Math 821 Problem Set #7
Due date: Friday, May 2

Problem #1 [Hatcher p.156 #9abc] Compute the homology groups of the following spaces:

(a) The quotient of $\mathbb{S}^2$ obtained by identifying the north and south poles to a point.

(b) $\mathbb{S}^1 \times (\mathbb{S}^1 \vee \mathbb{S}^1)$.

(c) The space obtained from D^2 by first deleting the interiors of two disjoint subdisks, and then identifying all three resulting circles together via homomorphisms preserving clockwise orientations of these circles.

Solution: (a) Let A consist of the north and south poles in $\mathbb{S}^2$ and let $X = \mathbb{S}^2/A$. The pair $(\mathbb{S}^2, A)$ is good (because A is a subcomplex in the standard cell structure on $\mathbb{S}^2$), so $\tilde{H}(X = H(\mathbb{S}^2, A)$. The interesting part of the LES is

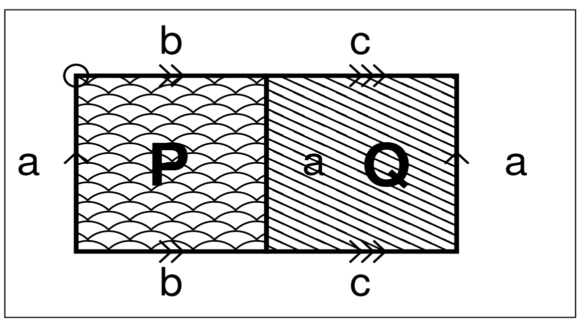
$$\underbrace{\tilde{H}_2(A)}_{0} \to \underbrace{\tilde{H}_2(\mathbb{S}^2)}_{\mathbb{Z}} \to \tilde{H}_2(X) \to \underbrace{\tilde{H}_1(A)}_{0} \to \underbrace{\tilde{H}_1(\mathbb{S}^2)}_{0} \to \tilde{H}_1(X) \to \underbrace{\tilde{H}_0(A)}_{\mathbb{Z}} \to \underbrace{\tilde{H}_0(\mathbb{S}^2)}_{0}$$

from which we get

$$\tilde{H}_2(X) = \mathbb{Z}, \quad \tilde{H}_1(X) = \mathbb{Z}, \quad \tilde{H}_0(X) = 0$$

(the last because X is path-connected).

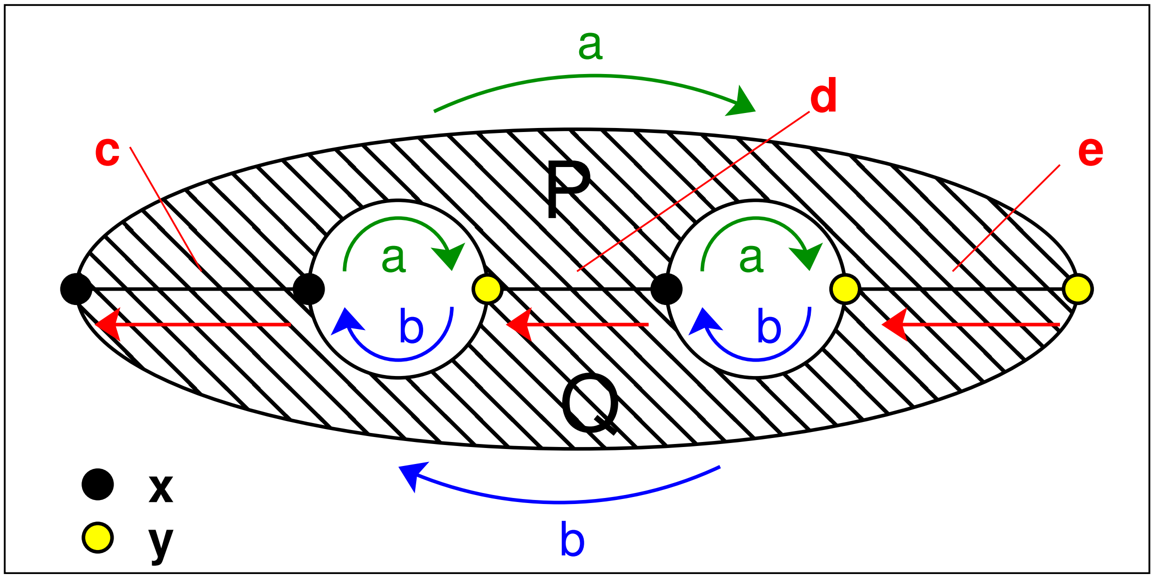
(b) Let's use cellular homology. The space $Y = \mathbb{S}^1 \times (\mathbb{S}^1 \vee \mathbb{S}^1)$ can be regarded as two tori joined along their meridional circles, so the cell structure looks like this:



