, \\ f(\min(2-2t, s)) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

which has $h_1 = f \cdot g$ and $h_0 = k_p$.

Associativity of multiplication is another irritating reparameterization argument. By the way, concatenation is associative up to homotopy even for non-closed paths — this fact will be useful soon. $\square$

Definition: A space X is **simply-connected** if it is path-connected and $\pi_1(X, p) = 0$. (Does this depend on the choice of basepoint? Hold that thought.)

Example: Any convex subspace of $\mathbb{R}^n$ is simply-connected, for the following reason. Let γ be a path and define $G : I \times I \to \mathbb{R}^n$ by $G(t, s) = t\gamma(s)$. Then $G_1 = \gamma$ and G_0 is the constant path at the origin. By suitably modifying this argument, one can show that every star-shaped space is simply-connected — provided the basepoint is a star point. But wait! There's more!

Proposition 1.4 (Change of Basepoint)**.** *Let X be path-connected and $p, q \in X$. Then $\pi_1(X, p) \cong \pi_1(X, q)$.*

Proof. Let h be any p, q-path. Then the map $\beta_h : \pi_1(X, q) \to \pi_1(X, p)$ defined by

$$[f] \mapsto [h \cdot f \cdot \overline{h}]$$

is a homomorphism because

$$[f \cdot g] \mapsto [h] \cdot [f] \cdot [g] \cdot [\overline{h}] = \big([h] \cdot [f] \cdot [\overline{h}]\big) \cdot \big([h] \cdot [g] \cdot [\overline{h}]\big) = [f] \cdot [g]$$

(thanks to associativity). It is invertible (swap p, q and $h, \overline{h}$), hence an isomorphism. $\square$

For this reason, we can talk about "the fundamental group $\pi_1(X)$"; this is the isomorphism class of any (hence every) group $\pi_1(X, p)$ for $p \in X$. (It is understood that X must be path-connected for this to make sense.)

How do you calculate $\pi_1(X)$?

It's not easy. We could prove at this point that the fundamental group of a convex subset of $\mathbb{R}^n$ is trivial. In fact, every contractible space is simply-connected; this is remarkably tricky to prove and will require a bit more machinery. How would one prove that some space is *not* simply-connected? The most important case is the circle S^1, which we will tackle explicitly.

2. Wednesday 2/12: Simply-Connected Spaces

Review from last time:

Let X be a path-connected space.

p, q-path: map $f : I \to X$ with $f(0) = p$, $f(1) = q$

p, q-path homotopy: map $F : I \times I \to X$ with

$$f_t(s) = F(t, s), \qquad f_t(0) = p, \quad f_t(1) = q \quad \forall t$$

Notation: $f_0 \simeq f_1$.

$[f]$ = path-homotopy equivalence class of f ("path class" for short)

Reparameterization lemma: if $\phi : I \to I$ is continuous with $\phi(0) = 0$, $\phi(1) = 1$, then $f \circ \phi \simeq f$.

Concatenation/composition: if f is a p, q-path and g is a q, r-path, let

$$f \cdot g(s) = \begin{cases} f(2s) & \text{if } s \in [0, \frac{1}{2}], \\ g(2s - 1) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

Then $f \cdot g$ is a q, r-path.

Concatenation is well-defined and associative on path-homotopy classes.

Fundamental group $\pi_1(X, p)$: set of all p, p-path classes.

- Group operation: Concatenation.
- Identity: $[k_p]$ where $k_p(s) = p$
- Inverse: $[f^{-1}] = [\bar{f}]$ where $\bar{f}(s) = f(1 - s)$

(Explain notation π_1. In general $\pi_k(X)$ = group of homotopy equivalence classes of basepointed maps $S^k \to X$.)

Change of basepoint lemma: If h is a p, q-path, then

$$\beta_h : \pi_1(X, q) \to \pi_1(X, p), \qquad [f] \mapsto [h \cdot f \cdot \bar{h}]$$

is an isomorphism.

(So $\pi_1(X)$ = isomorphism class of $\pi_1(X, p)$ for any $p \in X$.)

X is **simply-connected** if $\pi_1(X) = 0$. (E.g., convex, star-shaped.)

Proposition 2.1. *Let X be a path-connected space. Then X is simply-connected if, for every two p, q-paths f, g, we have $f \simeq g$.*

Proof. Suppose X is simply-connected. Then in particular $f \cdot \bar{g} \simeq k_q$, so $f \simeq f \cdot \bar{g} \cdot g \simeq k_q \cdot g = g$.

Suppose the other condition holds. Then there is only one homotopy class of p, p-paths. □

Fact: X contractible $\implies$ X simply-connected. But this is surprisingly hard to prove — we will need some more machinery. Contracting a space to a point will contract any closed loop to a point; the problem is that this last contraction need not be a path homotopy. (We could prove that any space that can be deformation-retracted to a point is simply-connected — but that is a stronger assumption than contractibility; see Exercise 6(b) on p.18.)

Proposition 1.14: For $n \geq 2$, S^n is simply-connected. (Note that it is not contractible — at least, we don't think so.)

Proof. Let $p \in S^n$ and $f \in \pi_1(S^n, p)$. Suppose f is not onto, say $f(I) \subset S^n \setminus \{q\}$ for some $q \in S^n$. Since $S^n \setminus \{q\} \cong \mathbb{R}^n$ is simply-connected, we can path-homotope f to k_p.

But what if f is a space-filling curve? In that case we need to homotope f to a non-space-filling curve. The key to doing this is compactness

Let $q \neq p$ and let B be a small neighborhood around q. Then $f^{-1}(B)$ is open in I, hence is the union of (possibly infinitely many) open intervals. Meanwhile, the set $f^{-1}(x)$ is closed in I, hence **compact**. Therefore finitely many of those intervals $J_1, \ldots, J_n \subset I$ cover $f^{-1}(x)$. We may as well assume these are disjoint (if two overlap, merge them). Then if, say, $J_1 = [a_1, b_1]$, we have $f(a_1), f(b_1) \in \partial B$; in particular $f(a_1, f(b_1) \neq q$. Homotope each section $f|_{J_i}$ to an arc on ∂B. maintaining continuity and staying in the same homotopy class (since $\partial B \cong D^2$ is convex, hence simply-connected). We wind up with a non-surjective closed path $f' \simeq f$. □

3. Friday 2/14: Fundamental Group of S^1

Theorem: For any $p \in S^1$, we have $\pi_1(S^1, p) \cong \mathbb{Z}$.

Before we go any further, let's describe the points of S^1. Identify S^1 with the unit circle in $\mathbb{C}$, and define

$$E(s) = e^{2\pi i s} \qquad \text{for } s \in \mathbb{R}.$$

Then $E : \mathbb{R} \twoheadrightarrow S^1$, and the preimage of any point in S^1 is a coset of $\mathbb{Z}$ in $\mathbb{R}$.

Lemma 3.1. *Every path $f : I \to S^1$ has a* ***lift*** *to $\mathbb{R}$, i.e., a function $\tilde{f} : I \to \mathbb{R}$ such that the diagram*

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "E"] \\ I \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S_1 \end{tikzcd}$$

is commutative.

In other words, every way f of wrapping I around a circle can be described by first mapping I to a line (that's $\tilde{f}$), then wrapping the line around the circle (that's E).

Proof. In order to lift f, we need to able to invert E. So, suppose that $J \subseteq I$ such that $f|_J$ is not surjective. Then there is a branch of the complex logarithm whose domain includes J, and so the function

$$\tilde{f}(s) \mapsto \frac{1}{2\pi i} \log(f(s))$$

is a lift of $f|_J$.

If $f : I \to S^1$ is surjective, the idea is to cover I with intervals on which f is not surjective, lift f on each one, then splice all the lifts together.

Specifically, let

$$L^+ = S^1 \setminus \{1\}, \qquad L^- = S^1 \setminus \{-1\}$$

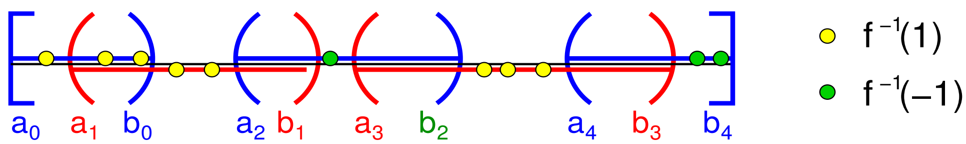
(L stands for "lacks"). I claim that we can cover the domain of f (namely I) with finitely many pieces:

$$I = \underbrace{[a_0 = 0, b_0)}_{I_0} \cup \underbrace{(a_1, b_1)}_{I_1} \cup \cdots \cup \underbrace{(a_{n-1}, b_{n-1})}_{I_{n-1}} \cup \underbrace{(a_n, b_n = 1]}_{I_n}$$

with

$$0 = a_0 < a_1 < b_0 < a_2 < b_1 < a_3 < \cdots < a_n < b_{n-1} < b_n = 1$$

such that each $f(I_j)$ lies inside either L^+ or L^-.



Given such a cover, we can construct an appropriate lift of f on each piece in succession, then splice all the lifts together. Specifically:

Let

$$\log_0 : f(I_0) \to \mathbb{R}$$

be a branch of the complex logarithm. Such a thing exists because $f(I_0) \subsetneq S^1$. We can therefore lift $f|_{I_0}$ to a function

$$\tilde{f}_0 = \log_0 \circ f|_{I_0} : I_0 \to \mathbb{R}.$$

Now define

$$\log_1 : f(I_1) \to \mathbb{R}$$

as the unique branch of the complex log extending $\log_0|_{f(I_0 \cap I_1)}$. Such a branch exists and is unique because $f(I_0 \cap I_1)$ is nonempty and $f(I_1)$ is a connected proper subset of S^1. We can therefore lift $f|_{I_1}$ to a function

$$\tilde{f}_1 = \log_1 \circ f|_{I_1} : I_1 \to \mathbb{R}$$

that agrees with $\tilde{f}_0$ on $I_0 \cap I_1$.

Pasting $\tilde{f}_0$ and $\tilde{f}_1$ together gives a lift $\tilde{f}_{01}$ of $f|_{I_0 \cup I_1}$.

Now define

$$\log_2 : f(I_2) \to \mathbb{R}$$

as the unique branch of the complex log extending $\log_1|_{f(I_1 \cap I_2)}$, and lift $f|_{I_2}$

$$\tilde{f}_2 = \log_2 \circ f|_{I_2} : I_2 \to \mathbb{R}.$$

Pasting $\tilde{f}_2$ and $\tilde{f}_{01}$ together gives a lift f_{012} of $f|_{I_0 \cup I_1 \cup I_2}$.

Continuing in this way, we eventually get a lift of f. Note also that the only choice we made was of $\log_0$.

Why can we do all this? Because of continuity and compactness. Continuity (in its metric $\epsilon - \delta$ formulation) says that for every $s \in I$, there is a open interval J containing s such that either $f(J) \subseteq L^+$ (if $f(s) \neq 1$) or $f(J) \subseteq L^-$ (if $f(s) \neq -1$). Compactness guarantees that finitely many of these intervals cover I. $\square$

We now return to the regularly scheduled proof of the theorem.

Define a map

$$\mathsf{w} : \pi_1(S^1, p) \to \mathbb{Z}, \qquad \mathsf{w}(f) = \tilde{f}(1) - \tilde{f}(0).$$

The number $\mathsf{w}(f)$ is called the *winding number* of f. We will prove that w is a group isomorphism.

We first have to prove that w is uniquely determined, because f can have (in fact, does have) many lifts. Specifically, if $\tilde{f} : I \to \mathbb{R}$ is a lift of $f : I \to S^1$, then so are the functions $\tilde{g}$ defined by

$$\tilde{g}(t) = \tilde{f}(t) + n$$

for $n \in \mathbb{Z}$. In fact, we'll show that these are *all* the lifts, for which it suffices to show that the lift $\tilde{f}$ is determined uniquely by the choice of $\tilde{f}(0)$.

Indeed, let $\tilde{f}$ and $\tilde{f}'$ be two lifts such that $\tilde{f}(0) = \tilde{f}'(0)$. Define $\tilde{h}(s) = \tilde{f}(s) - \tilde{f}'(s)$, and let

$$J = \{s \in I \mid \tilde{f}(s) = \tilde{f}'(s)\} = \tilde{h}^{-1}(0).$$

Then

- $0 \in J$.
- J is closed because it is the continuous preimage of the closed set $\{0\}$.
- J is open, for the following reason. By continuity, every $s \in J$ has some neighborhood $U \subseteq I$ such that $\tilde{h}(U) \subseteq (-\frac{1}{2}, \frac{1}{2})$. On the other hand, $E(\tilde{f}(u)) = f(u) = E(\tilde{f}'(u))$ for every $u \in U$ says that $\tilde{f}(u) - \tilde{f}'(u) \in \mathbb{Z} \cap (-\frac{1}{2}, \frac{1}{2}) = \{0\}$. Therefore $\tilde{f}(u) = \tilde{f}'(u)$ and $U \subseteq J$.

We've now shown that J is nonempty and clopen. Since I is connected, we have $J = I$, which says that $\tilde{f} = \tilde{f}'$.

Note that $\tilde{f}'(1) - \tilde{f}'(0) = \tilde{f}(1) - \tilde{f}(0)$, so we now know that the winding number of a path f is well-defined, and does not depend on a choice of lift. Moreover, if $f : I \to S^1$ is a *closed* path, then $E(\tilde{f}(1)) = E(\tilde{f}(0))$, so $\mathsf{w}(f)$ must be an integer.

We next need to show that $\mathsf{w}(f_0) = \mathsf{w}(f_1)$ whenever $f_0 \simeq f_1$.

Suppose we have a path homotopy $F : I \times I \to S^1$, $f_t(s) = F(t, s)$. We need to show that it lifts to a path homotopy $\tilde{F} : I \times I \to \mathbb{R}$ that is determined by the choice of a basepoint $\tilde{F}(0, 0)$. So, cover $I \times I$ with small open sets on which the image of F is contained in either L^+ or L^-. Choose a finite subcover $\{U_1, \ldots, U_n\}$, and assemble $\tilde{F}$ piece by piece, one U_i at a time, as before.

(We will need to reorder the U_j's so that $(0, 0) \in U_1$, and for $j > 1$, every U_j has a point (hence an open subset) in common with a previous U. (If we *can't* do this, then we'd have a clopen decomposition of $I \times I$, which would violate connectedness. Note that it is important to have the number of U's be finite!)

When we lift F to $\tilde{F}$ in this way, the property that F is a path homotopy (i.e., $f_t(0)$ and $f_t(1)$ are independent of t) implies that $\tilde{F}(0, t)$ and $\tilde{F}(1, t)$ are also independent of t), by an argument much like that for uniqueness of lifting a path up to translation. This says exactly that the winding numbers $\mathsf{w}(f_t) = f_t(1) - f_t(0)$ are independent of t. We have now shown that w is a well-defined **function** $\pi_1(S^1, p) \to \mathbb{Z}$.

