Math 821 Problem Set #6
Due date: Friday, April 18

Problem #1 [Hatcher p.131 #11] Show that if $A \subset X$ and there is a retraction $X \to A$, then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \hookrightarrow X$ is injective.

Solution: Call the inclusion i. A retraction is by definition a function $r : X \to A$ with $r|_A = r \circ i = \mathbb{1}_A$. By functoriality we have a commutative diagram of spaces and continuous maps that induces a commutative diagram of homology groups and homomorphisms:

$$A \xrightarrow{i} X \xrightarrow{r} H_n(A), \quad r \circ i = \mathbb{1}_A \qquad\qquad H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{r_*} H_n(A), \quad r_* \circ i_* = \mathbb{1}$$

From this we see that i_* is injective (and that r_* is surjective).

Problem #2 (a) [Hatcher p.132 #15] A homological algebra warmup: Prove that if

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{j} E$$

is exact with f surjective and j injective, then $C = 0$.

Solution: We have $\operatorname{im} f = B = \ker g$, so g is the zero map. Similarly, $\ker j = \operatorname{im} h = 0$, so h is also the zero map. It follows that $\operatorname{im} g = 0 = \ker h = C$, so $c = 0$.

(b) Prove the *Snake Lemma*: if the commutative diagram

$$\begin{array}{ccccccccc}
0 & \dashrightarrow & A & \xrightarrow{d} & B & \xrightarrow{e} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A' & \xrightarrow{d'} & B' & \xrightarrow{e'} & C' & \dashrightarrow & 0
\end{array}$$

has exact rows, then there is an exact sequence

$$0 \dashrightarrow \ker f \xrightarrow{\alpha} \ker g \xrightarrow{\beta} \ker h \xrightarrow{\gamma} \operatorname{coker} f \xrightarrow{\delta} \operatorname{coker} g \xrightarrow{\varepsilon} \operatorname{coker} h \dashrightarrow 0.$$

(The dashed arrows can be either included or omitted — both versions are often referred to as the Snake Lemma. In your solution, prove the version without the dashed arrows and then observe what happens if the arrows are included.)

Solution: First, we construct the maps and check that they are well-defined. Second, we check that the sequence is exact.

(1) For $a \in \ker f$, define $\alpha(a) = da$. We have $gda = d'fa = 0$, so $\alpha(a) \in \ker g$.

(2) For $b \in \ker g$, define $\beta(b) = eb$. We have $heb = e'gb = 0$, so $\beta(b) \in \ker h$.

(3) For $c \in \ker h$, we can write $c = e(b + \tilde{b})$ for some $b \in B$ and $\tilde{b} \in \ker e$. (All preimages of c can be obtained in this way by fixing b and letting $\tilde{b}$ vary over $\ker e$.) Since $\ker e = \operatorname{im} d$ and d is injective, there is a unique $a \in A$ such that $\tilde{b} = da$. So

$$c = e(b + \tilde{b}) = e(b + da) \qquad \text{and} \qquad e'g(b + da) = he(b + da) = hc = 0$$

so $g(b+da) \in \ker e' = \operatorname{im} d'$. Since d' is injective, there is a unique $a' \in A'$ such that $d'a' = g(b+da)$. Since

$$d'a' = g(b + da) = gb + gda = gb + d'fa \quad \therefore \qquad gb = d'a' - d'fa = d'(a' - fa).$$

That is, a' is determined by c modulo $\operatorname{im} f$, which means that $[a'] \in \operatorname{coker} f$ is uniquely determined by c. Therefore $\gamma(c) = [a']$ is a well-defined function $\ker h \to \operatorname{coker} f$, and the choice of $\tilde{b}$ was irrelevant (so we might as well have taken $\tilde{b} = 0$ and hence $a = 0$). This lets us rewrite the construction of $\gamma(c)$ more conveniently for future use:

$$\text{find } b \in B \text{ such that } c = eb, \text{ find } a' \in A' \text{ such that } d'a' = gb, \text{ and put } \gamma(c) = [a'] \in \operatorname{coker} f. \tag{0.1}$$

(4) Let $[a'] \in \operatorname{coker} f$, i.e., $[a'] = a' + \operatorname{im} f$ for some $a' \in A'$. Define

$$\delta[a'] = [d'a'] = d'a' + \operatorname{im} d'f = d'a' + \operatorname{im} gd = d'a' + g(\operatorname{im} d)$$

which is a well-defined element of $\operatorname{coker} g = B'/\operatorname{im} g$.