All the cellular chain maps are zero, so the unreduced homology groups are free of order equal to the number of cells of that dimension. Therefore

$$\tilde{H}_2(Y) = \mathbb{Z}^2, \quad \tilde{H}_1(Y) = \mathbb{Z}^2, \quad \tilde{H}_0(X) = 0.$$

(c) Again let's use cellular homology. Call this space W. It can be endowed with the cell structure below, with $E_2 = \{P\}$, $E_1 = \{a, b, c, d\}$, $E_0 = \{x, y\}$ and orientations as indicated:



Starting at the right-hand vertex and moving down, we see that $\partial P = b-c-b-d-b-a+d-a+c+a = -b-a$. Thus $\partial P = a+e-a+d-a+c = -a+c+d+e$ and $\partial Q = e+b+d+b+c-b = e+d+c+b$, so the full cellular chain complex $E_\bullet(W)$ is

$$\mathbb{Z}E^2 \xrightarrow{d_2} \mathbb{Z}E^1 \xrightarrow{\qquad\qquad d_1 \qquad\qquad} \mathbb{Z}E^0 \to 0$$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} P \\ \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 \end{pmatrix} \end{array}$$

We can then compute homology in Macaulay2, e.g.:

```
d2 = matrix {{-1}, {-1}, {0}, {0}};
d1 = matrix {{-1,1,0,-1}, {1,-1,0,1}};
C  = chainComplex(d1,d2);
prune HH C
```

The end result:

$$\tilde{H}_2(W) = \mathbb{Z}, \qquad \tilde{H}_1(W) = \mathbb{Z}^2, \qquad \tilde{H}_0(W) = 0.$$

Problem #2 [Hatcher p.157 #19] Compute $H_i(\mathbb{R}P^n/\mathbb{R}P^m)$ for $m < n$ by cellular homology, using the standard CW structure on $\mathbb{R}P^n$ with $\mathbb{R}P^m$ as its m-skeleton.

Solution: The relative cellular chain complex $E_\bullet(\mathbb{R}P^n, \mathbb{R}P^m)$ is the cokernel of the injection $E_\bullet(\mathbb{R}P^m) \hookrightarrow E_\bullet(\mathbb{R}P^n)$. Thus its groups are