Finally, we need to show that w is a **group isomorphism.**

$\underline{\mathsf{w}\text{ is a homomorphism}}$ because if f and g are two closed paths at x_0, then we can find lifts $\tilde{f}, \tilde{g}$ such that $\tilde{f}(1) = \tilde{g}(0)$. Then $\tilde{h} = \tilde{f} \cdot \tilde{g}$ is a lift of $f \cdot g$, and

$$\begin{aligned}\mathsf{w}(\tilde{h}) = \tilde{h}(1) - \tilde{h}(0) &= \tilde{h}(1) - \tilde{h}(\tfrac{1}{2}) + \tilde{h}(\tfrac{1}{2}) - \tilde{h}(0) \\ &= \tilde{g}(1) - \tilde{g}(0) + \tilde{f}(1) - \tilde{f}(0) \\ &= \mathsf{w}(g) + \mathsf{w}(f).\end{aligned}$$

$\underline{\mathsf{w}(f)\text{ is surjective}}$ because for any n, the path $f(s) = e^{2\pi i s n}$ lifts to $\tilde{f}(s) = ns$, with winding number $\tilde{f}(1) - \tilde{f}(0) = n - 0 = n$.

$\underline{\mathsf{w}(f)\text{ is injective}}$ for the following reason. If f is a path with winding number 0, then it it lifts to a **closed** path $\tilde{f} : I \to \mathbb{R}$. We know that $\mathbb{R}$ is contractible, so we have a homotopy $\tilde{F}$ between $\tilde{f}$ and a constant map. Then $E \circ \tilde{F}$ is a homotopy between f and a constant map. It follows that a homotopy class is in the kernel of w if and only if it is the identity in $\pi_1(S^1)$.

$\square$

4. **Monday 2/17: Applications of $\pi_1(S^1) = \mathbb{Z}$**

4.1. **Products.**

Proposition 4.1. *Let X, Y be path-connected spaces, and let $p \in X$, $q \in Y$. Then*

$$\pi_1(X \times Y, (p,q)) \cong \pi_1(X,p) \times \pi_1(Y,q).$$

This is quite easy to prove. If you try to fill in the details of the following sketchy proof on your own, you will find that you have to do little more than write down definitions. (This is a good sign — it means that π_1 is a "natural" construction.)

Proof. A loop f in $X \times Y$ with basepoint (p,q) is precisely a function $f(s) = (f_X(s), f_Y(s))$ such that f_X and f_Y are loops at p and q respectively. Moreover, $f \simeq g$ if and only if $f_X \simeq g_X$ and $f_Y \simeq g_Y$. So we have a bijection $\pi_1(X \times Y, (p,q)) \to \pi_1(X,p) \times \pi_1(Y,q)$, and it is again easy to check that it respects concatenation, hence is a homomorphism, hence an isomorphism. $\square$

Corollary 4.2. *The torus has fundamental group $\mathbb{Z} \times \mathbb{Z}$.*

More generally, the "n-dimensional torus" $(S^1)^n$ has fundamental group $\mathbb{Z}^n$. This can be viewed as the space — in fact, the cell complex — obtained from the n-dimensional unit cube $[0,1]^n$ by identifying opposite faces.

4.2. **The Fundamental Theorem of Algebra.**

Theorem 4.3. *Every nonconstant complex-valued polynomial has at least one complex root.*

Proof. Let $p(z) = z^n + a_n z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial that has no root.

For any real number $r \in \mathbb{R}$ and polynomial $q(z)$ such that $q(r) \neq 0$, we can define a path $\phi(q, r) : I \to S^1$ by

$$\boxed{\phi(q,r)(s) = \frac{q(re^{2\pi is})/q(r)}{|q(re^{2\pi is})/q(r)|}.} \tag{4.1}$$

Note that $\phi(q, 0) = k_1$. Also, if we restrict r to an interval on which q is nonzero, then $\phi(q, r)$ is a homotopy.

Now consider the polynomials

$$\boxed{p_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)}$$

for $t \in [0, 1]$. Thus $p_0(z) = z^n$ and $p_1(z) = p(z)$.

Choose r large enough that r^n clobbers all the other terms in $p(r)$. (For example, $|r| > \max(1, \sum_{j=0}^{n-1} |a_j|)$ is sufficient.) Then $p_t(r) \neq 0$ for $t \in [0, 1]$, and (4.1) gives a homotopy from $\phi(p_0, r)$ to $\phi(p_1, r)$.

We have shown that

$$\phi(p_0, r) \;\simeq\; \phi(p_1, r) = \phi(p, r) \;\simeq\; \phi(p, 0) = k_1.$$

On the other hand,

$$\phi(p_0, r)(s) = r^n e^{2\pi isn})/r^n = e^{2\pi isn}$$

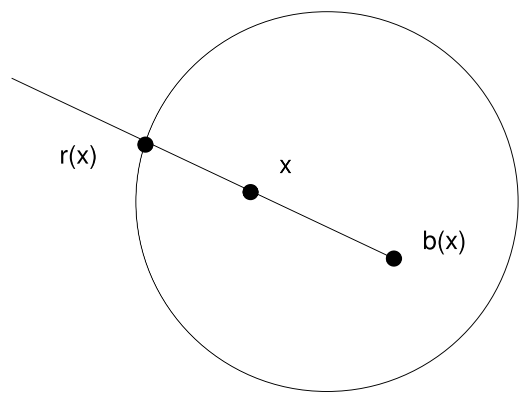
which has winding number n.

Therefore $n = 0$, i.e., $p(z)$ is a constant polynomial. $\square$

4.3. **Brouwer Fixed Point Theorem for D^2.**

Theorem 4.4. *Every continuous function $f : D^2 \to D^2$ has a fixed point.*

Proof. Suppose b is a fixed-point-free function $D^2 \to D^2$. For each $x \in D^2$, let $r(x)$ be the point on $S^1 = \partial D^2$ obtained by drawing a ray from $b(x)$ to x and extending it until it hits the boundary.



Note that $r(x) = x$ for $x \in S^1$, so r is a retraction $D^2 \to S^1$.

Now let $f : I \to S^1$ be any path. Considered as a path in D^2, certainly f is nullhomotopic; let $F : D^2 \times I \to D^2$ be a nullhomotopy. But then $r \circ F$ is a nullhomotopy of f. We have just shown that S^1 is simply-connected, a contradiction. $\square$

In fact the Brouwer Fixed Point Theorem holds for any D^n. For $n = 1$ it is easy (a consequence of the Intermediate Value Theorem). For $n \geq 3$, the argument above doesn't work because $\pi_1(S^{n-1}) = 0$, so we cannot use it to rule out the possibility of a retraction $D^n \to S^{n-1}$. However, one can use either higher homotopy groups or higher homology groups in place of π_1; we will eventually do the latter.

4.4. **Borsuk-Ulam Theorem for S^2.**

Theorem 4.5. *Let $f : S^2 \to \mathbb{R}^2$ be any continuous function. Then there exist two antipodal points $x, -x \in S^2$ such that $f(x) = f(-x)$.*

Proof. Suppose not. Then we can define $g : S^2 \to S^1 \subseteq \mathbb{R}^2$ by

$$\boxed{g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.}$$

Let η be a path around the equator of $S^2 \subseteq \mathbb{R}^3$, i.e., $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$ for $s \in I$. Then $h = g \circ \eta$ is a loop in S^1, which we can lift to $\mathbb{R}$. The diagram:

$$\begin{tikzcd} & & \mathbb{R} \arrow[d, "E"] \\ I \arrow[urr, "\tilde{h}"] \arrow[r, "\eta"'] \arrow[rr, bend right, "h"'] & S^2 \arrow[r, "g"'] & S^1 \end{tikzcd}$$

Note that

$$\begin{aligned} & g(x) = -g(x) && \forall x \\ \therefore \quad & h(s + \tfrac{1}{2}) = -h(s) && \forall s \in [0, \tfrac{1}{2}] \\ \therefore \quad & \tilde{h}(s + \tfrac{1}{2}) = \tilde{h}(s) + q/2 && \forall s \in [0, \tfrac{1}{2}] \end{aligned}$$

where q is some odd integer. In fact q is independent of s because it depends continuously on s, but must be an integer. Therefore

$$\mathsf{w}(h) = \tilde{h}(1) - \tilde{h}(0) = \tilde{h}(1) - \tilde{h}(\tfrac{1}{2}) + \tilde{h}(\tfrac{1}{2}) - \tilde{h}(0) = q \neq 0.$$

But on the other hand η is certainly nullhomotopic in S^2 (push the equator up to the North Pole), and composing with g gives a nullhomotopy for h. This is a contradiction. □

Remark 4.6. The general Borsuk-Ulam theorem says that for any continuous function $f : S^n \to \mathbb{R}^n$, there are points $x, -x$ with $f(x) = f(-x)$.

Corollary 4.7. *If S^2 is the union of three closed sets $A_1 \cup A_2 \cup A_3$, then one of the A_i must contain a pair of antipodal points.*

Sketch of proof: Let $d_i(x) = \inf\{\|x - y\| : y \in A_i\}$. Apply Borsuk-Ulam to the function $f(x) = (d_1(x), d_2(x))$. The same argument works in higher dimension for a cover of S^n by $n + 1$ closed sets.

5. **Wednesday 2/19: Induced Homomorphisms**

Theorem 5.1. *If two path-connected spaces X and Y are homotopy-equivalent, then $\pi_1(X) = \pi_1(Y)$.*

The main technical tool we need is **induced homomorphisms**.

Let $\phi : X \to Y$ be a continuous function. Then for any path $f : I \to X$, the composition $\phi_\sharp(f) = \phi \circ f$ is a path in Y.

Proposition 5.2. *Let $f, g : I \to X$. If $[f] = [g]$, then $[\phi \circ f] = [\phi \circ g]$. Therefore, for every $p \in X$, there is an induced map*

$$\boxed{\phi_* : \pi_1(X, p) \to \pi_1(X, \phi(p)), \qquad \phi_*[f] = [\phi \circ f].}$$

This map is a group homomorphism. Moreover,

$$\boxed{(\phi \circ \psi)_* = \phi_* \circ \psi_* \quad \text{and} \quad (\mathbb{1}_X)_* = \mathbb{1}.}$$

Proof. If $F : I \times I \to X$ is a path-homotopy from f to g, then $\phi \circ F$ is a path-homotopy from $\phi \circ f$ to $\phi \circ g$ — this can be verified directly (and we've probably already used it). Therefore the map ϕ_* is well-defined. It is a group homomorphism because

$$\phi_*[f] \cdot \phi_*[g] = [f \circ \phi \cdot g \circ \phi] = [(f \cdot g) \circ \phi] = \phi_*([f] \cdot [g]).$$

The other verifications are also straightforward. $\square$

The proposition says that π_1 is a **functor** from topological spaces to groups. In other words, it not only transforms spaces into groups; it also transforms morphisms of spaces (continuous functions) into morphisms of groups (homomorphisms).

Technically, the domain of π_1 is the category of *basepointed* spaces. A basepointed space (X, p) is a space X with a distinguished point p; a morphism of basepointed spaces is a continuous function $f : (X, p) \to (X', p')$ with $f(p) = p'$.

Proposition 5.3. *If A is a deformation retract of X, then $\pi_1(X,p) \cong \pi_1(A,p)$ for every $p \in A$.*

Proof. Let $F : X \times I \to X$ be a deformation-retraction. Recall this means that

$$f_t(x) = F(x,t), \quad f_0 = \mathbb{1}_X, \quad f_t|_A = \mathbb{1}_A, \quad f_1(X) \subseteq A.$$

Define a map $\alpha : \pi_1(X,p) \to \pi_1(A,p)$ as follows. Given any path $g : I \to X$ with basepoint p, consider the path homotopy

$$\boxed{g_t(s) = F(g(s),t) \qquad \text{for } t \in I.}$$

Note that

$$g_0(s) = F(g(s),0) = g(s),$$
$$g_1(s) = F(g(s),1) \in A.$$

Define $\alpha[g] = [g_1]$. The map α respects concatenation, hence is a group homomorphism. It is surjective because if f is a path in A then $\alpha(i_*[f]) = [f]$, and it is injective because any nullhomotopy in Y is a nullhomotopy in A. $\square$

Corollary 5.4. *If two spaces are homotopy-equivalent then their fundamental groups are isomorphic.*

Proof. Remember that if $X \simeq Y$, then there is a space containing both X and Y as deformation retracts (namely, the mapping cylinder of a homotopy equivalence). $\square$

This is fine as far as it goes, but you should not be satisfied with this! What we are really after is the following basepointed version of the theorem.

Theorem 5.5. *Let $\phi : X \to Y$ be a homotopy equivalence. Then for every $p \in X$, the induced map $\phi_* : \pi_1(X,p) \to \pi_1(Y,\phi(p))$ is an isomorphism of groups.*

Lemma 5.6 (Sliding Lemma)**.** *Let $\phi_t : X \to Y$ be a homotopy and $p \in X$. For short, write $q_t = \phi_t(p)$. Let h be the path from q_0 to q_1 given by $h(t) = q_t$. Then the following diagram commutes:*

$$\begin{tikzcd} & \pi_1(Y, q_1) \arrow[dd, "\beta_h", "\cong"'] \\ \pi_1(X, p) \arrow[ur, "(\phi_1)_*"] \arrow[dr, "(\phi_0)_*"'] & \\ & \pi_1(Y, q_0) \end{tikzcd} \qquad \textit{where} \quad \beta_h[f] = [h \cdot f \cdot \overline{h}].$$

Proof. Let h_t be the part of h from q_0 to q_t and let $[f] \in \pi_1(X, p)$. Then

$$\{h_t \cdot (\phi_t \circ f) \cdot \overline{h_t} : \quad t \in I\}$$

is a path-homotopy with basepoint q_0. Moreover,

$$\begin{aligned} [h_0 \cdot (\phi_0 \circ f) \cdot \overline{h_0}] &= \phi_0 \circ f = (\phi_0)_*[f], \\ [h_1 \cdot (\phi_1 \circ f) \cdot \overline{h_1}] &= \beta_h((\phi_1)_*[f]). \end{aligned}$$

□

Proof of Theorem 5.5. Consider the maps

$$\begin{tikzcd} \pi_1(X, p) \arrow[r, "\phi_*"] \arrow[rr, bend left, "\omega"] & \pi_1(Y, \phi(p)) \arrow[r, "\psi_*"] \arrow[rr, bend right, "\alpha"'] & \pi_1(X, \phi\psi(p)) \arrow[r, "\phi_*"] & \pi_1(X, \phi\psi\phi(p)) \end{tikzcd}$$

where ω, α are just the indicated compositions. By the Sliding Lemma we have $\omega = \psi_*\phi_* = [\psi \circ \phi]_* = \beta_h \circ \mathbb{1}_* = \beta_h$, where h is the path from p to $\psi(\phi(p))$ given by the homotopy $\psi \circ \phi \simeq \mathbb{1}_X$. In particular, ω is an isomorphism. The same argument shows that α is an isomorphism.

