Math 821 Problem Set #4
Due date: Friday, April 4

Problem #1 Verify that the simplicial boundary map defined by

$$\partial_n[v_0,\dots,v_n] = \sum_{i=0}^{n}(-1)^i[v_0,\dots,\widehat{v_i},\dots,v_n]$$

satisfies the equation $\partial_{n-1}\circ\partial_n = 0$ for all n. (Yes, this calculation is done explicitly in Hatcher. But it is so important that everyone should do it for themselves at least once.)

Solution: The summands in $\partial_{n-1}\partial_n[v_0,\dots,v_n]$ all have the form $\pm[v_0,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_n]$ for $i<j$. Each such summand arises twice; we need to check that the signs are opposite. If v_j is removed first, then the sign contribution is $(-1)^j(-1)^i$, because i is the i^{th} leftmost element of the list $[v_0,\dots,v_i,\dots,\widehat{v_j},\dots,v_n]$. On the other hand, if v_i is removed first, then the sign contribution is $(-1)^i(-1)^{j-1}$, because j is the $\underline{(j-1)^{th}}$ leftmost element of the list $[v_0,\dots,\widehat{v_i},\dots,v_j,\dots,v_n]$. Therefore all summands cancel.

Problem #2 Let X be an abstract simplicial complex on vertex set $[n]$ and let $|X|$ be a geometric realization of X (not necessarily the standard one — it doesn't matter). What invariant of $|X|$ corresponds to the dimension of $H_0^\Delta(X)$?

Solution: We have $H_0^\Delta(X) = \Delta^0(X)/\operatorname{im}\partial_1$. The group $\Delta_1(X)$ is free abelian on the 0-simplices, i.e., the vertices. The image of ∂_1 is generated by 0-chains $[v]-[w]$ whenever vw is an edge. If two vertices v_0, v_n are in the same component of X, then there is a path $v_0,v_1,\dots,v_n$ in the 1-skeleton, so

$$[v_0]-[v_n] = ([v_0]-[v_1]) + ([v_1]-[v_2]) + \cdots + ([v_{n-1}-v_n]) \in \operatorname{im}\partial_1.$$

In other words, any two 0-chains representing vertices in the same component are equal modulo $\operatorname{im}\partial_1$. On the other hand, the chain $\partial[v,w]=[v]-[w]$ has the property that the sum of coefficients of vertices in any given component is even (because v,w are certainly in the same component by virtue of the existence of the 1-simplex $[v,w]$), and this property extends $\mathbb{Z}$-linearly to all of $\operatorname{im}\partial_1$. Therefore no two vertices in different components are equal modulo $\operatorname{im}\partial_1$. We conclude that $H_0^\Delta(X)\cong\mathbb{Z}^c$, where c is the number of components, and any selection of one vertex from each component gives a natural basis for $H_0^\Delta(X)$.

Problem #3 Consider the matrix

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Describe $\operatorname{coker} M$ (i) if M is regarded as a linear transformation over $\mathbb{Q}$; (ii) if M is regarded as a linear transformation over $\mathbb{Z}$; (iii) if M is regarded as a linear transformation over $\mathbb{F}_q$ (the finite field with q elements).

Solution: (i) Over $\mathbb{Q}$, the matrix is nonsingular, hence represents an isomorphism $\mathbb{Q}^3\to\mathbb{Q}^3$. Therefore $\operatorname{coker} M = \mathbb{Q}^3/\operatorname{im} M = 0$.

(ii) Over $\mathbb{Z}$, the matrix is still nonsingular, but is not invertible. Since $\det M = 2$, the cockerel must be an abelian group of order 2, so must be $\mathbb{Z}_2$. More explicitly, performing $\mathbb{Z}$-invertible column operations (replacing the first column with the sum of all three) gives the matrix

$$\underbrace{\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}}_{M}\underbrace{\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}}_{B}=\begin{bmatrix}2&1&0\\2&0&1\\2&1&1\end{bmatrix}=\underbrace{\begin{bmatrix}1&1&0\\1&0&1\\1&1&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{S}$$

with A also $\mathbb{Z}$-invertible. The matrix S is the Smith normal form of M, from which we can read off $\operatorname{coker} M \cong \mathbb{Z}_2$.

(iii) Let $q = p^a$. If $p \neq 2$, then $\det M = 2$ is a unit in $\mathbb{F}_q$, so the transformation is invertible and $\operatorname{coker} M = 0$ just as in (i).

If $p = 2$ then the matrix is singular. The rank is still 2 (since any two columns are linearly independent) so $\operatorname{coker} M = \mathbb{F}_q$. (Note that the cokernel must be a vector space, so the only invariant we need is its rank.)