(5) Let $[b'] \in \operatorname{coker} g$, i.e., $[b'] = b' + \operatorname{im} g$ for some $b' \in B'$. Define

$$\varepsilon[b'] = [e'b'] = e'b' + \operatorname{im} e'g = e'b' + \operatorname{im} he = e'b' + h(\operatorname{im} e)$$

which is a well-defined element of $\operatorname{coker} h = C'/\operatorname{im} h$.

Now that we have all the maps defined, we check exactness.

(1) Exactness at $\ker f$ (in the dashed-arrow case): If d is injective then so is α.

(2) Exactness at $\ker g$:

$$\begin{aligned}
\operatorname{im}\alpha &= \alpha(\ker f) = d(\ker f) = \{da \mid fa = 0\} \\
&= \{da \mid d'fa = 0\} && \text{by injectivity of } d' \\
&= \{da \mid gda = 0\} && \text{by commutativity} \\
&= \ker g \cap \operatorname{im} d = \ker g \cap \ker e && \text{by exactness of the top row} \\
&= \ker \beta.
\end{aligned}$$

(3) Exactness at $\ker h$: observe that

$$\operatorname{im}\beta = \beta(\ker g) = e(\ker g) = \{eb : \ gb = 0\} \tag{0.2}$$

and

$$\begin{aligned}
\ker\gamma &= \{c \in \ker h \mid \exists b \in B : \ c = eb, \exists a' \in A' : \ d'a' = gb, \ \exists a \in A : a' = fa\} \\
&= \{c \in \ker h \mid \exists b \in B : \ c = eb, \exists a \in A : \ d'fa = gb\} \\
&= \{eb \mid b \in B, \ heb = 0, \ \exists a \in A : \ gda = gb\}.
\end{aligned} \tag{0.3}$$

Comparing (0.2) with (0.3) shows that $\operatorname{im}\beta \subset \ker\gamma$, because if $c = eb \in \operatorname{im}\beta$, then $gb = 0$ and we may take $a = 0$ in (0.3). On the other hand, if $c = eb \in \ker\gamma$, then let $\tilde{b} = b - da$. Then $c = e\tilde{b}$ (since $ed = 0$) and $\tilde{b} \in \ker g$ by (0.3), which by (0.2) says that $c \in \operatorname{im}\beta$.

(4) Exactness at $\operatorname{coker} f$: Let $c \in \ker h$. Adopting the notation of the construction of γ (0.1), we have

$$\delta\gamma(c) = \delta a' = [d'a'] = [gb] = 0 \in \operatorname{coker} g$$

so $\operatorname{im}\gamma \subseteq \ker\delta$. On the other hand, if $[a'] \in \ker\delta$ then $d'a' \in \operatorname{im} g$, i.e., there exists $b \in B$ such that $d'a' = gb$. But then $\gamma(eb) = [a']$, so $[a'] \in \operatorname{im}\gamma$.

(5) Exactness at $\operatorname{coker} g$: First, observe that

$$\begin{aligned} \operatorname{im}\delta &= \{\delta[a'] \mid a' \in A'\} = \{[d'a'] \mid a' \in A'\} \\ &= \{[b'] \mid b' \in \operatorname{im} d'\} \\ &= \{[b'] \mid b' \in \ker e'\} \subset eq \ker\varepsilon. \end{aligned}$$

On the other hand, if $\varepsilon[b'] = 0$ then $e'b' = hc$ for some $c \in C$. Since e is surjective, there exists $b \in B$ such that $eb = c$, whence $e'b' = heb = e'gb$, i.e., $b' - gb \in \ker e' = \operatorname{im} d'$. That is, there is some $a' \in A'$ such that

$$b' - gb = d'a']$$

whence $\delta[a'] = [d'a'] = [b' - gb] = [b'] \in \operatorname{coker} g$. So we have shown that $\ker\varepsilon \subseteq \operatorname{im}\delta$.