Therefore, the proposition reduces to the purely algebraic statement that if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ are group homomorphisms such that $g \circ f$ and $h \circ f$ are isomorphisms, then all three maps are isomorphisms. Indeed.

- $g \circ f$ isomorphism $\therefore$ f injective, g surjective.
- $h \circ g$ isomorphism $\therefore$ g injective, h surjective.
- Therefore g is an isomorphism.
- So are $f = g^{-1} \circ (g \circ f)$ and $h = (h \circ g) \circ g^{-1}$.

□

Corollary 5.7. *If X is contractible then it is simply-connected.*

Proof. "Contractible" means "homotopy-equivalent to a point", and the fundamental group of a point is certainly trivial. ☐

Corollary 5.8. $S^1 \not\simeq S^m$ *for* $m \geq 2$, *and* $\mathbb{R}^2 \not\cong \mathbb{R}^n$ *for* $n \geq 3$.

Proof. The first statement follows because $\pi_1(S^1) = \mathbb{Z} \neq \pi_1(S^m) = 0$. If there were a homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^n$, it would restrict to a homeomorphism $\phi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^n \setminus \{\phi(0)\}$. But $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$ is simply-connected and $\mathbb{R}^2 \setminus \{0\} \simeq S^1$ isn't. ☐

(But could it be possible that $\mathbb{R}^3 \cong \mathbb{R}^4$?)

Corollary 5.9. *The annulus and the Möbius strip both have fundamental group* $\mathbb{Z}$.

Proof. Both of them deformation-retract onto their central circles. ☐

(What about the projective plane and the Klein bottle? What about the wedge of two circles?)

6. **Friday 2/21: Van Kampen's Theorem — Motivation and Examples**

6.1. **Motivating examples: The torus and the figure-8.**

The torus $T = S^1 \times S^1$ has fundamental group $\mathbb{Z}^2$. We can see this by expressing T as a quotient $\mathbb{R}^2/\mathbb{Z}^2$, with quotient map q, and by talking about paths in T in terms of their lifts to $\mathbb{R}^2/\mathbb{Z}^2$.

The group $\pi_1(T, q(\underline{0}))$ is generated by path classes $[a], [b]$, where

$$\tilde{a}(s) = (s, 0), \qquad \tilde{b}(s) = (0, s).$$

Note that $[a \cdot b] = [b \cdot a]$ for the following reason. Since $\mathbb{R} \times \mathbb{R}$ is simply connected, we can find a homotopy $F : I \to \mathbb{R} \times \mathbb{R}$ between $\tilde{a} \cdot \tilde{b}$ and $\tilde{b} \cdot \tilde{a}$, and then $q \circ F$ is a homotopy between $a \cdot b$ and $b \cdot a$. Equivalently, the path $\overline{\tilde{a}} \cdot \overline{\tilde{b}} \cdot \tilde{a} \cdot \tilde{b}$ is nullhomotopic, because it is the boundary of the unit square, which can be path-homotoped to k_0 by shrinking the square.

This is an example of a *universal covering space*. The idea is that to understand the fundamental group of an arbitrary space X (in this case $X = T$), we can find a simply connected space $\tilde{X}$ which covers it (whatever that means; in this case $\tilde{X} = \mathbb{R} \times \mathbb{R}$), so that we can lift every loop in X to a loop in $\tilde{X}$, then hope to extract algebraic information about $\pi_1(X)$ from the behavior of the covering map q.

Here is another space whose fundamental group has two generators: the figure-eight $\infty = S^1 \vee S^1$. Take the middle point p to be the basepoint.

The group $\pi_1(\infty, p)$ has two "obvious" generators: the paths a, b that wrap once clockwise around the top and bottom circles.

Difference from torus: a and b do not commute. E.g., $aba^{-1}b^{-1}$ is not nullhomotopic. In fact, a and b are about as non-commuting as you can get.

In fact, $\pi_1(\infty) = \mathbb{Z} * \mathbb{Z}$, where $*$ (pronounced "smash") is the **free product** of groups. That is, the elements of $\pi_1(\infty)$ include all words of the form

$$a^{n_1} b^{m_1} a^{n_2} b^{m_2} \cdots$$

of finite length, where $n_i, m_i \in \mathbb{Z}$ for all i. Multiplication of these elements is given by concatenation. We are allowed to make the simplifications $a^n a^m = a^{n+m}$ and $b^n b^m = b^{n+m}$, but no others.

This is a much bigger group than $\mathbb{Z}\times\mathbb{Z}$. Not in the set-theoretic sense (they're both countably infinite), but in a group-theoretic sense: $\mathbb{Z}\times\mathbb{Z}$ is in fact the **abelianization** of $\mathbb{Z} * \mathbb{Z}$, its largest abelian quotient. Details to come.

We'd like to know that $\pi_1(\infty)$ is in fact this group for sure that the group we have just described is indeed $\pi_1(\infty)$. That's where Van Kampen's Theorem is going to come in, but first we need to understand the operation $*$.

6.2. **Free Products of Groups.** Let $\{G_\alpha\}_{\alpha\in A}$ be a family of groups.

Definition: The *free product* $*_\alpha G_\alpha$ is the group whose elements are the words of finite length

$$g_1 g_2 \cdots g_m$$

where each letter g_i belongs to one of the G_α, and the relations are given by the individual groups themselves. I.e., if g_i, g_{i+1} both belong to G_α for some α, then the two-letter subword g_i, g_{i+1} can be replaced with the single letter $g_i \cdot g_{i+1}$, where $\cdot$ means multiplication in G_α.

• The identity element is the empty word, and inversion is given by $(g_1 \cdots g_m)^{-1} = g_m^{-1} \cdots g_1^{-1}$. I will skip the verification of associativity.

• If $|A| = n < \infty$, particularly if $n = 2$, then we'll often write $G_1 * \cdots * G_n$.

• If G, H are groups and you want to make a group that has G and H sitting inside it, then the direct product is the "most commutative"/"smallest" way to do so and the free product is the "least commutative"/"biggest" way.

• Universal properties: There is a natural inclusion $G \hookrightarrow G * H$ (or more generally into any free product involving G). For any group X, any pair of homomorphisms $g : G \to X$, $h : H \to X$ factor through $G * H$ in the sense that there is a unique homomorphism given by the dotted line in the diagram on the left:

$$
\begin{tikzcd}
G \arrow[dr, "i_G"] \arrow[ddr, "g"'] & & H \arrow[dl, "i_H"'] \arrow[ddl, "h"] \\
& G * H \arrow[d, dashed] & \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
G & & H \\
& G \times H \arrow[ul, "\pi_G"'] \arrow[ur, "\pi_H"] & \\
& X \arrow[uul, "g"] \arrow[uur, "h"'] \arrow[u, dashed] &
\end{tikzcd}
$$

This is precisely what you get by reversing the arrows for the universal property of the direct product $G \times H$, where π_G and π_H are projection maps (shown on the right).

- Modding out by all the commutators, i.e., by all elements of $*_\alpha G_\alpha$ of the form

$$[g, h] = ghg^{-1}h^{-1}$$

where g, h belong to <u>different</u> G_α's, gives the direct sum $\oplus_\alpha G_\alpha$.

In the case that the G_α were abelian to begin with, these elements generate the commutator subgroup of $*_\alpha G_\alpha$, and so the direct sum is the abelianization of the free product.

6.3. **Van Kampen's Theorem.** Now suppose we have a basepointed space (X, p) and an open cover $X = \bigcup_\alpha A_\alpha$, with $p \in A_\alpha$ for every α.

We have inclusions of spaces $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ and $A_\alpha \hookrightarrow X$ which induce group homomorphisms

$$i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta, p) \to \pi_1(A_\alpha, p), \qquad j_\alpha : \pi_1(A_\alpha, p) \to \pi_1(X, p).$$

By the universal property of free product, we have a group homomorphism

$$\Phi : *_\alpha \pi_1(A_\alpha, p) \to \pi_1(X, p).$$

This just says if you concatenate a series of loops at p in individual A_α's — that is, if you write down something in $*_\alpha \pi_1(A_\alpha, p)$ — then what you have written down can be regarded as a loop at p in X.

We can summarize all this in a commutative diagram which is easiest to write in the case that the open cover has just two subspaces A_α, A_β:

(6.1)

$$
\begin{tikzcd}
 & & \pi_1(A_\alpha \cap A_\beta) \arrow[dll, "i_{\alpha\beta}"'] \arrow[drr, "i_{\beta\alpha}"] & & \\
\pi_1(A_\alpha) \arrow[drr, "\subseteq"] \arrow[ddrr, "j_\alpha"'] & & & & \pi_1(A_\beta) \arrow[dll, "\subseteq"'] \arrow[ddll, "j_\beta"] \\
 & & \pi_1(A_\alpha) * \pi_1(A_\beta) \arrow[d, "\Phi"] & & \\
 & & \pi_1(X) & &
\end{tikzcd}
$$

$\pi_1(-) = \pi_1(-, p)$

For a path class $[f] \in \pi_1(A_\alpha \cap A_\beta, p)$, note that

$$j_\alpha i_{\alpha\beta}[f] = j_\beta i_{\beta\alpha}[f]$$

which says that

$$i_{\alpha\beta}[f] * i_{\beta\alpha}[\bar{f}] \in \ker \Phi.$$

Van Kampen's Theorem — General Case

Let $p \in X$, and let $\{A_\alpha : \alpha \in \mathcal{A}\}$ be a cover of X by path-connected open sets such that $p \in A_\alpha$ for every α.

1. If every pairwise intersection $A_\alpha \cap A_\beta$ is path-connected, then the map Φ is surjective.

2. If in addition every triple intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then

$$\ker \Phi = \left\langle\!\left\langle i_{\alpha\beta}(\omega) * i_{\beta\alpha}(\bar{\omega}) \right\rangle\!\right\rangle \ : \ \alpha, \beta \in \mathcal{A},\ \omega \in \pi_1(A_\alpha \cap A_\beta).$$

The notation in Case 2 means the smallest normal subgroup containing these elements, or equivalently the subgroup generated by these elements and their conjugates. This is in general larger than the subgroup $\langle i_{\alpha\beta}[f] * i_{\beta\alpha}[\bar{f}] \rangle$.

Case 2 gives an explicit description of $\pi_1(X,p)$ as a quotient of a free product. Note that the triple-intersection condition is vacuously true for covers of X by *two* open path-connected sets — a very common case.

Corollary 6.1. *If $X = \bigcup_\alpha A_\alpha$ such that every A_α is simply-connected and every $A_\alpha \cap A_\beta$ is connected, then X is simply-connected.*

Proof. Van Kampen's theorem says that there is a surjection

$$*_\alpha \pi_1(A_\alpha, p) = *_\alpha 0 = 0 \twoheadrightarrow \pi_1(X,p).$$

□

Corollary 6.2. *Suppose $X = A \cup B$ with B simply-connected, then $\pi_1(B) = 0$ and $\pi_1(A) = \pi_1(A) * \pi_1(B)$. We are in the second case of Van Kampen's Theorem, and the map $i_{\beta\alpha}$ is zero, so* $\ker \Phi$ *is simply the image of $i = i_{\alpha\beta}$. Therefore the diagram* (6.1) *simplifies to a short exact sequence of groups*

$$0 \to \pi_1(A \cap B, p) \xrightarrow{i_*} \pi_1(A,p) \xrightarrow{\Phi} \pi_1(X,p) \to 0.$$

Example: Let A_1, A_2 be the closed north and south hemispheres of S^n with $n \geq 2$. Each one is contractible, hence simply-connected. Their intersection is the "equator," which looks like S^{n-1} and is connected. Therefore S^n is simply-connected. We already knew that but it's nice to have it confirmed! Note that this argument theorem fails for $n = 1$ because S^0 is not connected.

Example: Let $X = \mathbb{R}P^2$, which has a cell decomposition $e^2 \cup e^1 \cup e^0$. Let $B = e^2$ and let A be an open mapping cylinder neighborhood of the circle $e^1 \cup e^0$. Then $\{A, B\}$ is a path-connected open cover of $\mathbb{R}P^2$, and $A \cap B$ is open (if we think of B as the Euclidean plane, then $A \cap B$ is the complement of a closed disc). By Corollary (6.2), since A is simply-connected, the diagram (6.1) simplifies to a short exact sequence of groups

$$0 \to \pi_1(A \cap B, p) \xrightarrow{i_*} \pi_1(B) \xrightarrow{\Phi} \pi_1(X) \to 0$$

where i is inclusion. The first two groups are $\mathbb{Z}$ (because $A \cap B \simeq S^1$). But the map i_* sends a generator of $\pi_1(A \cap B)$ to **twice** a generator of $\pi_1(B)$, because the attaching map is two-to-one. Therefore

$$\boxed{\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}.}$$

More generally, if X_k is the cell complex obtained by attaching a 2-cell to S^1 by wrapping its boundary k times around the circle, then $\pi_1(X_k) = \mathbb{Z}/k\mathbb{Z}$.

7. Monday 2/24: Van Kampen's Theorem — The Proof

Recall the statement of Van Kampen's Theorem.

Let $p \in X$, and let $\{A_\alpha : \alpha \in \mathscr{A}\}$ be a cover of X by path-connected open sets such that $p \in A_\alpha$ for every α. We have a commutative diagram of groups, which looks in part like this (where the i's and j's are the group homomorphisms induced by inclusions of spaces).

$$
\begin{tikzcd}
& \pi_1(A_\alpha \cap A_\beta) \arrow[dl, "i_{\alpha\beta}"'] \arrow[dr, "i_{\beta\alpha}"] & \\
\pi_1(A_\alpha) \arrow[dr, "\subseteq"] \arrow[ddr, "j_\alpha"'] & & \pi_1(A_\beta) \arrow[dl, "\subseteq"'] \arrow[ddl, "j_\beta"] \\
& F = *_\alpha \pi_1(A_\alpha) \arrow[d, "\Phi"] & \\
& \pi_1(X) &
\end{tikzcd}
\tag{7.1}
$$

$$\pi_1(-) = \pi_1(-, p)$$

Van Kampen's Theorem:

(1) If every pairwise intersection $A_\alpha \cap A_\beta$ is path-connected, then the map Φ is surjective.