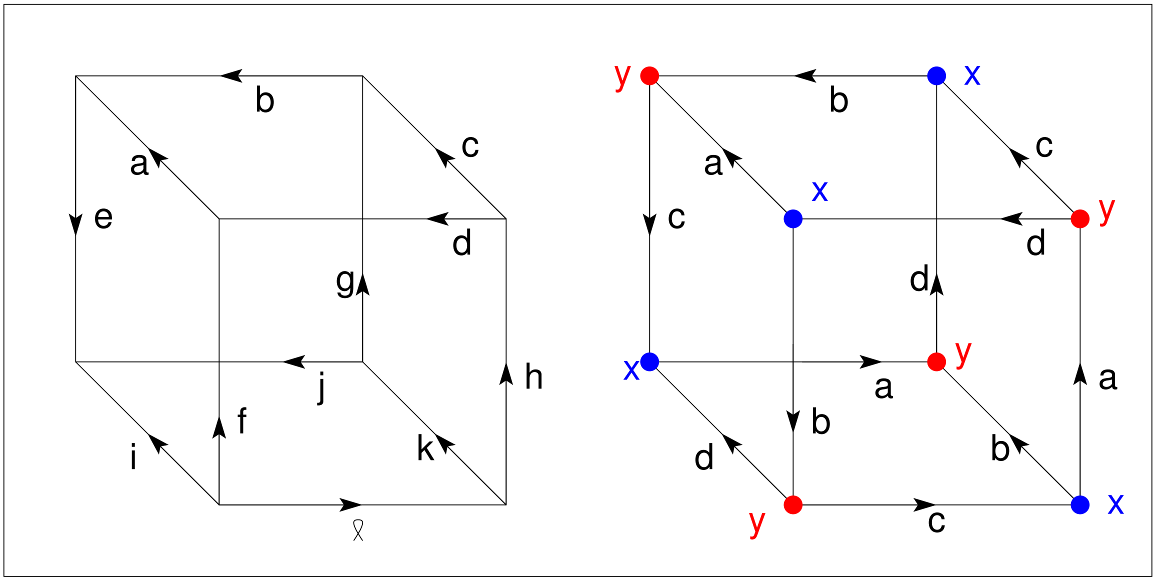
$$E_k(\mathbb{R}P^n, \mathbb{R}P^m) = \begin{cases} \mathbb{Z} & \text{if } m+1 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

and its maps are the same as for $\mathbb{R}P^n$: i.e., the map $d_k : E_k \to E_{k-1}$ is multiplication by 2 if k is even, or zero if k is odd. Therefore,

$$\boxed{H_k(\mathbb{R}P^n, \mathbb{R}P^m) = \begin{cases} \mathbb{Z} & \text{if } k = n \text{ and } n \text{ is odd,} \\ \mathbb{Z}_2 & \text{if } m+1 \le k < n \text{ and } k \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}}$$

Problem #3 [Hatcher p.156 #11] Let K be the 3-dimensional cell complex obtained from the cube I^3 by identifying each pair of opposite faces via a one-quarter twist. (See exercise #14 on p.54.) Compute the homology groups $\tilde{H}_n(K;\mathbb{Z})$ and $\tilde{H}_n(K;\mathbb{Z}_2)$ for $n > 0$.

Solution: Call this complex K. We start with the oriented labeled cube as shown.



Identifying opposite faces with the twist brings us down to two vertices x, y (indicated by their colors in the diagram) and forces the edge identifications

$$(b, -a, -d, c) \sim (k, j, -i, -\ell), \quad (e, -a, -f, i) \sim (-c, -h, k, g), \quad (\ell, f, -d, -h) \sim (e, -b, -g, j).$$

Calling the top, left and front 2-faces respectively A, B, C and the solid cube Q., we get the cellular chain complex

$$\mathbb{Z}E^3 \xrightarrow{d_3} \mathbb{Z}E^2 \xrightarrow{\qquad d_2 \qquad} \mathbb{Z}E^1 \xrightarrow{\qquad d_1 \qquad} \mathbb{Z}E^0 \to 0.$$

$$\begin{array}{c|c} & Q \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{array} \qquad \begin{array}{c|c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \end{array}$$

(It is perhaps not clear geometrically whether the coefficients of ∂Q should be 0, 2 or -2, but it is clear from the algebra — the condition $d_2 d_1 = 0$ says that d_2 must be orthogonal to every row of d_1, and d_1 has full row rank so the only possibility for d_2 is the zero vector.) We can compute the homology groups in Macaulay:

```
d3 = matrix {{0}, {0}, {0}};
d2 = matrix {{-1,1,1}, {1,-1,1}, {1,1,1}, {-1,-1,1}};
d1 = matrix {{-1,-1,1,1},{1,1,-1,-1}};
C  = chainComplex(d1,d2,d3);
prune HH C
```

The result is

$$\boxed{\tilde{H}_3(K) = \mathbb{Z}, \quad \tilde{H}_2(K) = 0, \quad \tilde{H}_1(K) = \mathbb{Z}_2^2.}$$