(6) Exactness at $\operatorname{coker} h$ (in the dashed-arrow case): Assume e' is onto. For every $[c'] \in \operatorname{coker} h$, we can find $b' \in B'$ such that $e'b' = c'$, and then $\varepsilon[b'] = [e'b'] = [c']$. So ε is onto as well.

Problem #3 Recall that the *torsion subgroup* $T(G)$ of an abelian group is the subgroup consisting of all elements of finite order. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} \mathbb{Z}^n \to 0$ be a short exact sequence of finitely generated $\mathbb{Z}$-modules. Show that $T(A) \cong T(B)$. In particular, if A is free abelian then so is B.

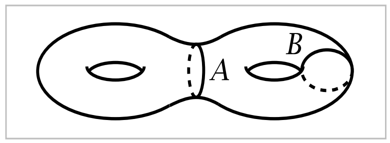
Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of finitely generated $\mathbb{Z}$-modules. Show that if C is free abelian, then $T(A) \cong T(B)$, but that A free abelian does not necessarily imply that $T(B) = T(C)$.

Solution: Suppose C is free. If $x \in A$, $k \in \mathbb{Z}$, and $kx = 0$, then $kf(x) = f(kx) = 0$, so $f(x)$ is torsion. Therefore, f restricts to a one-to-one map $T(A) \to T(B)$. On the other hand, if $b \in B$ is torsion then it is certainly in the kernel of g (because its image is torsion in C, hence zero), hence in the image of f by exactness. So the map $T(A) \to T(B)$ is an isomorphism.

For the other direction, the short exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$ is a counterexample.

Problem #4 [Hatcher p.132 #17] (a) Compute the homology groups $H_n(X, A)$ when X is $\mathbb{S}^2$ or $\mathbb{S}^1 \times \mathbb{S}^1$ and A is a set of k points in X with $k < \infty$. You may use the computation of the homology groups of X from §2.1.

(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$, where X is a closed orientable surface of genus two with A and B the circles shown. (What are X/A and X/B?)



Solution: (a) Recall that

$$\begin{aligned} &H_2(\mathbb{S}^2) = \mathbb{Z}, && H_2(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z}, && H_2(A) = 0,\\ &H_1(\mathbb{S}^2) = 0, && H_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z}^2, && H_1(A) = 0,\\ &H_0(\mathbb{S}^2) = \mathbb{Z}, && H_0(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z}, && H_0(A) = \mathbb{Z}^k \end{aligned}$$

with everything vanishing in dimension ≥ 3. So the long exact sequence for relative homology

$$0 \to H_2(A) \to H_2(X) \to H_2(X, A) \to H_1(A) \to H_1(X) \to H_1(X, A) \to H_0(A) \to H_0(X) \to H_0(X, A) \to 0$$

becomes

$$0 \to \mathbb{Z} \to H_2(X, A) \to 0 \to H_1(X) \to H_1(X, A) \to \mathbb{Z}^k \to \mathbb{Z} = H_0(X) \to H_0(X, A) \to 0$$

Split up the sequence at the 0. The first piece tells us that $\mathbf{H_2(X, A) = \mathbb{Z}}$.

Now focus on the second piece, which is

$$0 \to H_1(X) \xrightarrow{j} H_1(X, A) \xrightarrow{\partial} \mathbb{Z}^k \xrightarrow{i} \mathbb{Z} = H_0(X) \xrightarrow{j'} H_0(X, A) \to 0.$$

The map i is surjective (it maps the class of any point in A to the class of a point in X). Therefore $\operatorname{im} i = H_0(X) = \ker j'$, so j' is the zero map. It follows that $\mathbf{H_0(X, A) = 0}$.

Since i is surjective, its kernel must be a copy of $\mathbb{Z}^{k-1}$. Replacing the target of ∂ with $\operatorname{im} \partial = \ker i$ gives a short exact sequence

$$0 \to H_1(X) \xrightarrow{j} H_1(X, A) \xrightarrow{\partial} \mathbb{Z}^{k-1} \to 0.$$

If $X = \mathbb{S}^2$ then $j = 0$ and so ∂ is an isomorphism, and we get $\mathbf{H_1(\mathbb{S}^2, A) = \mathbb{Z}^{k-1}}$.