(2) If in addition every triple intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then

$$\ker \Phi = N := \langle\!\langle i_{\alpha\beta}(\omega) * i_{\beta\alpha}(\bar{\omega}) \rangle\!\rangle \; : \; \alpha, \beta \in \mathcal{A}, \; \omega \in \pi_1(A_\alpha \cap A_\beta)$$

and so

$$\pi_1(X) = F/N.$$

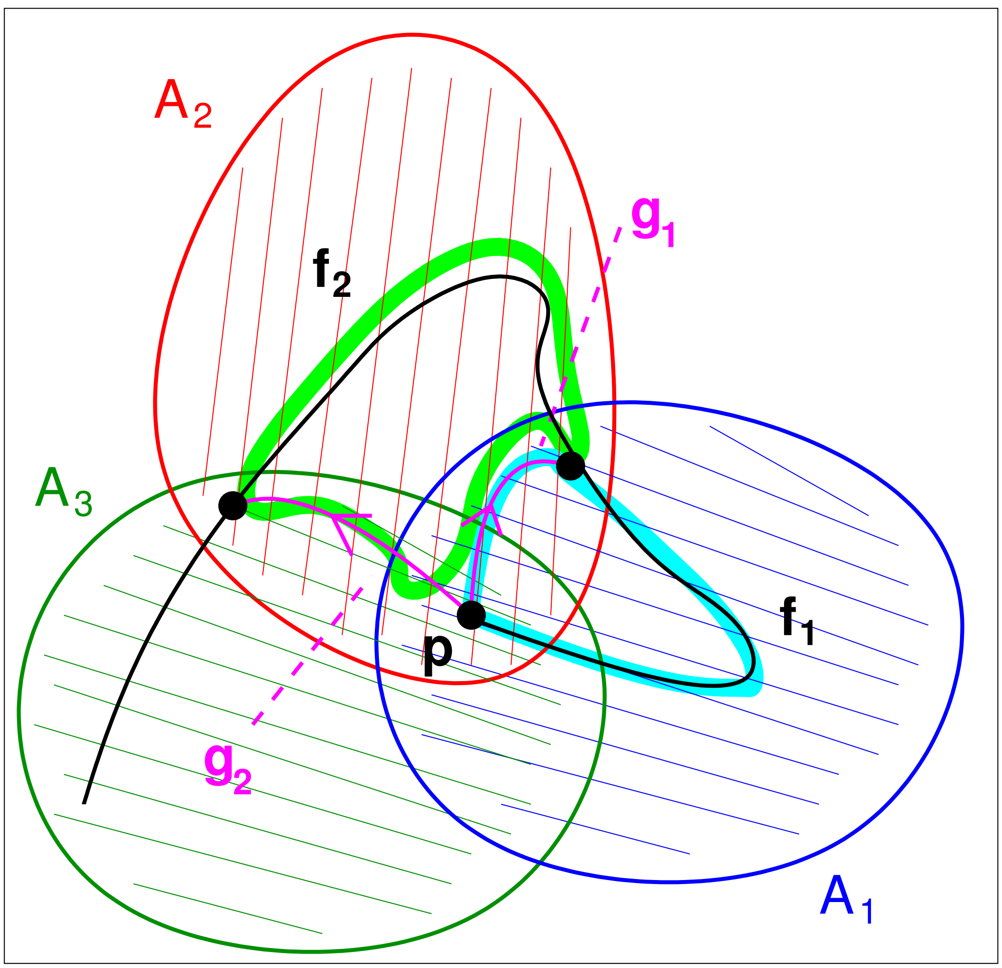
Proof of (1). Let $f : I \to X$ be a loop based at p. Every $s \in I$ has a neighborhood mapped by f into some U_α. By compactness of I, there exist numbers $0 = s_0 < s_1 < \cdots < s_m = 1$ and indices $\alpha_1, \ldots, \alpha_m$ such that

$$f([s_{i-1}, s_i]) \subseteq A_{\alpha_i} \quad \forall i \in [m].$$

Let $f_i = f|_{[s_{i-1}, s_i]}$, so that $f = f_1 \cdot f_2 \cdots f_m$. For each $i \in [m]$, the set $A_i \cap A_{i+1}$ is path-connected, hence contains a path g_i from p to $f(s_i)$. Therefore

$$\begin{aligned} f &= f_1 \cdot f_2 \cdots f_m \\ &= (f_1 \cdot \overline{g_1}) \cdot (g_1 \cdot f_2 \cdot \overline{g_2}) \cdots (g_{m-2} \cdot f_{m-1} \cdot \overline{g_{m-1}}) \cdot (g_m \cdot f_m) \\ &\in \pi_1(A_1, p) * \pi_1(A_2, p) * \cdots * \pi_1(A_m, p) \\ &\in \operatorname{im} \Phi. \end{aligned}$$

□

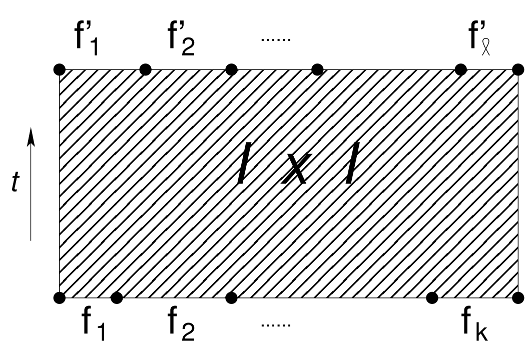


Proof of (2). Let $[f] \in \pi_1(X)$. Say that a **factorization** of $[f]$ is an expression $[f_1] * [f_2] * \cdots * [f_n]$ that maps to $[f]$ via Φ. Here I am using $*$ to denote concatenation of letters to make a word in $*_\alpha \pi_1(A_\alpha)$. That is, each $[f_i]$ belongs to some $\pi_1(A_\alpha)$, and $f \simeq f_1 \cdot f_2 \cdots f_n$.

We want to show that any two factorizations of $[f]$ are related by operations of the following forms:

- "Type A": If $f_i : I \to A_\alpha \cap A_\beta$, then we can regard the letter $[f_i]$ as coming either from $\pi_1(A_\alpha)$ or from $\pi_1(A_\beta)$. This amounts to inserting an element of N into f, namely
$$i_{\alpha\beta}[f_i] * i_{\beta\alpha}[\overline{f_i}].$$
- "Type B": If two consecutive letters in the factorization come from the same A_α, we can multiply them. This, of course, doesn't change the element of F we're talking about.

So, suppose we have two factorizations
$$[f] \;=\; \Phi\left([f_1] * \cdots * [f_k]\right) \;=\; \Phi\left([f_1'] * \cdots * [f_\ell']\right).$$
In particular, there is a path-homotopy of p-loops $H : I \times I \to X$, $h_t(s) = H(s,t)$, such that
$$h_0 = f_1 \cdots f_k \quad \text{and} \quad h_1 = f_1' \cdots f_\ell'.$$
Schematically, here's what this looks like:

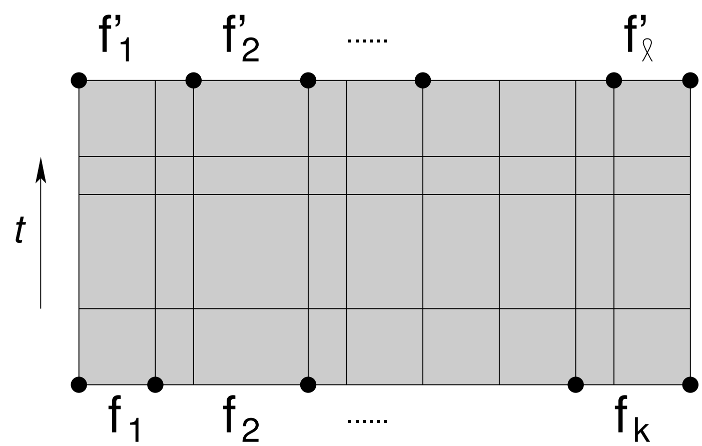


The dots on the top and bottom lines are the breakpoints between successive f_i's or f_i''s.

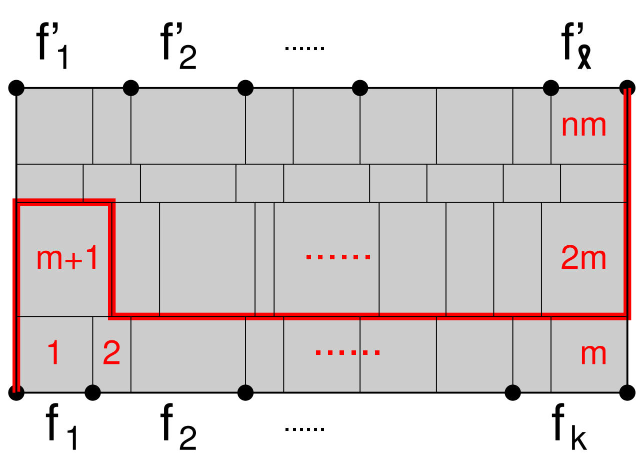
Now, we do something clever. Partition $I \times I$ into a finite grid of finitely many little rectangles R_i such that

$$\forall R_i : \ \exists i \in \mathscr{A} : \ H(R_i) \subset A_i. \tag{7.2}$$

(By continuity of H, we can put such a rectangle around each point in $I \times I$, then choose a finite subcover, then subdivide if necessary.) Subdivide more by adding vertical lines at all the breakpoints, and at least two horizontal lines.



Now, we do something *exceedingly* clever. For all of the vertical lines not in the first or last row, give them a little nudge to one side so they don't match up. We can do this while still retaining the condition (7.2). Number the rectangles $R_1, \ldots, R_{mn}$ as shown, where m is the number of columns and n is the number of rows.



Let γ_k be the path from $(0,0)$ to $(1,1)$ along the cell walls that separates rectangles $R_1, \ldots, R_k$ from $R_{k+1}, \ldots, R_{mn}$. (For example, the thick red path shown in the figure above is R_{m+1}.) Thus $H \circ \gamma_k$ is a closed path in X with basepoint p, and all the paths $H \circ \gamma_k$ are path-homotopic.

Each γ_k can be written as

$$\gamma_k = e_1 \cdot e_2 \cdots e_N$$

where each e_i is the path in X given by part of a side of one rectangle, say from v_{i-1} to v_i.

For each v_i, choose some path g_i in X from p to $F(v_i)$. Each v_i belongs to at most three rectangles, so we can require g_i to stay in the intersection of the corresponding three A's. (Wasn't that clever of us?)

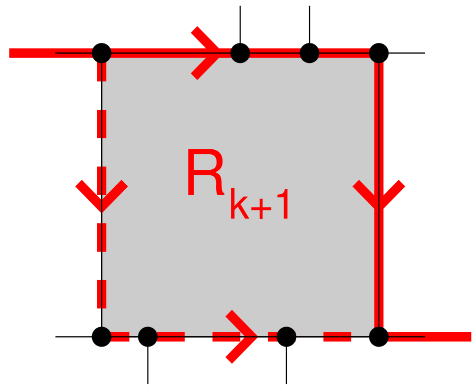
Then each γ_k can be factored as

$$\begin{aligned}\gamma_k &= e_1 \cdots e_N \\ &= \Phi(e_1 * \cdots * e_N) \\ &= \Phi\Big([e_1 \cdot g_1] * [\overline{g_1} \cdot e_2 \cdot g_2] * \cdots * [\overline{g_{N-2}} \cdot e_{N-1} \cdot g_{N-1}] * [\overline{g_{N-1}} \cdot e_N]\Big)\end{aligned}$$

Recall that $*$ means concatenation of letters in the free product F, while $\cdot$ means concatenation within one of its free factors.

To pass from the factorization for γ_k to that of γ_{k+1}, we have to trade the south and west sides of R_{k+1} for the north and east sides. We can do this by

- regarding the letters in the south and west sides as now coming from $\pi_1(A_{k+1})$ instead of wherever they came from in the factorization of γ_k (this is a type-A move);
- using the group structure of $\pi_1(A_{k+1})$ to trade the letters in the south and west sides for the north and east ones (this is a type-B move).



Now let's look at the path γ_0, which consists of the bottom and right edges of $I \times I$. The right edge is a stationary path, so forget about it. For each vertex v_i on the bottom edge of $I \times I$, we have so far only required g_i to lie in **two** of the A's. Let's also require it to lie in the same one whose fundamental group contains the letter f_i (which came from the factorization of f given in advance). That says that

For example, if $f_3 = e_1 \cdot e_2 \cdot e_3$, then the factorization begins

$$[e_1 \cdot g_1] * [\overline{g_1} \cdot e_2 \cdot g_2] * [\overline{g_2} \cdot e_3 \cdot g_3] * \cdots$$

where g_1 is a path in $A_1 \cap A_2$ and g_2 is a path in $A_2 \cap A_3$. But in fact we can require g_1 and g_2 to be paths in $A_1 \cap A_2 \cap A_\alpha$ and $A_2 \cap A_3 \cap A_\alpha$, where $\pi_1(A_\alpha)$ is the group containing the letter f_3. We also may as well assume that g_3 is the stationary path. So the partial factorization shown above can be replaced (with type-A moves) with one in $\pi_1(A_\alpha)$, and then simplified to the single letter $[e_1 \cdot e_2 \cdot e_3] = [f_1] \in \pi_1(A_\alpha)$.

More generally, if v_i is a breakpoint then we take g_i to be the constant path, and if v_i is not a breakpoint then we require g_i to lie in A_α for whichever $\pi_1(A_\alpha)$ contains the letter f_j to which the edges at v_i contribute. Then parenthesizing the factorization of f at the breakpoints shows that it is equivalent to $[f_1] * \cdots * [f_k]$.

Playing the same game at the top of the square shows that the factorization of γ_{nm} is equivalent to $[f'_1] * \cdots * [f'_\ell]$. $\square$

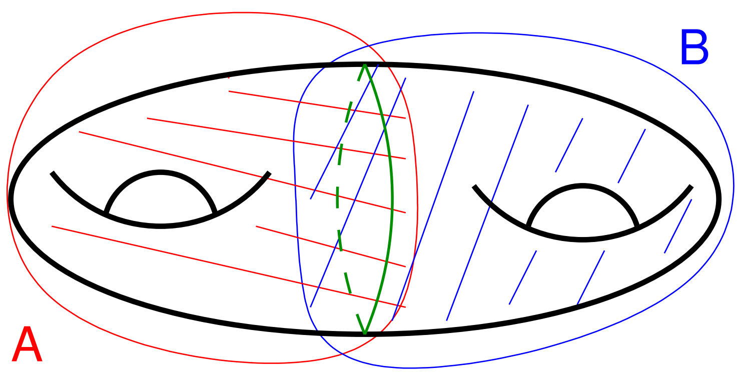
$\square$

8. **Wed 2/26: Applications of Van Kampen's Theorem**

The following is the most useful case of Van Kampen's Theorem.