Now for homology with $\mathbb{Z}_2$-coefficients. Note that it is **not** sufficient simply to tensor these groups with $\mathbb{Z}_2$. Instead, we have to tensor the chain complex with $\mathbb{Z}_2$ and then recompute homology. The Macaulay symbol for tensor product is $**$, so all you need to do is this:

```
C2  = C ** ZZ/2
prune HH C2
```

The result is

$$\boxed{\tilde{H}_3(K;\mathbb{Z}_2) = \mathbb{Z}_2, \quad \tilde{H}_2(K;\mathbb{Z}_2) = \mathbb{Z}_2^2, \quad \tilde{H}_1(K) = \mathbb{Z}_2^2.}$$

Indeed, note that $\tilde{H}_2(K;\mathbb{Z}_2) \neq \tilde{H}_2(K;\mathbb{Z}) \otimes \mathbb{Z}$.

It is also possible to use the Universal Coefficient Theorem for Homology [Hatcher, p.264], which says that there are split short exact sequences $0 \to H_n(K) \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to H_n(K;\mathbb{Z}_2) \to \operatorname{Tor}(H_{n-1}(K), \mathbb{Z}_2) \to 0$. Using general facts about tensor product — in particular, tensoring a module with itself or the ground ring does nothing to it, and tensor is distributive over direct sum — we get

$$\begin{gathered} 0 \to \mathbb{Z}_2 \to H_3(K;\mathbb{Z}_2) \to \operatorname{Tor}(0, \mathbb{Z}_2) \to 0, \\ 0 \to H_2(K;\mathbb{Z}_2) \to \operatorname{Tor}(\mathbb{Z}_2^2, \mathbb{Z}_2) \to 0, \\ 0 \to \mathbb{Z}_2^2 \to H_1(K;\mathbb{Z}_2) \to \operatorname{Tor}(\mathbb{Z}, \mathbb{Z}_2) \to 0. \end{gathered}$$

Using standard properties of Tor [Hatcher, Prop. 3A.5, p.265], we can rewrite these short exact sequences:

$$\begin{gathered} 0 \to \mathbb{Z}_2 \to H_3(K;\mathbb{Z}_2) \to 0, \\ 0 \to H_2(K;\mathbb{Z}_2) \to \mathbb{Z}_2^2 \to 0, \\ 0 \to \mathbb{Z}_2^2 \to H_1(K;\mathbb{Z}_2) \to 0 \end{gathered}$$

so the middle maps are all isomorphisms.

Problem #4 [Hatcher p.157 #20,22] In this problem χ denotes Euler characteristic.

(a) Let X, Y be finite CW-complexes. Show that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

(b) Let X be a finite CW complex and let $\tilde{X} \xrightarrow{p} X$ be an n-sheeted covering space. Show that $\chi(\tilde{X}) = n \cdot \chi(X)$.

Note: Part (a) is easy, but part (b) is much harder and relies on facts about Lebesgue numbers of coverings, which I pushed under the rug when I talked about subdivision. So I awarded full credit for doing (a) correctly and saying something intelligent about (b).

Solution: (a) Define the f-polynomial of X to be $f(X,q) = \sum_{e_\alpha \in X} q^{\dim e_\alpha}$. Then $\chi(X) = f(X,-1)$ and $f(X \times Y, q) = f(X,q) \cdot f(Y,q)$ (because the product of a k-cell and an ℓ-cell is a $(k+\ell)$-cell), so

$$\chi(X \times Y) = f(X \times Y, -1) = f(X,-1)f(Y,-1) = \chi(X) \cdot \chi(Y).$$

(b) Let $\mathcal{U} = \{U_i\}_{i \in \mathscr{I}}$ be an open cover of X such that each $p^{-1}(U_i)$ is a disjoint union of n homeomorphic copies of U_i; we may assume that $\mathscr{I}$ is finite, say $\{1, \ldots, m\}$, since X is compact. Also, abbreviate $\tilde{U}_i = p^{-1}(U_i)$.