Problem #4 [Hatcher p.131 #4] Compute by hand the simplicial homology groups of the "triangular parachute" obtained from $\boldsymbol{\Delta}^2$ by identifying its vertices to a single point.

Call the complex P (for "parachute"). Call the triangle T and the edges a, b, c. It doesn't matter how we orient them — say $\partial T = a + b + c$. There is only one vertex v, so all edges are loops. So the simplicial chain complex is

$$\Delta_2 = \mathbb{Z}\{T\} \xrightarrow[\begin{bmatrix}1\\1\\1\end{bmatrix}]{\partial_2} \Delta_1 = \mathbb{Z}\{a,b,c\} \xrightarrow[\begin{bmatrix}0&0&0\end{bmatrix}]{\partial_1} \Delta_0 = \mathbb{Z}\{v\}$$

and

$$\begin{aligned}
H_2^\Delta(P) &= \ker \partial_2 && = 0,\\
H_1^\Delta(P) &= \ker \partial_1 / \operatorname{im} \partial_2 = \mathbb{Z}\{a,b,c\}/\mathbb{Z}\{a+b+c\} && \cong \mathbb{Z}^2,\\
H_0^\Delta(P) &= \Delta_0 / \operatorname{im} \partial_1 && \cong \mathbb{Z} \quad \text{(or cite Problem 2).}
\end{aligned}$$

Problem #5 [Hatcher p.131 #5] Compute by hand the simplicial homology groups of the Klein bottle using the $\boldsymbol{\Delta}$-complex structure on p.102 (with two triangles).

Using Hatcher's labeling of the simplices, the simplicial chain complex is

$$\Delta_2 = \mathbb{Z}\{U,L\} \xrightarrow[\begin{bmatrix}1&1\\1&-1\\-1&1\end{bmatrix}]{\partial_2} \Delta_1 = \mathbb{Z}\{a,b,c\} \xrightarrow[\begin{bmatrix}0&0&0\end{bmatrix}]{\partial_1} \Delta_0 = \mathbb{Z}\{v\}$$

The columns of ∂_2 are linearly independent, so $H_2^\Delta(K) = 0$, and yet again $H_0^\Delta(K) = \mathbb{Z}$. To calculate $H_1^\Delta(K)$, observe that

$$\{v_1 = (1,1,-1),\ v_2 = (1,0,0),\ v_3 = (0,1,0)\}$$

generates $\mathbb{Z}^3$ as a $\mathbb{Z}$-module, and that

$\operatorname{im} \partial_2 = \mathbb{Z}\{(1,1,-1),\ (1,-1,1)\} = \mathbb{Z}\{(1,1,-1),\ (1,-1,1)+(1,1,-1\} = \mathbb{Z}\{(1,1,-1),\ (2,0,0)\} = \mathbb{Z}\{v_1, 2v_2\}.$

Therefore $H_1^{\Delta}(K) = \mathbb{Z}^3 / \operatorname{im} \partial_2 = \mathbb{Z} \oplus \mathbb{Z}_2$.

Problem #6 Check your answers on the last two problems using `Macaulay2` or your favorite computer algebra system.

Here is one efficient way of doing it:

```
D2 = matrix{{1},{1},{1}}; D1 = matrix{{0,0,0}};
Parachute = chainComplex (D1,D2);
prune HH Parachute

D2 = matrix{{1,1},{1,-1},{-1,1}}; D1 = matrix{{0,0,0}};
Klein = chainComplex (D1,D2);
prune HH Klein
```

Problem #7 Let $\boldsymbol{\Delta^{n,d}}$ denote the $\boldsymbol{d}$-skeleton of the $\boldsymbol{n}$-simplex. As an abstract simplicial complex, $\boldsymbol{\Delta}$ is generated by all $\boldsymbol{(d+1)}$-element subsets of $\boldsymbol{\{0,\dots,n\}}$. Use `Macaulay2` (or another computer algebra system) to compute the homology groups of $\boldsymbol{\Delta^{n,d}}$ for various values of $\boldsymbol{n}$ and $\boldsymbol{d}$. Conjecture a general formula for $\boldsymbol{H_k(\Delta^{n,d})}$ in terms of $\boldsymbol{n}$, $\boldsymbol{d}$ and $\boldsymbol{k}$. (Prove it, if you want.)

The answer is

$$\tilde{H}_k(\Delta^{n,d}) = \begin{cases} \mathbb{Z}^{\binom{n}{d+1}} & \text{if } k = d, \\ 0 & \text{if } k < d. \end{cases}$$

I gave full credit for making the correct conjecture. With the tools we have available, one probably needs an induction argument (e.g., using the fact that the chain complexes of $\Delta^{n,d}$ and $\Delta^{n,d+1}$ are identical except in dimension $d+1$). This problem will appear at a later date.