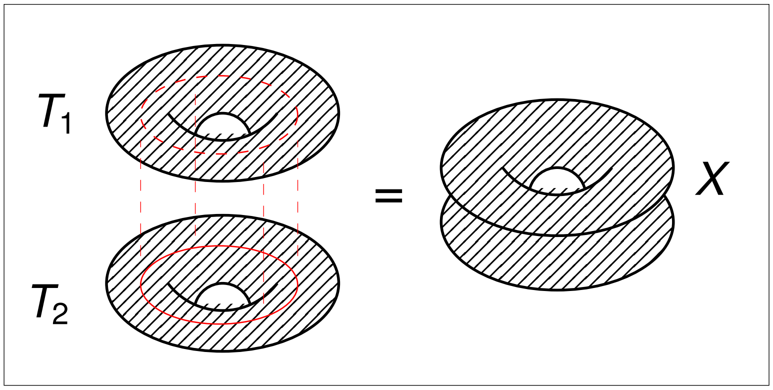
Let X be a path-connected space. Let $A \cup B$ be an open cover of X with $A \cap B$ path-connected. Then
$$\pi_1(X) = \pi_1(A) * \pi_1(B)/\langle (i_A)_*[f](i_B)_*[f]^{-1} : [f] \in \pi_1(A \cap B)\rangle$$
where all fundamental groups are taken with a common basepoint.

Example 8.1 (The fundamental group of the genus-2 torus). Let X be the genus-2 torus, and let A and B be the open subsets of X shown. Then $A \cup B = X$ and $A \cap B \cong S^1 \times (0,1) \simeq S^1$. So we can hope to calculate $\pi_1(X)$ using Van Kampen's Theorem. The key step is to calculate $(i_A)_*[f]$, where f is a generator of the fundamental group of $A \cap B$ and i_A is the inclusion $A \cap B \hookrightarrow A$.



The space A is homotopy-equivalent to a punctured torus, which deformation-retracts to $S^1 \vee S^1$. So $\pi_1(A)$ is the free group $\langle a, b\rangle$ on two generators a, b as shown. The path f wraps once around the puncture; homotoping it to the boundary of the square shows that $(i_A)_*[f] = aba^{-1}b^{-1} = [a, b]$. Similarly, $\pi_1(A) = \langle c, d\rangle$ is the free group on two generators c, d and $(i_B)_*[f] = [d^{-1}, c^{-1}]$. Why this choice of orientation? Van Kampen's Theorem says that
$$\begin{aligned}\pi_1(X) &= \pi_1(A) * \pi_1(B)/\langle (i_A)_*[f](i_B)_*[\bar{f}]\rangle \\ &= \langle a, b, c, d | [a, b][d^{-1}, c^{-1}]^{-1}\rangle \qquad = \langle a, b, c, d | [a, b][c, d]\rangle.\end{aligned}$$
We'll have a more efficient way of calculating this group soon.

Example 8.2 (A stack of donuts). The space $S^1 \times (S^1 \vee S^1)$ looks like two tori T_1, T_2 identified along longitudinal circles a_1, a_2:



The map $\Phi : \pi_1(T_1) \times \pi_1(T_2) \to \pi_1(X)$ identifies the corresponding elements of $\pi_1(T_1)$ and $\pi_1(T_2)$, so

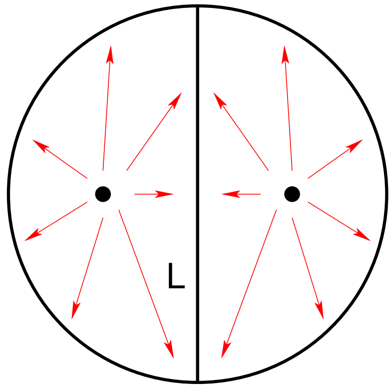
$$\begin{aligned}
\pi_1(T_1 \cap T_2) &= \langle a \rangle \\
\pi_1(T_1) &= \langle a_1, b_1 \mid [a_1, b_1] \rangle \\
\pi_1(T_2) &= \langle a_2, b_2 \mid [a_2, b_2] \rangle \\
\pi_1(X) &= \langle a, b_1, b_2 \mid [a, b_1],\ [a, b_2] \rangle = \mathbb{Z} \times (\mathbb{Z} * \mathbb{Z})
\end{aligned}$$

This is #8 on p.53 of Hatcher.

8.1. **Detecting Linking of Circles.** Suppose A and B are two disjoint circles in $\mathbb{R}^3$. We can tell if they are linked by computing the fundamental group of $X = \mathbb{R}^3 \setminus (A \cup B)$.

First, let's calculate $\pi_1(\mathbb{R}^3 \setminus A)$. I claim that this space is homotopy-equivalent to $S^1 \vee S^2$. Draw a sphere S^2 containing A in its interior. Everything outside this sphere deformation-retracts onto it by shrinking, so we just need consider $Y = S^2 \setminus A$.

Inside the sphere, let L be the diameter along the z-axis. Every cross-section of a plane containing L looks like this, where the two "eyes" are points on A:



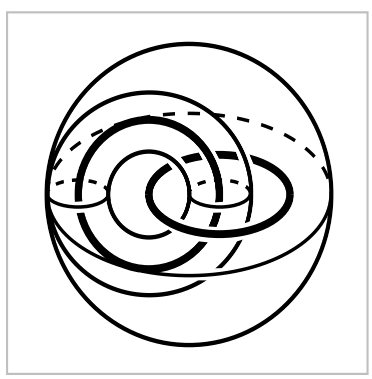
Putting together all the cross-sections, the red arrows give a deformation-retraction of Y onto $S^2 \cup L$, which is homotopy-equivalent to $S^2 \vee S^1$ by sliding the two points of $S^2 \cap L$ together.

In particular, $\pi_1(\mathbb{R}^3 \setminus S^1) = \mathbb{Z}$. If A and B are two unlinked circles, then

$$X \simeq (\mathbb{R}^3 \setminus A) \vee (\mathbb{R}^3 \setminus B)$$

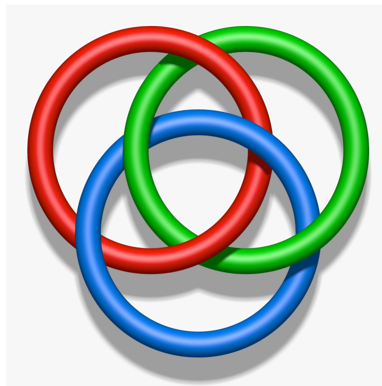
and so $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$ by VKT.

Now suppose A and B are linked. Draw a torus T around A. In the picture on Hatcher, p.47 (reproduced below), A is the left-hand boldface circle and B is the one on the right; T is the torus around A.



The interior of T deformation-retracts to T (pull it away from A), while the stuff between T and S^2 deformation-retracts to $T \cup S^2$ (because as far as this stuff is concerned, we might as well replace T with a solid torus $\mathbf{T}$, and $S^2 \cup \mathbf{T}$ is homotopy-equivalent to $S^2 \cup L$ from the previous example, and the deformation-retraction we are looking for is just like the deformation-retraction $Y \to S^2 \cup L$). Therefore $X \simeq T \vee S^2$ and $\pi_1(X) = \mathbb{Z} \times \mathbb{Z}$ rather than $\mathbb{Z} * \mathbb{Z}$.

Now suppose we have three Borromean rings $A, B, C \subset \mathbb{R}^3$ (image from Wikipedia; marked as public domain)



We have already shown that the fundamental group of the complement of two unlinked circles is $\mathbb{Z} * \mathbb{Z}$. If a, b are the generators of $G = \pi_1(\mathbb{R}^3 \setminus (A \cup B))$ corresponding respectively to path classes going through A, B, then the path around the circle C corresponds to the path class $[a, b]$, which is nontrivial in G. Therefore the three rings cannot be pulled apart, even though any two are unlinked.

9. **Friday 2/28: More Van Kampen Examples**

9.1. **The Hawaiian earring.** In this example, let $\mathbb{N}$ denote the positive integers. For $n \in \mathbb{N}$, let A_n denote the circle in $\mathbb{R}^2$ of radius $1/n$ and center $(1/n, 0)$. The **Hawaiian earring** is the space

$$H = \bigcup_{n \in \mathbb{N}} A_n$$

with the subspace topology inherited from $\mathbb{R}^2$. It looks like a countably infinite wedge of circles (whose fundamental group is free on countably infinitely many generators), but in fact the fundamental group of H is much larger.

For every function $w : \mathbb{N} \to \mathbb{Z}$, there is a closed loop $f_w : I \to H$ that starts at the origin O and winds $w(n)$ times clockwise around A_n in the time interval $[\frac{n-1}{n}, \frac{n}{n+1}]$. (Of course, $f_w(1) = O$.) This function is continuous with respect to the subspace topology, and varying w gives uncountably many different elements of $\pi_1(H) = \pi_1(H, O)$. We can think of w as the "vector of winding numbers" of the loop f_w.

More explicitly, I think that the fundamental group $\pi_1(H, O)$ consists of words of infinite length in countably infinitely many letters (and their inverses), with the property that any particular letter occurs only finitely many times. In particular, every loop f has a finite winding number around any single circle, hence a well-defined vector of winding numbers $w(f) \in \mathbb{Z}^{\mathbb{N}}$. If we declare two loops to be equivalent if they have the same vectors of winding numbers, we obtain a quotient of $\pi_1(H, O)$ that is isomorphic to the direct product (not the direct sum!) of countably infinitely many copies of $\mathbb{Z}$. This quotient is called the *strong abelianization* and is a proper quotient of the usual abelianization (which would declare two loops f, g equivalent if they can be related by a *finite* sequence of interchanging adjacent letters — for which it is necessary, but not sufficient, that $w(f) = w(g)$).

9.2. **Fundamental groups of 2-dimensional cell complexes.** X = path-connected space
Y = space formed from X by attaching 2-cells $\{e_\alpha\}$ via maps $\{\phi_\alpha\}$
i = inclusion $X \hookrightarrow Y$

If $I \to \bar{e}_\alpha$ wraps once around the circle ∂e_α, then the composition $I \to \partial e_\alpha \xrightarrow{\phi_\alpha} X$ is a loop in X which we'll also call ϕ_α. Let p_α be its basepoint.

Now fix a basepoint p and a path γ_α in X from p to p_α for each α. Observe that

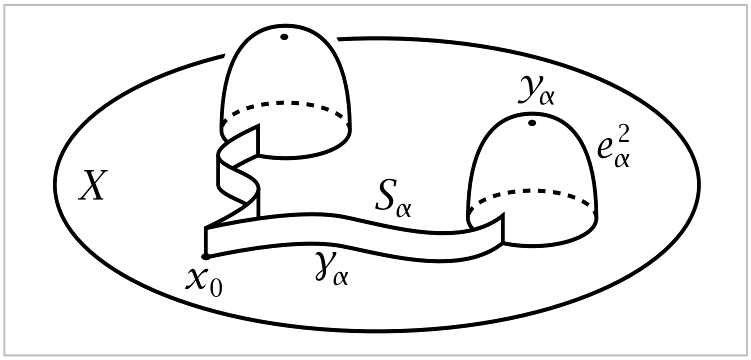
$$[\gamma_\alpha \cdot \phi_\alpha \cdot \overline{\gamma_\alpha}] \in \ker i_* : \pi_1(X, p) \to \pi_1(Y, p) \tag{9.1}$$

because ϕ_α becomes nullhomotopic once e_α is attached.

Proposition: The map i_* is surjective, and its kernel N is generated by loops of the form (9.1).

Proof. Surjectivity is easy. Given any loop $f : I \to Y$ and a cell e_α with a point in $\mathrm{im}(f)$, first homotope f to a path whose image omits at least one point from e_α, then push it onto the boundary circle by deformation-retracting away from the omitted point. (This is just like the argument that S^2 is simply-connected.) Doing this simultaneously for all 2-cells produces a path $f' : I \to X$ with $[f] = [f'] = i_*[f']$.

Consider the space Z shown below (figure taken from Hatcher, p.50)



- The point labeled x_0 is what I am calling p.
- The gumdrop-shaped things are the 2-cells e_α, attached along their boundaries in X.
- The ribbon-shaped thin S_α is a copy of $I \times I$, with the bottom edge attached to γ_α in X, the right edge attached to a little arc in the gumdrop e_α, and all the left edges identified together.

Thus Z deformation-retracts to Y by simultaneously squashing each S_α's onto its bottom and right edges.

Now we will use Van Kampen. For each α, pick a point y_α that is not in the arc along which S_α is attached. Let

$$A = Z \setminus \{y_\alpha\}, \qquad B = Z \setminus X.$$

Then $Z = A \cup B$. Moreover, A deformation-retracts onto X, and B is contractible (push the disks onto the arcs and then push everything to p). So Corollary 6.2 gives us a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(A \cap B, p) \arrow[r, "i_*"] & \pi_1(A, p) \arrow[r, "\Phi"] \arrow[d, "\cong"'] & \pi_1(Z, p) \arrow[r] \arrow[d, "\cong"'] & 0. \\ & & \pi_1(X, p) \arrow[r, "i_*"] & \pi_1(Y, p) & \end{tikzcd}$$

As for $A \cap B$, it looks like a bunch of punctured open disks held together by a wedge of strips. It deformation-retracts to a wedge of S^1's at p, and by the easy case of Van Kampen's Theorem the fundamental group of that wedge is the free group generated by the loops ϕ_α, which come from the loops $\gamma_\alpha \cdot \phi_\alpha \cdot \overline{\gamma_\alpha}$ in $A \cap B$. Therefore $\ker \Phi = N$. $\square$

10. Mon 3/3: Fundamental groups of 2-D cell complexes

Last time, we proved:

Proposition: Let X be path-connected and $p \in X$. Form Y from X by attaching 2-cells $\{e_\alpha\}$.
Fix points $p_\alpha \in \partial e_\alpha$ and paths γ_α from p to p_α.
Let ϕ_α generate $\pi_1(\partial e_\alpha, p_\alpha)$.

Then the inclusion $i : X \hookrightarrow Y$ induces a surjection $i_* : \pi_1(X, p) \twoheadrightarrow \pi_1(Y, p)$, and $\ker i_* = \langle\!\langle \gamma_\alpha \cdot \phi_\alpha \cdot \overline{\gamma_\alpha} \rangle\!\rangle$, where γ_α is a path from p to p_α. $\square$

A corollary is that we can write down the fundamental group of any 2-dimensional cell complex:

Theorem 10.1. *Let X be a 2-dimensional cell complex with a single vertex p and loops $\{e_1^\alpha\}$, so that $\pi_1(X^1, p)$ is the free group $*_\alpha \mathbb{Z}_\alpha$, with one generator ϕ_α for each loop. (Note that we have to fix the orientation of ϕ_α.) Then every 2-cell has its attaching map given by a sequence*

$$\phi_{\alpha_1}^{\pm 1} \phi_{\alpha_2}^{\pm 1} \cdots \phi_{\alpha_n}^{\pm 1}.$$

Then $\pi_1(X, p)$ is the quotient of the free group $*_\alpha \mathbb{Z}_\alpha$ by these words.