Repeatedly barycentrically subdivide the cells of X until every cell is contained in some U_i. (This is possible by a Lebesgue number argument; see p.123 of Hatcher.) Call the resulting Δ-complex Δ. We can now endow $\tilde{X}$ with the structure of a Δ-complex $\tilde{\Delta}$ whose simplices are the components of the preimages of the simplices in Δ. We then have $f(\tilde{X}, q) = n \cdot f(X,q)$ and so

$$\chi(\tilde{X}) = \chi(\tilde{\Delta}) = f(\tilde{\Delta}, -1) = n \cdot f(\Delta, q) = n \cdot \chi(X).$$

Note: The reason this subdivision argument is necessary is that there is no guarantee that $p^{-1}(\sigma)$ is homeomorphic to n copies of σ for every cell σ. Another possible avenue of attack, which I cannot get to work, is as follows. The covering map p induces a map of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{i=1}^m C_{k+1}(\tilde{U}_i) \arrow[r, "\partial"] \arrow[d, "p_\sharp"'] & \bigoplus_{i=1}^m C_k(\tilde{U}_i) \arrow[r, "\partial"] \arrow[d, "p_\sharp"'] & \bigoplus_{i=1}^m C_{k-1}(\tilde{U}_i) \arrow[r] \arrow[d, "p_\sharp"'] & \cdots \\
\cdots \arrow[r] & \bigoplus_{i=1}^m C_{k+1}(U_i) \arrow[r, "\partial"] & \bigoplus_{i=1}^m C_k(U_i) \arrow[r, "\partial"] & \bigoplus_{i=1}^m C_{k-1}(U_i) \arrow[r] & \cdots
\end{tikzcd}$$

where the horizontal maps are the direct sums of the boundary maps on the summands. Since $\{\tilde{U}_i\}$ and $\{U_i\}$ are open covers of $\tilde{X}$ and X respectively, the Subdivision Lemma says that the rows compute their homology groups. Certainly $C_k(\tilde{U}_i) = C_k(U_i)^{\oplus n}$ and $H_k(\tilde{U}_i) = H_k(U_i)^{\oplus n}$. However, while the **groups** in this diagram break as direct sums, the **chain complexes** don't — if $U_i \cap U_j \neq \emptyset$ then the map $\partial : C_k(U_i) \to C_{k-1}(U_j)$ is not zero. Indeed, it is not true that $H_k(\tilde{X}) \cong H_k(X)$ in general; for example, take $X = \mathbb{S}^1$ and $\tilde{X}$ to be its connected n-fold covering space, which is also homeomorphic to $\mathbb{S}^1$.

Problem #5 [Hatcher p.155 #2, modified] Given a map $f : \mathbb{S}^{2n} \to \mathbb{S}^{2n}$, show that there is some point $x \in \mathbb{S}^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. (Hint: Use the fact that $\mathbb{S}^{2n}$ is a covering space of $\mathbb{R}P^{2n}$.)

Solution: For the first part, suppose that there exists a function f without the given property. Then we can construct a nonzero tangent vector field $T(x)$ on $\mathbb{S}^{2n}$ as follows: let $g(x)$ be the projection of $f(x)$ onto the tangent hyperplane to $\mathbb{S}^{2n}$ at x, and let $T(x)$ be the vector from x to $g(x)$. Note that $f(x) \notin \{x, -x\}$ implies that $g(x) \neq x$, so $T(x) \neq 0$. We have constructed an everywhere-nonzero tangent vector field on $\mathbb{S}^{2n}$, which is impossible by the Hairy Bowling Ball Theorem.

(An equally good solution is to incorporate the argument of the Hairy Bowling Ball Theorem explicitly. Consider

$$F(x,t) = \frac{x \cos t + f(x) \sin t}{\|x \cos t + f(x) \sin t\|}$$

which is well-defined because x and $f(x)$ are linearly independent. Then $F(x,0) = x/\|x\| = x$ and $F(x,\pi) = -x/\|x\| = -x$, so F is a homotopy between the identity and antipodal maps, a contradiction because their degrees are 1 and -1 respectively. This is essentially the argument for property (g) of degree (Hatcher, p.134); you could even just cite that property.)