If $X = \mathbb{S}^1 \times \mathbb{S}^1$ then the exact sequence is

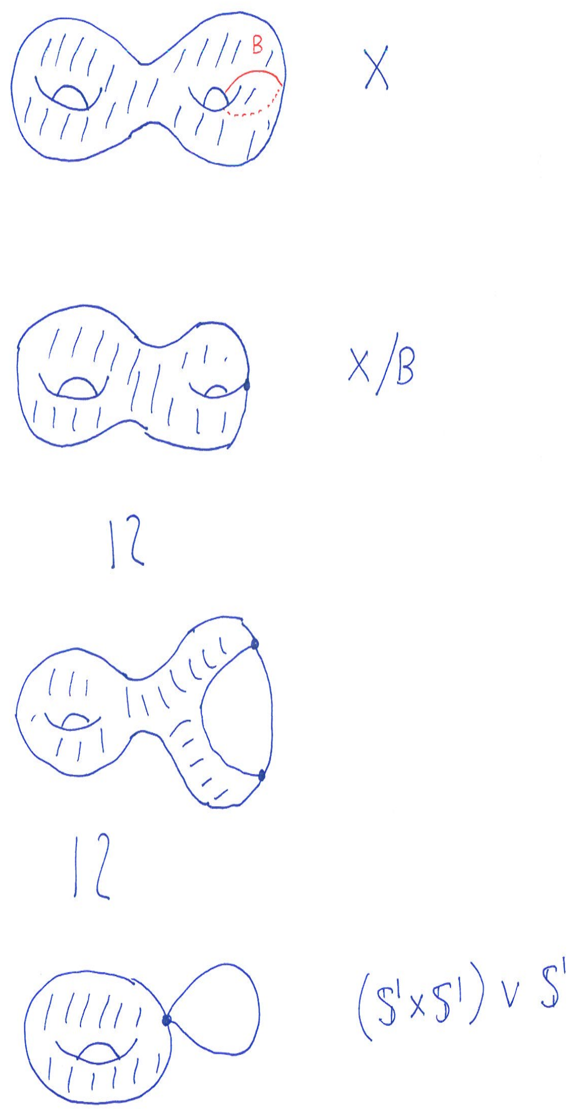
$$0 \to \mathbb{Z}^2 \xrightarrow{j} H_1(X, A) \xrightarrow{\partial} \mathbb{Z}^{k-1} \to 0$$

which by the result of a previous problem says that $\mathbf{H_1(\mathbb{S}^2, A) = \mathbb{Z}^{k+1}}$.

By the way, note that $\mathbb{S}^2/A$ is $\mathbb{R}^2$ with k points removed, which is homotopy-equivalent to the wedge of $k-1$ circles. In summary,

	$H_k(X, A)$			
	$k > 2$	$k = 2$	$k = 1$	$k = 0$
$X = \mathbb{S}^2$	0	$\mathbb{Z}$	$\mathbb{Z}^{k-1}$	0
$X = \mathbb{S}^1 \times \mathbb{S}^1$	0	$\mathbb{Z}$	$\mathbb{Z}^{k+1}$	0

Solution: (b) These pairs are good, so $H_n(X, A) = \tilde{H}_n(X/A)$ and $H_n(X, B) = \tilde{H}_n(X/B)$. Contracting A gives $X/A \cong (\mathbb{S}^1 \times \mathbb{S}^1) \wedge (\mathbb{S}^1 \times \mathbb{S}^1)$ (with the contracted point as the wedge point). Meanwhile, X/B is homotopy-equivalent to $(\mathbb{S}^1 \times \mathbb{S}^1) \wedge \mathbb{S}^1$ by the following figure.