More generally, if X has finitely many vertices (and maybe even if it doesn't), we can contract a maximal tree in X to produce a complex with one vertex. Therefore, the corollary is sufficient to describe the fundamental groups of all 2-dimensional cell complexes with finitely many vertices (and possibly even all 2-dimensional cell complexes). In fact we can get any group this way:

Corollary 10.2. *For every group G, there exists a space X with $\pi_1(X) = G$.*

Proof. Fix a presentation of G via generators and relations. Construct a 2-dimensional cell complex with one vertex; one loop for each generator, and one 2-cycle attached along the loop corresponding to each relation. $\square$

Example: To get $\pi_1(X) = \mathbb{Z}/n\mathbb{Z}$, wrap a disk n times around a circle. To get the Klein four-group, start with a torus and wrap one disk twice around a meridional circle, then wrap another disk twice around a longitudinal circle. (Another space with the same fundamental group is $\mathbb{R}P^2 \times \mathbb{R}P^2$.)

Recall that a *topological n-manifold* is a space that is locally homeomorphic to $\mathbb{R}^n$, that is, every point has an open neighborhood U with $U \cong \mathbb{R}^n$. In a 2-dimensional cell complex X, this condition is certainly true for any point in the interior of a 2-cell. For points in the interior of a 1-cell, we can ensure a local homeomorphism to $\mathbb{R}^2$ by making sure that the attaching maps hit the interior exactly twice. For example, if X has one vertex, then we want to attach a 2-cell via a word using each loop (or its inverse) twice. (Although I'm not sure what a neighborhood of the vertex looks like in general.)

Example 10.3 (**The fundamental group of the torus).** Let T_g be the g-holed torus (also known as the *orientable surface of genus n*). As you proved last week (and as in the picture on p.5 of Hatcher), this surface has a cell structure with one vertex, $2g$ loops $a_1, b_1, \ldots, a_g, b_g$, and one 2-cell, attached along the loop $[a_1, b_1] \cdots [a_g, b_g]$. Therefore

$$\pi_1(T_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle .$$

In the special case $g = 1$, the relation $[a_1, b_1]$ generates the commutator subgroup and the quotient is $\mathbb{Z} \times \mathbb{Z}$. In general, the abelianization of $\pi_1(T_g)$ is $\mathbb{Z}^{2g}$, which shows that tori with different numbers of holes are not homotopy-equivalent, hence not homeomorphic.

Example 10.4 (**The Klein bottle and other nonorientable surfaces).** The standard construction of the Klein bottle K says immediately that its fundamental group is

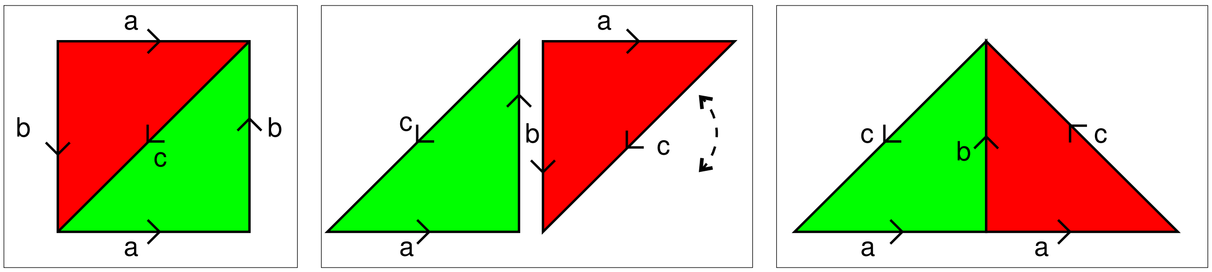
$$\pi_1(K) = \left\langle a, b \mid aba^{-1}b \right\rangle .$$

In the abelianization of this group, we have $aba^{-1}b = aba^{-1}b^{-1} = e$, which implies $b = b^{-1}$. Also, $ba = ab^{-1} = ab$, so a and b commute. So in fact

$$\mathrm{Ab}(\pi_1(K)) = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

It follows that K is not homotopy-equivalent to any torus T_n.

More generally, let N_g be the *nonorientable surface of genus g*, which is obtained by attaching a 2-cell to the wedge of g circles by the word $a_1^2 a_2^2 \cdots a_g^2$. So $N_1 = \mathbb{R}P^2$ and N_2 is the Klein bottle, although with a different presentation. Draw a diagonal across the usual square-with-sides-identified, cut along it, and reassemble the triangles along b, as shown on the right. This gives rise to the same space, and we can read off the fundamental group as $\langle a, c \mid a^2c^2\rangle$.



The equation $abc = e$ (from the green triangle) means that any two of the three loops a, b, c generate $\pi_1(K)$. Algebraically we have $b = ac$ (looking at the boundary of the red triangle), so

$$\left\langle a, b \mid aba^{-1}b \right\rangle = \left\langle a, c \mid a(ac)a^{-1}(ac) \right\rangle = \left\langle a, c \mid a^2c^2 \right\rangle.$$

Returning to arbitrary genus, we have

$$\pi_1(N_g) = \langle a_1, \ldots, a_g \mid a_1^2 \cdots a_g^2 \rangle,$$
$$\mathrm{Ab}(\pi_1(N_g)) = \mathbb{Z}^g/(2, 2, \ldots, 2) = \mathbb{Z}^{g-1} \times \mathbb{Z}/2\mathbb{Z}.$$

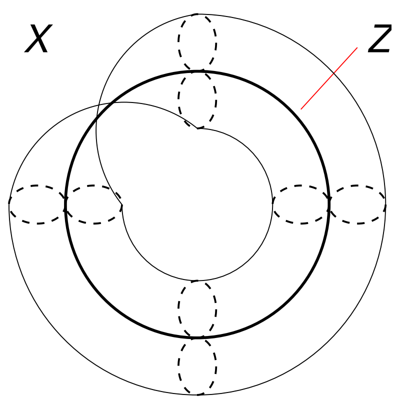
So the surfaces N_g are not homotopy-equivalent to each other or to the surfaces M_g.

This appearance of *torsion* (finite-order elements) is characteristic of nonorientable things. We'll see it again in the section on homology.

Example 10.5 (The "Möbius tube"). Let $Y = S^1 \vee S^1$, with p the wedge point. Let $\phi : Y \to Y$ be the map that swaps the two circles (preserving orientations), and let

$$X = Y \times I/(x, 0) \sim (\phi(x), 1).$$

What is $\pi_1(X, p)$?



Let $Z = \{p\} \times I \subset X$, so Z is a copy of S^1. We can put a cell structure on X with $X^1 = Y \cup Z$. Let a, b be the generators of $\pi_1(Y)$ and let c be the generator of $\pi_1(Z)$. Then X is formed from X^1 by attaching two 2-cells along the words $acbc^{-1}$ and $bcac^{-1}$. We have just computed

$$\pi_1(X,p) = \langle a,b,c \mid acbc^{-1},\ bcac^{-1}\rangle.$$

Notice that abelianizing this group would give

$$\mathbb{Z}\{a,b,c\}/\langle a+b\rangle \cong \mathbb{Z}^2.$$

If instead ϕ swapped the two circles but reversed the orientation of one of them, we would get a space with fundamental group

$$\langle a,b,c \mid acbc,\ bcac^{-1}\rangle$$

whose abelianization is

$$\mathbb{Z}\{a,b,c\}/\langle a+b+2c,\ a+b\rangle = \mathbb{Z}\{a,b,c\}/\langle 2c,\ a+b\rangle \cong \mathbb{Z}\oplus\mathbb{Z}_2.$$

And if it swapped the orientation on both circles we would get

$$\langle a,b,c \mid acbc,\ bcac\rangle$$

whose abelianization is

$$\mathbb{Z}\{a,b,c\}/\langle a+b+2c\rangle \cong \mathbb{Z}^2$$

(raising the question of whether these two spaces are homotopy-equivalent).

This is an instance of a *mapping torus* (see exercise #whatever in Hatcher). If we have a map $\phi : X \to X$, the mapping torus T_ϕ is defined as

$$T_\phi = X \times I/\Big((x,0) \sim (\phi(x),1)\ \forall x \in X\Big).$$

In general, if ϕ fixes some basepoint p, then $\pi_1(T_\phi)$ can be calculated from $\pi_1(X)$, since it is homotopy-equivalent (I think) to the space formed by first attaching a circle $Z \cong S^1$ (with fundamental group generated by z) to X, then attaching 2-cells to $X \vee S^1$ along the words $az\phi_*(a)z^{-1}$ for each generator a of $\pi_1(X)$.

11. Wed 3/5 – Fri 3/7: Covering Spaces: Definitions, Examples

Definition: Let X be a space. A **covering space** of X is a space $\tilde{X}$ together with a map $p : X \to \tilde{X}$ such that: X has an open cover $\{U_\alpha\}$ such that for every α, the preimage $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each one of which is mapped homeomorphically to U_α by p.

Technically p need not be onto — $p^{-1}(U_\alpha)$ could be a disjoint union of zero open sets — but that's fairly silly. In practice the interesting case is that p is onto and that X and $\tilde{X}$ are both path-connected.

Example: What are some covering spaces of the circle $X = S^1$?

- $C_1 = \tilde{X} = X$, $p = \mathbb{1}_X$. This is trivially a covering space.
- $C_\infty = \tilde{X} = \mathbb{R}$, $p(r) = e^{2\pi i r}$. The preimage of any little open arc U in S^1 is the union of countably infinitely many disjoint little open intervals in $\mathbb{R}$, each of which is mapped homeomorphically to U by p.
- $C_n = \tilde{X} = S^1$, $p(z) = z^n$. The preimage of an arc U is the disjoint union of n open intervals, each mapped homeomorphically to U by p.

It turns out that these are all the covering spaces. Note that some of the covering spaces are covering spaces of each other:

$$\begin{tikzcd}
C_{nm} = S^1 \arrow[dr, "z \mapsto z^n"] \arrow[dd, "z \mapsto z^{mn}"'] & \\
& C_m = S^1 \arrow[dl, "z \mapsto z^m"] \\
S^1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
C_\infty = \mathbb{R} \arrow[dr, "r \mapsto e^{2\pi i r/m}"] \arrow[dd, "r \mapsto e^{2\pi i r}"'] & \\
& C_m = S^1 \arrow[dl, "z \mapsto z^m"] \\
S^1 &
\end{tikzcd}$$

In fact, C_a covers C_b iff $b|a$. So the covering spaces of S^1 form a partially ordered set that is identical to the lattice of subgroups of $\mathbb{Z}$ — and of course $\mathbb{Z} = \pi_1(S^1)$. This is not an accident: the covering spaces of every space X form a lattice isomorphic to the lattice of subgroups of $\pi_1(X)$, with the top element (the *universal covering space*) corresponding to the unique simply-connected covering space.

1

X = topological space

Defn Covering space of X: space $\tilde{X}$, together with a map $p: \tilde{X} \to X$, such that:

(*) there exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U_α by p.

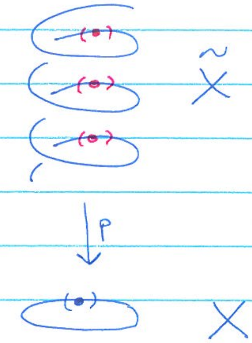
Fundamental Example 1

$X = S^1$ (coordinatized in $\mathbb{R}^2$)

$\tilde{X} = \mathbb{R}$

$p = ~~\exp~~$

$p(t) = (\cos 2\pi t, \sin 2\pi t)$



e.g. $p^{-1}(1,0) = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

and for any open arc $U \supseteq (1,0)$, $p^{-1}(U)$ is a disjoint, ~~countably infinite~~ union of countably infinitely many homeomorphic arcs.

In the definition of covering space:
$\{U_\alpha\}$ = any family of open arcs (two will do) whose union is S^1.

(2)

Another Fundamental Example (2)

$X = S^1$
$\tilde{X} = S^1$
p = "wrap around k times" for some $k \in \mathbb{Z}$:

$$p\big((\cos 2\pi t, \sin 2\pi t)\big) = (\cos 2\pi k t, \sin 2\pi k t)$$

or ~~instead~~ if we think of $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$
then $p(z) = z^k$.

Here, for an open arc U_α, $p^{-1}(U_\alpha) = k$ disjoint copies of U_α..

~~Note: In Ex. 1~~:

	$\pi_1(\tilde{X})$	$p_* : \pi_1(\tilde{X}) \to \pi_1(X)$	$\lvert \pi^{-1}(\cdot)$
Example 1	0	0	$\aleph_0$
Example 2	$\mathbb{Z}$	$x \mapsto kx$	k

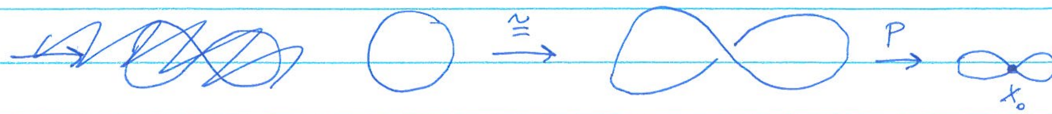
~~Questions? • Classify covering~~

3

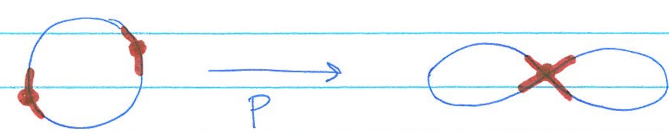
Yet Another Example

$X = 8$. What are the possible $\tilde{X}$?

- Not S^1! There's a well-defined map

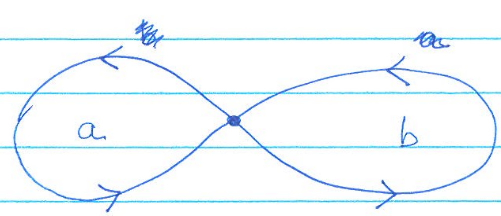


but x_0 does not have a neighborhood U ~~mapped~~ satisfying the condition of the $(*)$ defn of covering space.



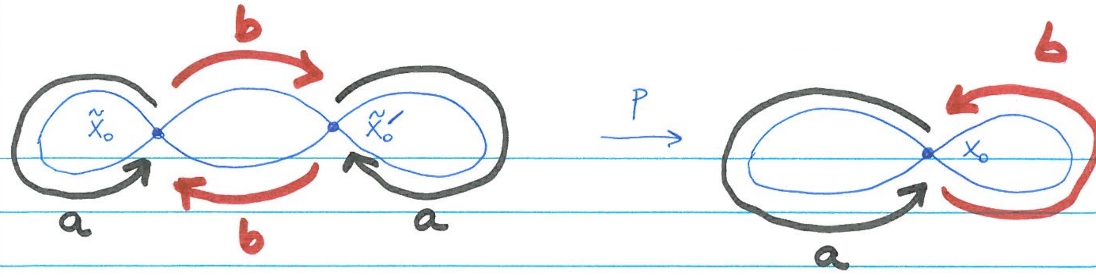
~~Possible covering spaces~~ To construct a covering space:

- Label circles as a, b, with orientation.