For the second part, let $p : \mathbb{S}^{2n} \to \mathbb{R}P^{2n}$ be the usual two-to-one covering space map that identifies antipodal points of $\mathbb{S}^{2n}$, so that $p(x) = p(-x)$. For any $g : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$, we have a commutative diagram

$$\begin{tikzcd} \mathbb{S}^{2n} \arrow[r, "f"] \arrow[d, "p"] \arrow[dr, "g \circ p"] & \mathbb{S}^{2n} \arrow[d, "p"] \\ \mathbb{R}P^{2n} \arrow[r, "g"] & \mathbb{R}P^{2n} \end{tikzcd}$$

where f is obtained by lifting $g \circ p$ (using the Lifting Criterion, Prop. 1.33, p.61 of Hatcher). (Note that we do not have to construct f explicitly!) By the first part of the problem, there is a point $x \in \mathbb{S}^{2n}$ with $f(x) = x$ or $-x$, so $p(x)$ is a fixed point of g.

Problem #6 [Hatcher p.157 #28(a), modified] (a) Use a Mayer-Vietoris sequence to compute the homology groups of the space X obtained from a torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ by attaching a Möbius band M via a homeomorphism from the boundary circle C of M to the circle $\mathbb{S}^1 \times \{x_0\}$ in the torus.

(b) How does the answer change if C is attached to a closed loop that wraps k times around the first circle (i.e., via the path $f : I \to \mathbb{S}^1 \times \mathbb{S}^1$ given by $f(t) = (e^{2\pi ikt}, e^{2\pi it})$)?

Mea culpa! Part (b) is inconsistent. The formula should be $f(t) = (e^{2\pi kit}, 1)$ (or replace 1 with any complex number on $\mathbb{S}^1 \subset \mathbb{C}$). This caused some of you to write $\gamma(1) = (k, 1, 2)$ in part (b), in which case the value of k is irrelevant — this vector is always part of a $\mathbb{Z}$-basis.

Solution: (a) Certainly $H_0(X) = \mathbb{Z}$ because X is path-connected. We know the homology groups of $T \cap M = \mathbb{S}^1$, of $M \simeq \mathbb{S}^1$, and of T, so the nonzero part of the reduced Mayer-Vietoris sequence is

$$0 \to \mathbb{Z} \xrightarrow{\alpha} H_2(X) \xrightarrow{\beta} \underbrace{H_1(T \cap M)}_{\mathbb{Z}} \xrightarrow{\gamma} \underbrace{H_1(T) \oplus H_1(M)}_{\mathbb{Z}^3} \xrightarrow{\delta} H_1(X) \to 0.$$

The map γ is given by $\gamma(1) = (1, 0, 2)$ (since C is a double cover of the central circle, which generates $H_1(M)$). Certainly γ is one-to-one, so $\beta = 0$, hence α is an isomorphism. That is, $H_2(X) \cong \mathbb{Z}$. The rest of the diagram is a short exact sequence

$$0 \to \underbrace{H_1(T \cap M)}_{\mathbb{Z}} \xrightarrow{\gamma} \underbrace{H_1(T) \oplus H_1(M)}_{\mathbb{Z}^3} \xrightarrow{\delta} H_1(X) \to 0.$$

Evidently $H_1(X)$ has free rank 2. The vector $\gamma(1)$ is part of a basis for $\mathbb{Z}^3$ (say $\{(1, 0, 2), (0, 1, 0), (0, 0, 1)\}$. Therefore $H_1(X) = \operatorname{coker} \gamma$ is torsion-free and $H_1(X) = \mathbb{Z}^2$. In conclusion,

$$\boxed{H_2(X) = \mathbb{Z}, \quad H_1(X) = \mathbb{Z}^2.}$$

(b) In this case we have $\gamma(1) = (k, 0, 2)$. If k is odd then nothing changes. If k is even, then $\gamma(1)$ is not part of a basis but $\gamma(1)/2$ is, and we will wind up with $H_1(X) = \mathbb{Z}^2 \oplus \mathbb{Z}_2$. In short,

$$\boxed{H_2(X) = \mathbb{Z}, \qquad H_1(X) = \begin{cases} \mathbb{Z}^2 & \text{if } k \text{ is odd,} \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2 & \text{if } k \text{ is even.} \end{cases}}$$

Problem #7 [Hatcher p.158 #29] The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a compact closed 3-manifold X. Compute the homology groups of X using a Mayer-Vietoris sequence. Also compute the relative groups $H_i(R, M_g)$.