Therefore

$$\begin{array}{lll} H_2(X, A) = \mathbb{Z}^2, & H_1(X, A) = \mathbb{Z}^4, & H_n(X, A) = 0 \quad (n \neq 1, 2), \\ H_2(X, A) = \mathbb{Z}, & H_1(X, A) = \mathbb{Z}^3, & H_n(X, A) = 0 \quad (n \neq 1, 2). \end{array}$$

Problem #5 [Hatcher p.132 #20] The suspension SX of a space X is obtained by taking two copies of the cone $CX = X \times [0,1]/X \times \{1\}$ and attaching them along their bases. Equivalently, take a prism over X and contract each of the top and bottom faces to points:

$$SX = X \times [0,1] \;/\; X \times \{0\} \;/\; X \times \{1\}.$$

For example, the suspension of $\mathbb{S}^n$ is $\mathbb{S}^{n+1}$.

Prove that $\tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X)$ for all $n > 0$. More generally, for any integer k, compute the reduced homology groups of the union of k copies of CX with their bases identified. (The suspension is the case $k = 2$.)

Solution: First we handle the suspension SX. Observe that $SX = CX/X \times \{0\}$, so the long exact sequence for the pair $(CX, X \times \{0\})$ is

$$\cdots \to \tilde{H}_n(CX) \to \tilde{H}_n(CX, X \times \{0\}) \to \tilde{H}_{n-1}(X \times \{0\}) \to \tilde{H}_{n-1}(CX) \to \cdots$$

but the two outer terms are zero because CX is contractible (recall that it deformation-retracts onto the cone point). So the middle arrow is the desired isomorphism.

More generally, let $X^{[k]}$ denote the union of k copies of CX with their bases identified (so in particular $X^{(1)} = CX$ and $X^{(2)} = SX$). For $k > 1$, we can form $X^{[k]}$ from $X^{[k-1]}$ by a two-step process: first attach a cylinder $X \times I$ to the base of $X^{[k-1]}$ along $X \times \{0\}$ to get a space Y, then contract $X \times \{1\}$ to a point. This identifies $X^{[k]}$ with $Y/X \times \{1\}$, and meanwhile Y deformation-retracts to $X^{[k-1]}$ by shrinking $X \times [0,1]$ to $X \times \{0\}$. Therefore, the inclusion and quotient

$$X \times \{1\} \xrightarrow{i} Y \xrightarrow{j} Y/(X \times \{1\})$$

gives rise to a long exact sequence

$$\cdots \to H_n(X \times \{1\}) \xrightarrow{i_*} H_n(Y) \xrightarrow{j_*} H_n(Y/(X \times \{1\})) \xrightarrow{\partial} H_{n-1}(X \times \{1\}) \to \cdots$$

The map i_* is zero, since any map $\Delta^n \to X \times \{1\}$ is homotopic in Y to a map to the cone point. Therefore the LES breaks up into short exact sequences

$$0 \to H_n(Y) \xrightarrow{j_*} H_n(Y/(X \times \{1\})) \xrightarrow{\partial} H_{n-1}(X \times \{1\}) \to 0.$$

Replacing these spaces with their homotopy equivalents, we can rewrite this as

$$\boxed{0 \to H_n(X^{[k-1]}) \xrightarrow{j_*} H_n(X^{[k]}) \xrightarrow{\partial} H_{n-1}(X) \to 0.} \tag{0.4}$$

Note: There are other way to obtain (0.4) is to consider the good pair $(X^{[k]}, X^{[k-1]})$. The quotient space $X^{[k]}/X^{[k-1]}$ can be identified with SX, and the long exact sequence for the pair, together with the isomorphism $\tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X)$, consists of the sequences (0.4), spliced together (although one still has to argue that the connecting homomorphism is zero). One can also use Corollary 2.24, or a Mayer-Vietoris sequence (which you will learn soon).

Obtaining (0.4) **was as far as you had to get to obtain full credit.** To complete the problem, we show that the sequence *splits*, i.e., that $H_n(X^{[k]}) \cong H_n(X^{[k-1]}) \oplus H_{n-1}(X)$. By the Splitting Lemma (Hatcher, p.147), we can do this by constructing a homomorphism

$$p : H_n(X^{[k]}) \to H_n(X^{[k-1]}) =: Q$$

such that $p \circ j_* = \mathbb{1}_Q$.