- $\tilde{X}$ should look like a cell complex in which ~~each~~ the edges are labeled a, b, and each vertex has ~~the same number of~~ two incident half-edges labeled a and two labeled b.

4



$\tilde{X}$ $\quad\quad\quad\quad$ X

Note that

- for every $x \in X$, $|p^{-1}(x)| = 2$
- the condition $(*)$ is satisfied.

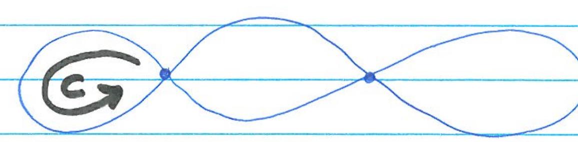
$x = x_0$: $U = \times$ $\quad$ $p^{-1}(U) = \times\times$

$x \neq x_0$: $U = \frown$ $\quad$ $p^{-1}(U) = \frown\frown$

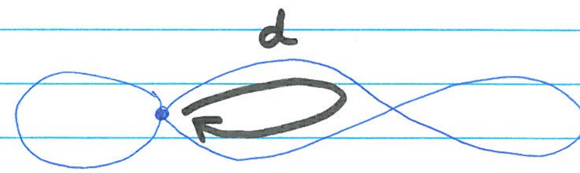
What is the map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$?

$\pi_1(\tilde{X}, \tilde{x}_0)$ = free group on c, d, e

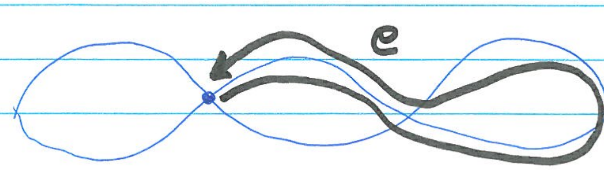
~~$p_*(c) = a$~~
~~$p_*(d) = b^2$~~
~~$p_*(e) = a$~~



c



d



e

5

~~the~~

$\mathbb{Z}\ \mathbb{Z}*\mathbb{Z} \cong \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{p_*} \pi_1(X, x_0) \cong \mathbb{Z}*\mathbb{Z}$

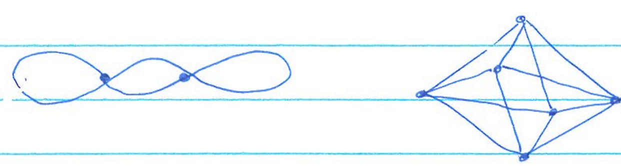
$$c \longmapsto a$$
$$d \longmapsto b^2$$
$$e \longmapsto bab^{-1}$$

Odd fact: p_* is 1-1.

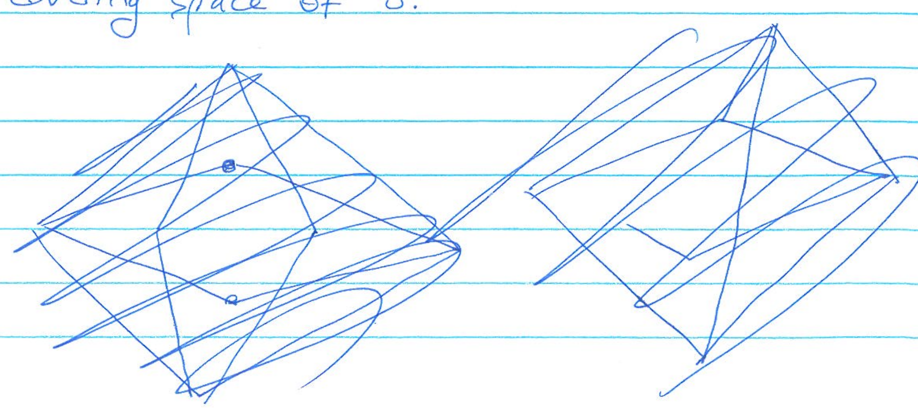
I.e., $\langle a, b^2, bab^{-1} \rangle \cong \mathbb{Z}^{*3}$ is a subgroup of $\mathbb{Z}^{*2}$. (Bizarre!)

Combinatorial side trip:

Let G be any finite graph with n vertices in which each vertex has degree 4.



Then G has a labeling that makes it into a covering space of 8.



12. Monday 3/10: Covering Spaces: Lifting Properties

Let $p : \mathbb{R} \to S^1$ be the covering map $p(r) = e^{2\pi i r}$. In the proof of $\pi_1(S^1) = \mathbb{Z}$, we showed that

1. For each path $f : I \to S^1$ with $f(0) = x_0$, and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \to \mathbb{R}$ with $\tilde{f}(0) = \tilde{x}_0$ and $f = p \circ \tilde{f}$.

2. For each path homotopy $F : I \times I \to S^1$, $f_t(s) = F(t,s)$, with $f_t(0) = x_0$ $(\forall t)$, and each $\tilde{x}_0 \in p^{-1}(x_0)$, the unique lift $\tilde{f}_0$ of f_0 extends to a unique lift $\tilde{f}_t : I \to \mathbb{R}$ with $\tilde{f}_t(0) = \tilde{x}_0$ $(\forall t)$ and $f_t = p \circ \tilde{f}_t$.

$\boxed{1}$
$$\begin{tikzcd} & \mathbb{R} \arrow[dd, "p"] \\ I \arrow[ur, dashed, "\tilde{f}"] \arrow[dr, "f"'] & \\ & S^1 \end{tikzcd}$$

$\boxed{2}$
$$\begin{tikzcd} & \mathbb{R} \arrow[dd, "p"] \\ I \arrow[ur, dashed, "\tilde{f}"] \arrow[dr, "f"'] & \\ & S^1 \end{tikzcd}$$

$\boxed{3}$
$$\begin{tikzcd} & X \arrow[dd, "p"] \\ Y \times I \arrow[ur, dashed, "\tilde{F}"] \arrow[dr, "F"'] & \\ & X \end{tikzcd}$$

Idea of proof: cover I or $I \times I$ with open sets U_α that are small enough, and then lift f one U_α at a time.

> **Key observation:** For "small enough", the covering-space condition suffices—if $p^{-1}(f(U))$ consists of disjoint homeomorphic copies of $f(U)$, then the argument goes through.

So the same proof carries over to covering spaces in general, giving the:

> **Proposition (Homotopy Lifting Property/HLP)**
>
> $(\tilde{X}, p)$ = covering space of X
> $f_t : Y \to X$ = homotopy
> $\tilde{f}_0 : \tilde{Y} \to X$ = lift of f_0.
>
> *Then* there is a unique homotopy $\tilde{f}_t$ extending $\tilde{f}_0$ and lifting f_t. (See $\boxed{3}$ above.)

Corollary (Path Lifting Property/PLP)

Take $Y = \{\bullet\}$ in HLP. Then f_t is a path in X and $\tilde{f}_0$ represents a choice of basepoint for the lift $\tilde{f}_t$. That is:

For every path $f : I \to X$ with basepoint x_0 and every $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \to \tilde{X}$ with $\tilde{f}(0) = \tilde{x}_0$.

(I'm abusing notation by writing $f(t) = f_t(\bullet)$.)

Corollary: Let $(\tilde{X}, p)$ be a covering space of X. Let $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$. Then the induced homomorphism

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$$

is **injective**.

Proof. Suppose $[\tilde{f}_0] \in \ker p_* \subseteq \pi_1(\tilde{X}, \tilde{x}_0)$. I.e.,

$$p_*[\tilde{f}_0] = [p \circ \tilde{f}_0] \simeq [k_{x_0}].$$

By the HLP, this lifts uniquely to a homotopy $\tilde{f}_t : I \to \tilde{X}$.

The map $\tilde{f}_1$ is the lift of a constant map, namely k_{x_0}. Therefore it is locally constant, and its domain I is connected. So in fact $\tilde{f}_1$ is constant, and $\tilde{f}_0$ is nullhomotopic. We have shown that $\ker p_*$ is trivial. $\square$

Problem: Understand the inclusion $H \subseteq G$, where

$$G = \pi_1(X, x_0), \quad H = p_*\big(\pi_1(\tilde{X}, \tilde{x}_0)\big).$$

Definition/Proposition: The function $x \mapsto |p^{-1}(x)|$ is also locally constant on X. If X is connected then this number is constant; it is called the **number of sheets** of $\tilde{X}$.

Proposition: The number of sheets of $\tilde{X}$ equals the index $[G : H]$.

Proof. Every $\omega \in \pi_1(X, x_0)$ lifts uniquely to a path class $\tilde{\omega}$ with $\tilde{\omega}(0) = \tilde{x}_0$. The subgroup H consists of loops ω such that $\tilde{\omega}(1) = \tilde{x}_0$. Two loop classes ω, ψ are in the same coset of H if and only if $\omega \cdot \bar{\psi} \in H$; lifting this loop to

$\tilde{X}$, we see that $\tilde{\omega}(1) = \tilde{\psi}(1)$. So the cosets of H are in bijection with the possible values of $\tilde{\omega}(1)$, i.e., the points in $p^{-1}(x_0)$. $\square$

Proposition (Unique Lifting Property/ULP):

$(\tilde{X}, p)$ = covering space of X
Y = connected space
$f : Y \to X$: continuous
$\tilde{f}, \tilde{f}'$: lifts of f

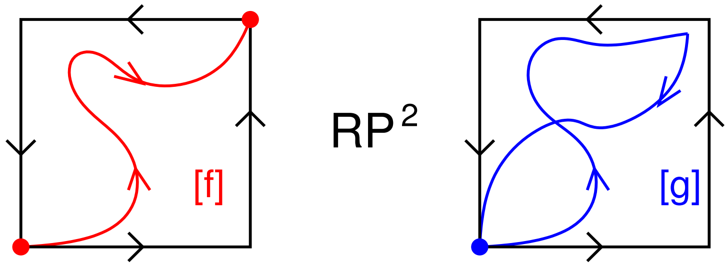
Then: If $\tilde{f}(y) = \tilde{f}'(y)$ for at least one point $y \in Y$, then $\tilde{f} \equiv \tilde{f}'$.

Fix (X, x_0). So far we have described a function

$$\Big\{\text{connected covering spaces } (\tilde{X}, \tilde{x}_0) \text{ of } X\Big\} \to \Big\{\text{subgroups of } \pi_1(X, x_0)\Big\}.$$

Big Result: This function is a bijection.

12.1. **The Lifting Criterion.** In general, when can a map $f : Y \to X$ be lifted to a map $Y \to \tilde{X}$? Consider these two maps $f, g : S^1 \to \mathbb{R}P^2$.



The red loop $[f]$ in $\mathbb{R}P^2$ will lift to a path between two antipodal points in S^2. If we regard the red loop as a function $f : S^1 \to \mathbb{R}P^2$, **it can't be lifted** to a function $\tilde{f} : S^1 \to S^2$. On the other hand, the black square lifts to a closed loop around (say) the equator in S^2.

We can detect the problem with the red path algebraically. Recall that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$. The red loop is the generator of this group; in terms of the function f, we have

$$f_*(\pi_1(S^1)) = \pi_1(\mathbb{R}P^2).$$

On the other hand, if $p : S^2 \to \mathbb{R}P^2$ is the covering map, then

$$p_*(\pi_1(S^2)) = p_*(0) = 0.$$

The obstruction to lifting is that

$$f_*(\pi_1(S^1)) \not\subseteq p_*(\pi_1(S^2)).$$

In other words, there's no appropriate path class in $\pi_1(S^2)$ to project down to $[f]$.

On the other hand, the blue path g is nullhomotopic in $\mathbb{R}P^2$, so that

$$g_*(\pi_1(S^1)) = 0 \subseteq p_*(\pi_1(S^2))$$

and it *is* possible to lift to a map $\tilde{g} : S^1 \to S^2$.

13. **Wed 3/12: Universal Covering Spaces**

Proposition (Lifting Criterion)

$(\tilde{X}, \tilde{x}_0, p)$ = covering space of (X, x_0)
(Y, y_0) = path-connected, locally path-connected
$f : (Y, y_0) \to (X, x_0)$: continuous

Then there exists a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ if and only if

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

$$\begin{tikzcd}
 & X \arrow[dd, "p"] \\
Y \arrow[ur, dashed, "\tilde{f}"] \arrow[dr, "f"'] & \\
 & X
\end{tikzcd}$$

Proof. ($\Longrightarrow$) If $\tilde{f}$ exists, then functoriality implies

$$f_*\pi_1(Y) = p_*\tilde{f}_*\pi_1(Y) \subseteq p_*\pi_1(\tilde{X}).$$

($\Longleftarrow$) If the inclusion condition holds, we can construct a lift $\tilde{f}$ as follows. For each $y \in Y$, let $\gamma = \gamma_y$ be a path in Y from y_0 to y. (Which path? Hold that thought.)

Then $f \circ \gamma$ is a path in X from x_0 to $f(y)$. By PLP, it lifts uniquely to a path $\widetilde{f \circ \gamma}$ in $\tilde{X}$ such that

$$\widetilde{f \circ \gamma}(0) = \tilde{x}_0,$$
$$\widetilde{f \circ \gamma}(1) = \text{some point in } p^{-1}(f(y)).$$

We now define $\tilde{f} : Y \to \tilde{X}$ by

$$\boxed{\tilde{f}(y) = \widetilde{f \circ \gamma_y}(1).}$$

I claim that for each y, this definition is independent of the choice of γ. If γ and γ' are two different paths, then $\gamma \cdot \overline{\gamma'} \in \pi_1(Y, y_0)$. Let

$$h_0 = f \circ (\gamma \cdot \overline{\gamma'}) = (f \circ \gamma) \cdot (f \circ \overline{\gamma'}).$$

By hypothesis, $[h_0] \in f_* \pi_1(Y) \subseteq p_* \pi_1(\tilde{X})$. I.e., it is path-homotopic to some $h_1 = p \circ \tilde{h}_1$.

Apply the PLP/ULP to lift h_1 to $\tilde{h}_1$, and then apply the HLP to lift h_0 to some $\tilde{h}_0 \simeq \tilde{h}_1$. Thus

$$\tilde{h}_0 = \left(\widetilde{f \circ \gamma}\right) \cdot \left(\widetilde{f \circ \overline{\gamma'}}\right)$$

is a loop at $\tilde{x}_0$, and

$$\tilde{h}_0(1/2) = \left(\widetilde{f \circ \gamma}\right)(1) = \left(\widetilde{f \circ \overline{\gamma'}}\right)(0) = \left(\widetilde{f \circ \gamma'}\right)(1)$$

which verifies the claim.