Solution: From Example 2.36, we have $\tilde{H}_1(M_g) = \mathbb{Z}^{2g}$, $\tilde{H}_2(M_g) = \mathbb{Z}$, and $\tilde{H}_n(M_g) = 0$ for $n \geq 3$.

$$\tilde{H}_2(M_g) = \mathbb{Z}\{M\}, \qquad \tilde{H}_1(M_g) = \mathbb{Z}\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$$

where the a_i are the longitudinal circles and the b_i are the meridional circles.

Meanwhile, R is homotopy equivalent to the wedge of the g longitudinal circles, so $\tilde{H}_1(R) = \mathbb{Z}^g$ and $\tilde{H}_n(R) = 0$ for $n \geq 2$.

For $n \geq 3$, the Mayer-Vietoris sequence includes the piece

$$\tilde{H}_n(R)^2 = 0 \to \tilde{H}_n(X) \to \tilde{H}_{n-1}(M_g) \to 0 = \tilde{H}_{n-1}(R)^2$$

whence $\tilde{H}_n(X) \cong \tilde{H}_{n-1}(M_g)$; this is zero for $n \geq 4$ and $\mathbb{Z}$ for $n = 3$.

That leaves the end of the sequence, which is

$$\tilde{H}_2(R)^2 = 0 \to \tilde{H}_2(X) \to \underbrace{\tilde{H}_1(M_g)}_{\mathbb{Z}^{2g}} \xrightarrow{\partial} \underbrace{\tilde{H}_1(R)^2}_{\mathbb{Z}^{2g}} \to \tilde{H}_1(X) \to 0$$

The map ∂ kills all the meridional circles and maps the longitudinal circles a_i to $(a_i, a_i) \in \tilde{H}_1(R)^2$; in particular $\tilde{H}_2(X) = \ker \partial \cong \mathbb{Z}^g$ and $\tilde{H}_1(X) = \operatorname{coker} \partial \cong \mathbb{Z}^g$.

In summary:

$$\boxed{\tilde{H}_3(X) = \mathbb{Z}, \qquad \tilde{H}_2(X) = \mathbb{Z}^g, \qquad \tilde{H}_1(X) = \mathbb{Z}^g, \qquad \tilde{H}_0(X) = \mathbb{Z}}$$

and $\tilde{H}_n(X) = 0$ for $n > 3$.

Now for relative homology. Observe that R deformation-retracts onto the wedge sum of g circles, so $\tilde{H}_1(R) \cong \mathbb{Z}^g$ and $\tilde{H}_k(R) = 0$ for $k \neq 1$. Also, the inclusion $M_g \hookrightarrow R$ is good (since we can certainly cellulate in such a way that M_g is a CW-subcomplex of R). So the long exact sequence of reduced relative homology breaks into pieces

$$H_3(R) = 0 \to H_3(R, M_g) \to H_2(M_g) = \mathbb{Z} \to 0$$

and

$$H_2(R) = 0 \to H_2(R, M_g) \to H_1(M_g) = \mathbb{Z}^{2g} \xrightarrow{\psi} \tilde{H}_1(R) = \mathbb{Z}^g \to \tilde{H}_1(R, M_g) \to \tilde{H}_0(M_g) = 0.$$

The first of these gives immediately $H_3(R, M_g) = \mathbb{Z}$.

For the second sequence, regard M_g as the connected sum of g tori. The longitudinal circles of these tori are mapped by ψ to the generators of $H_1(R)$, while their meridional circles are all mapped to 0. Therefore,

$$\tilde{H}_2(R, M_g) \cong \ker \psi \cong \mathbb{Z}^g$$

and since ψ is surjective, the arrow following it is the zero map. But then the group $\tilde{H}_1(R, M_g)$ is pinched by two zero maps, so it itself must be zero.