Regard $X^{[k]}$ as the union of k copies $Z_1, \ldots, Z_k$ of the cone CX, identified along their bases $X_i \times \{0\}$, and let U_i be an open deformation of Z_i in $X^{[k]}$: e.g., $U_i = Z_i \cup \bigcup_{j \neq i} X_j \times [0, 1/2)$. Note that $U_2 \cup \cdots \cup U_k$

deformation-retracts to $X^{[k-1]}$. Let $\mathcal{U} = \{U_1, \dots, U_k\}$ and $\mathcal{U}' = \{U_2, \dots, U_k\}$. By the Subdivision Lemma (Hatcher, Prop. 2.21, p.119) we have

$$C_\bullet^{\mathcal{U}} \simeq C_\bullet(X^{[k]}) \qquad \text{and} \qquad C_\bullet^{\mathcal{U}'} \simeq C_\bullet(X^{[k-1]}).$$

An element of $C_n^{\mathcal{U}}$ is a chain of the form $\tau = \sum_{i=1}^n \sigma_i$, where $\sigma_i \in C_n(U_i)$. The obvious homeomorphism $U_1 \to U_2$ induces an isomorphism $\phi : C_n(U_1) \to C_n(U_2)$, so replacing σ_1 with $\phi(\sigma_1)$ gives a map $C_n^{\mathcal{U}} \to C_n^{\mathcal{U}'}$, which induces a map on homology:

$$p : H_n(X^{[k]}) \to H_n(X^{[k-1]}).$$

On the other hand, the image of j_* consists of chains of the form τ with $\sigma_1 = 0$, so it follows that $p \circ j_*$ is the identity on $H_n(X^{[k-1]})$ as desired. This completes the proof that the short exact sequence (0.4) splits. Therefore

$$H_n(X^{[k]}) \cong H_n(X^{[k-1]}) \oplus H_{n-1}(X)$$

and by induction we now know the homology groups of a generalized suspension:

$$\boxed{\boldsymbol{H_n(X^{[k]}) \cong H_{n-1}(X)^{\oplus(k-1)}}.} \tag{$*$}$$

A better solution, submitted by Nick and others: Consider the effect of starting with the space $X^{[k]}$, regarded as the union of k copies $Z_1, \dots, Z_k$ of the cone CX identified along their bases, and then contracting Z_k. This is a deformation retraction since Z_k is contractible. On the other hand, each Z_i $(i < k)$ turns into a copy of SX, and the Z_i's are wedged together at the point coming from the base. Therefore

$$X^{[k]} \simeq (SX)^{\wedge(k-1)}.$$

Using the first part of the problem, together with there fact that wedge sum is additive on reduced homology, we get the formula $(*)$ much more easily.

Problem #6 Let $n \leq d \geq 0$ and let $X = \Delta^{n,d}$ denote the d-skeleton of the n-dimensional simplex (whose vertices are $v_0, v_1, \dots, v_n$). Most of you conjectured last time that the reduced homology groups of X are given by

$$\tilde{H}_k(X) = \begin{cases} \mathbb{Z}^{\binom{n}{d}} & \text{if } k = d, \\ 0 & \text{if } k < d. \end{cases}$$

This conjecture is correct. Prove it without writing down any explicit simplicial boundary matrices.

Solution: If $k < d$, then $\tilde{H}_k(X) = \tilde{H}_k(\Delta^n) = 0$ because Δ^n is contractible, hence acyclic. Otherwise, consider the subcomplex

$$\Gamma = \langle \sigma \in X \mid 0 \in \sigma.$$

This complex is contractible, because it is the cone with apex v_0 and base

$$\Lambda = \{\sigma \in X \mid 0 \notin \sigma,\ \sigma \cup \{v_0\} \in X\}.$$

(The complex Γ is known as the *star* of v_0, and Λ is its *link*.) Since Γ is contractible, we have $X \simeq X/\Gamma$. But this latter space is a CW-complex with a single vertex and $\binom{n}{d+1}$ cells of dimension d (corresponding to the simplices supported on $d+1$ of the vertices $v_1, \dots, v_n$). That is, X/Γ is the wedge of $\binom{n}{d+1}$ copies of $\mathbb{S}^n$, hence has the desired homology.