We still have to check that $\tilde{c}$ is continuous — I will skip that part. $\square$

Example: Let m, n be positive integers and consider the m- and n-fold coverings of $S^1 \subset \mathbb{C}$ by itself:

$$\begin{tikzcd} S^1 \arrow[dr, "z\mapsto z^n"'] & & S^1 \arrow[dl, "z\mapsto z^m"] \\ & X = S^1 & \end{tikzcd}$$

When does one of these maps lift to the other? In other words, if we think of $f(z) = z^n$ as an arbitrary map into X and think of $p(z) = z^m$ as a covering space map, then when does there exist a map $\tilde{f}$ such that $p \circ \tilde{f} = f$?

Complex-analytically, this means finding a function $\tilde{f} : S^1 \to S^1$ such that $(\tilde{f}(z))^m = z^n$ for all $z \in S^1$. If m divides n, then $\tilde{f}(x) = z^{n/m}$ does the trick; if not, no lift exists since we cannot define non-integer powers on S^1.

The Lifting Criterion agrees. If we identify $\pi_1(X)$ with $\mathbb{Z}$, then we have

$$f_* \pi_1(S^1) = n\mathbb{Z}, \qquad p_* \pi_1(S^1) = m\mathbb{Z}$$

and $n\mathbb{Z} \subseteq m\mathbb{Z}$ iff m divides n.

More generally, we have the following corollary of the Lifting Criterion:

Corollary: Let $p:(Y,y_0)\to(X,x_0)$ and $q:(Z,z_0)\to(X,x_0)$ be covering spaces. Then p lifts to a map $\tilde{p}:(Y,y_0)\to(Z,z_0)$ if and only if $p_*\pi_1(Y,y_0)\subseteq q_*\pi_1(Z,z_0)$.

$$\begin{tikzcd}
(Y,y_0) \arrow[rr, dashed, "\tilde{p}"] \arrow[dr, "p"'] & & (Z,z_0) \arrow[dl, "q"] \\
& (X,x_0) &
\end{tikzcd}$$

In this case, $\tilde{p}$ itself is a covering space map.

What happens if $p_*\pi_1(Y,y_0)=q_*\pi_1(Z,z_0)$? In this case *both* maps lift:

$$\begin{tikzcd}
(Y,y_0) \arrow[rr, "\tilde{p}"'] \arrow[dr, "p"'] & & (Z,z_0) \arrow[dl, "q"] \arrow[ll, bend right, "\tilde{q}"'] \\
& (X,x_0) &
\end{tikzcd}$$

In this case $\tilde{p}$ and $\tilde{q}$ are in fact homeomorphisms, determined by the choice of basepoints y_0 and z_0. Therefore:

Proposition: For two basepointed covering spaces (Y,y_0,p) and (Z,z_0,q) of (X,x_0), the following conditions are equivalent:

(1) There is a homeomorphism $\phi:(Y,y_0)\to(Z,z_0)$ such that $p=q\circ\phi$.
(2) $p_*\pi_1(Y,y_0)=q_*\pi_1(Z,z_0)$.

In this case we say that Y and Z are **isomorphic** as (basepointed) covering spaces.

We have proven that every basepointed covering space of X, up to isomorphism, is determined by a subgroup of $\pi_1(X,x_0)$.

Definition: A *universal cover of* X is a covering space Y that is simply-connected. In this case, the condition of the corollary is always met. Therefore, Y covers every covering space of X (hence the name).

Space	Universal cover
S^1	$\mathbb{R}$
$\mathbb{R}P^2$	S^2
$S^1 \vee S^1$	The infinite fractal space
Anything simply-connected	Itself

Question: Does every subgroup of $\pi_1(X, x_0)$ determine a covering space?

Theorem: Let X be path-connected, locally path-connected, and semilocally simply-connected. Then X has a universal covering space that is unique up to covering space isomorphism.

Proof. We construct a universal cover of (X, x_0) as follows. Let

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path } I \to X \text{ with } \gamma(0) = x_0\}$$

where as usual $[\gamma]$ denotes the path-homotopy class of γ. As defined, this is merely a set. Let $p : \tilde{X} \to X$ be the surjective map given by

$$p([\gamma]) = \gamma(1).$$

Suppose $U \subseteq X$ is simply-connected and path-connected, and let $x_1 \in U$. Consider the set

$$U_{[\gamma]} = \{[\gamma \cdot \alpha] \mid \alpha : I \to U,\ \alpha(0) = x_1\} \ \subset \ \tilde{X}.$$

Then p restricts to a bijection $U|_{[\gamma]} \to U$.

Lemma 13.1. *If* $[\gamma'] \in U_{[\gamma]}$ *then* $U_{[\gamma']} = U_{[\gamma]}$.

Proof. Say $[\gamma'] = [\gamma \cdot \alpha]$. Then any $[\beta] \in U_{[\gamma']}$ is of the form $[\beta] = [\gamma' \cdot \alpha'] = [\gamma \cdot (\alpha \cdot \alpha')] \in U_{[\gamma]}$ and any $[\delta] \in U_{[\gamma]}$ is of the form $[\delta] = [\gamma \cdot \eta] = [\gamma' \cdot (\bar{\alpha} \cdot \eta)] \in U_{[\gamma']}$. $\square$

If X is semilocally simply-connected, then it has a basis consisting of simply-connected sets U. We can then define a topology on $\tilde{X}$ by declaring all the sets $U_{[\gamma]}$ to be open; in fact they form a basis for a topology on $\tilde{X}$ (proof omitted). The map p is continuous. In fact it is a covering space map, because the Lemma says that $\tilde{X}$ can be partitioned into open connected subsets, each of which is homeomorphic to U.

More next time. $\square$

14. **Fri 3/12: Deck Transformations**

Theorem: Let X be path-connected, locally path-connected, and semi-locally simply-connected (i.e., it has a basis consisting of simply-connected open sets). Then X has a universal covering space $\tilde{X}$ that is unique up to covering space isomorphism.

Proof. Last time we defined

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path } I \to X \text{ with } \gamma(0) = x_0\}$$

where $[\gamma]$ denotes the path-homotopy class of γ. We put a topology on $\tilde{X}$ that made the map

$$p : \tilde{X} \to X, \qquad p([\gamma]) = \gamma(1)$$

into a covering space map.

<u>$\tilde{X}$ is path-connected.</u>

If $[\gamma]$ and $[\gamma']$ are paths in $\tilde{X}$, then let α be a path in X from $\gamma(0)$ to $\gamma(1)$, and let α_t be the restriction of α to $[0, t]$. Then $\Gamma(t) = [\gamma \cdot \alpha_t]$ is a path in $\tilde{X}$ from $[\gamma]$ to $[\gamma']$.

<u>$\tilde{X}$ is simply-connected.</u>

For the basepoint, use $\tilde{x}_0 = [k_{x_0}]$.

Loop in $\tilde{X}$: a function $\Gamma : I \to \tilde{X}$ such that $\Gamma(0) = \Gamma(1) = [k_{x_0}]$.

Note that $p \circ \Gamma : I \to X$ is a loop γ based at x_0. Again, let γ_t be the restriction of γ to $[0, t]$, i.e., $\gamma_t(s) = \gamma(ts)$. Then

$$t \mapsto [\gamma_t]$$

is a path in $\tilde{X}$ that lifts γ (since $p(\gamma_t) = \gamma_t(1) = \gamma(t)$) and starts at $\tilde{x}_0$. By the ULP, we have $\Gamma(t) = [\gamma_t]$. Therefore

$$\gamma_1 = \gamma = p \circ \Gamma \simeq \gamma_0 = k_{x_0}$$

which says that $p_*[\Gamma] = 0$ in $\pi_1(X)$. But p_* is injective, so $[\Gamma] = 0$ in $\pi_1(\tilde{X})$. $\square$

Theorem: Under the same conditions, the isomorphism classes of covering spaces of X are in bijection with the subgroups of $\pi_1(X)$.

Proof. We need only show that for every subgroup $G \subseteq \pi_1(X)$, there is a covering space of $\pi_1(X)$ having G as its fundamental group. Let $\tilde{X}$ be the universal cover of X and

$$X_G \;=\; \tilde{X} \;/\; [\gamma] \sim [\gamma'] \text{ if } p(\gamma) = p(\gamma') \text{ and } \gamma\overline{\gamma'} \in G.$$

This is an equivalence relation because G is a group (details omitted), and the quotient maps $\tilde{X} \to X$ factors through X_G. Moreover, closed loops in X_G push forward to loops in G — i.e., $q_*\pi_1(X_G) = G \subseteq \pi_1(X)$. □

A quick application:

Proposition 14.1. *$\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ for every $n \geq 2$.*

Proof. The map $p : S^n \to \mathbb{R}P^n$ that identifies antipodal points is a 2-sheeted covering space map. Meanwhile, S^n is simply-connected (hence the universal cover of $\mathbb{R}P^n$) and so

$$[\pi_1(\mathbb{R}P^n) : p_*\pi_1(S^n)] = |\pi_1(\mathbb{R}P^n)| = 2.$$ □

14.1. **Deck Transformations.** Throughout, let $(\tilde{X}, \tilde{x}_0, p)$ be a covering space of (X, x_0), and let

$$G = \pi_1(X, x_0), \quad H = p_*\pi_1(\tilde{X}, \tilde{x}_0).$$

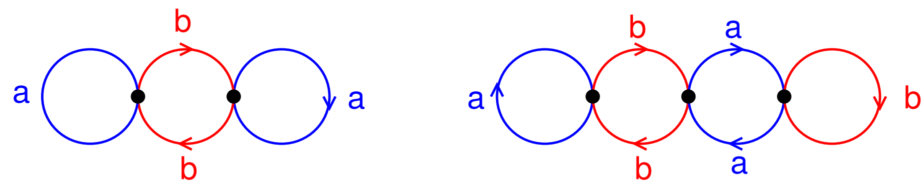
Definition: A *deck transformation* of $\tilde{X}$ is an automorphism as a covering space. The deck transformations form a group denoted $G(\tilde{X})$. Since an automorphism is determined by what it does to a single point, we can regard $G(\tilde{X})$ as a subgroup of the group of permutations of $p^{-1}(x_0)$.

Example: Let $X = S^1$. For $\tilde{X} = \mathbb{R}$ and $p(r) = e^{2\pi i r}$, the deck transformations of $\tilde{X}$ are the maps $r \mapsto r+n$ for $n \in \mathbb{N}$. These are autohomeomorphisms of $\mathbb{R}$ that restrict to permutations of each fiber of the covering map (namely the cosets of $\mathbb{Z}$ in $\mathbb{R}$). Here we have $\tilde{G} = \mathbb{Z}$.

If $\tilde{X} = S^1$ and $p(z) = z^n$, then the deck transformations are the maps $z \mapsto ze^{2\pi i j/n}$ for $j \in \{0, \ldots, n-1\}$. In particular $\tilde{G} = \mathbb{Z}_n$.

In both these cases, the deck transformations act *transitively* on $p^{-1}(x_0)$.

Example: Let $X = S^1 \vee S^1$, so $G = \langle a, b \rangle$, and consider these two covering spaces.



In the first case, $\tilde{G} = \mathbb{Z}_2$ — its two elements are the identity and rotation by $180°$. Also, we have

$$H = \langle a, b^2, bab \rangle, \quad G/H = \langle b | b^2 \rangle = \mathbb{Z}_2$$

no matter which point we choose as $\tilde{x}_0$.

In the second case, again $\tilde{G} = \mathbb{Z}_2$. But this is a three-sheeted covering, and the points of $p^{-1}(x_0)$ fall into two orbits — the center vertex and the pair of non-central ones. Choosing the center point as $\tilde{x}_0$ gives

$$H = \langle a^2, b^2, aba, bab \rangle$$

while choosing the left-hand point as $\tilde{x}_0$ gives

$$H = \langle a, b^2, ba^2b, babab \rangle.$$

These are distinct conjugate subgroups in G.

On the other hand, consider the cover of $S^1 \vee S^1$ by the infinite square grid. Here the deck transformations are clearly translations by vectors in $\mathbb{Z}^2$, and we have

$$H = \langle aba^{-1}b^{-1} \rangle, \qquad \tilde{G} \cong G/H = \mathrm{Ab}(\mathbb{Z} * \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}.$$

What we are seeing is that symmetry in the covering space corresponds to H being normal. Specifically:

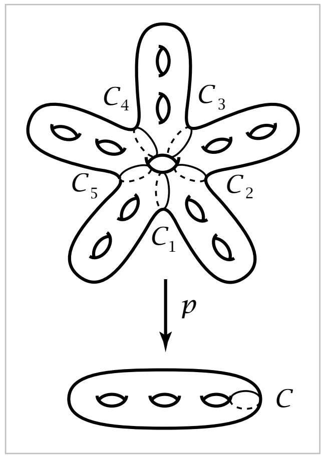
Theorem: For every covering space $(\tilde{X}, p)$ of X, we have

$$G(\tilde{X}) \cong N(H)/H$$

where $H = p_* \pi_1(\tilde{X})$ and $N(H)$ is its *normalizer*, i.e., the largest subgroup of $\pi_1(X)$ in which H is normal. We say that $\tilde{X}$ is *normal* if $N(H) = G$, i.e., H is normal in G. This is equivalent to condition that the action of $\tilde{G}$ on each fiber of p is *transitive*.

This machinery can be used to reduce the study of covering space actions to the study of concrete geometric symmetries. For example, here's a question: Which orientable surfaces cover each other?

Example: Let q, r be positive integers. Consider the orientable surface M_{qr+1}, drawn as a "starfish" with q arms with r holes each, plus one center hole. The figure below (taken from Hatcher, p.73) is the case $r = 2$ and $q = 5$.



The rotations about the center form a group G of autohomeomorphisms that is cyclic of order q. We can view these maps as deck transformations. Quotienting by that group — i.e., identifying each orbit of G with a single point — produces the orientable surface M_{r+1}, as shown. This means that we have injective maps

$$p_* : \pi_1(M_{qr+1}) \hookrightarrow \pi_1(M_{r+1})$$

for every $q, r \in \mathbb{N}$. Note that these are not normal covering spaces!

For a real simple example we can try to write down, take $q = 2$ and $r = 1$. We get a map

$$\pi_1(M_3) \hookrightarrow \pi_1(M_2).$$

The presentations for these groups, remember, are

$$\pi_1(M_2) = \langle a, b, c, d | [a, b][c, d] \rangle, \qquad \pi_1(M_3) = \langle u, v, w, x, y, z | [u, v]]w, x][y, z] \rangle$$