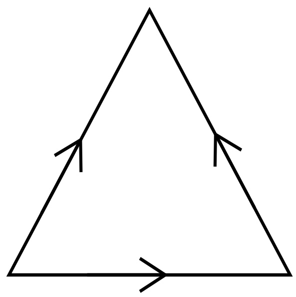
Math 821 Problem Set #4
Problem Set #4
Due date: Friday, March 14

Problem #1 The *dunce hat* is the space D obtained from a triangle by identifying all three edges with each other, with the orientations indicated below.

(a) Prove that D is simply-connected using Van Kampen's theorem.

(b) Find a different, one-line proof that D is simply-connected.



Solution:

(a) Decompose D into two pieces A, B as follows: A is the interior of the 2-cell, and $B = D \setminus \{p\}$, where $p \in A$.

Then:

- A is an open disk, hence contractible.
- B deformation-retracts onto, hence is homotopy-equivalent to, the boundary triangle, which is just a circle (the edge a becomes one loop around the circle).
- $A \cap B$ is an (open) annulus, whose fundamental group is generated by a path γ winding once around p. Note that $\gamma \simeq aaa^{-1}$ in B.

Now, since A is contractible, Van Kampen's Theorem says that

$$\pi_1(D) = \pi_1(B)/i_*\pi_1(A \cap B)$$

and

$$i_*\gamma = aaa^{-1} = a$$

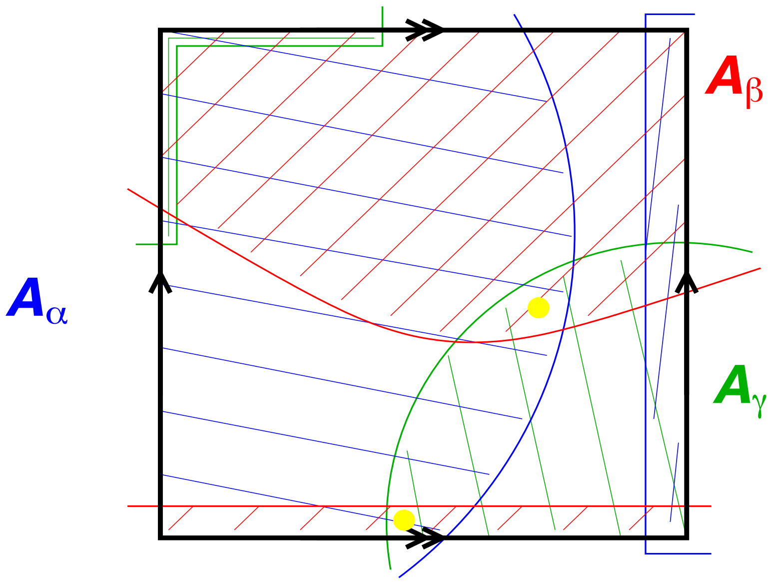
so this quotient is in fact trivial.

(b) By our theorems on 2-dimensional cell complexes, $\pi_1(D) = \langle g \mid ggg^{-1} \rangle = \langle g|g \rangle = 0$.

Problem #2 Consider the standard picture of the torus $T = S^1 \times S^1$ as a quotient space of the square. Explain what is wrong with the following "proof" (whose conclusion is certainly false):

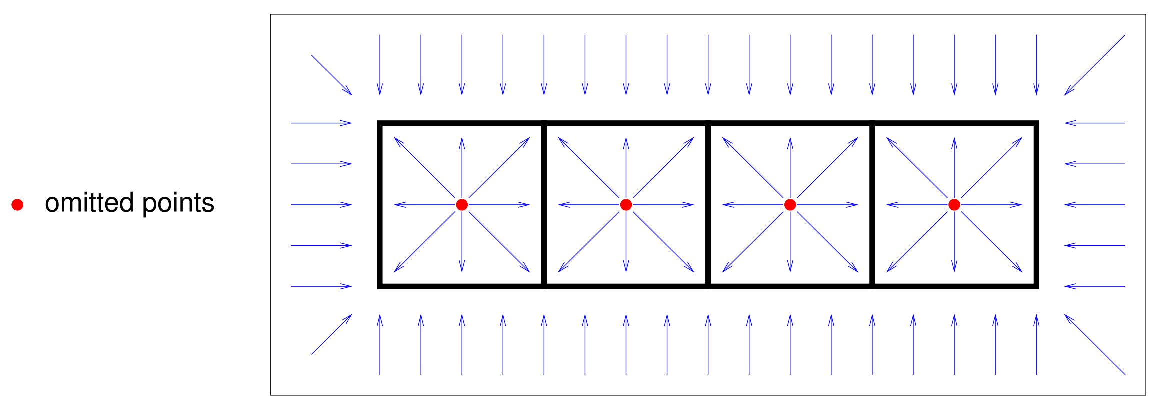
> Consider the open cover $A_\alpha \cup A_\beta \cup A_\gamma$ shown below. Each one is path-connected and simply-connected, and their intersection is path-connected. Therefore, by Van Kampen's theorem, the torus is simply-connected.

Solution: It is true that the sets $A_\alpha, A_\beta, A_\gamma$ are all simply-connected. However, the intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is not path-connected. The picture is misleading (which was the idea of the problem); it actually must look something like this, and the two yellow splotches denote different components of $A_\alpha \cap A_\beta \cap A_\gamma$.



Problem #3 (Hatcher, p.53, #4, modified) Let $n \geq 1$ be an integer, and let $X \subset \mathbb{R}^3$ be the union of n distinct rays emanating from the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.

Solution: The map $f_t(\mathbf{x}) = (1-t)\mathbf{x} + t\frac{\mathbf{x}}{\|\mathbf{x}\|}$ gives a deformation retraction from $\mathbb{R}^3 \setminus X$ to the unit sphere minus n points. We can regard the deleting the first point as giving a copy of $\mathbb{R}^2$, so we now have $\mathbb{R}^2$ minus $n-1$ points. This space deformation-retracts to the wedge of $n-1$ squares ($\cong (S^1)^{\vee(n-1)}$), as shown below. Therefore $\pi_1(X) \cong \pi_1((S^1)^{\vee(n-1)})$ is free on $n-1$ generators.



Problem #4 Let $a_1, \dots, a_n$ be nonzero integers. Construct a cell complex X from S^1 as follows: For each $j = 1, \dots, n$, attach a 2-cell to S^1 by wrapping it around the circle a_j times. Compute $\pi_1(X)$.

Solution: The theorem on 2-dimensional cell complexes (from pp.50–52 of Hatcher) implies that

$$\pi_1(X) \;=\; \langle g \mid g^{a_1}, g^{a_2}, \dots, g^{a_n} \rangle = \langle g \mid g^k \rangle \;=\; \mathbb{Z}_k$$

where $k = \gcd(a_1, \dots, a_k)$.

Problem #5 (Hatcher, p.53, #6, modified) Let X be a path-connected cell complex, and let Y be a cell complex obtained from X by attaching an n-cell for some $n \geq 3$. Show that the inclusion $X \hookrightarrow Y$ induces an isomorphism $\pi_1(X) \cong \pi_1(Y)$.

Solution: The proof of Prop. 1.26 goes through, changing e^2_α to e^n_α. At the very end, we have that A_α deformation-retracts onto a circle in $e^n_\alpha \setminus \{y_\alpha\}$, i.e., an n-ball minus a point. But such a thing is simply-connected (as we know, it is homotopy-equivalent to S^{n-1}) and therefore $\pi_1(A_\alpha) = 0$, and the group N in the statement of the proposition is trivial.

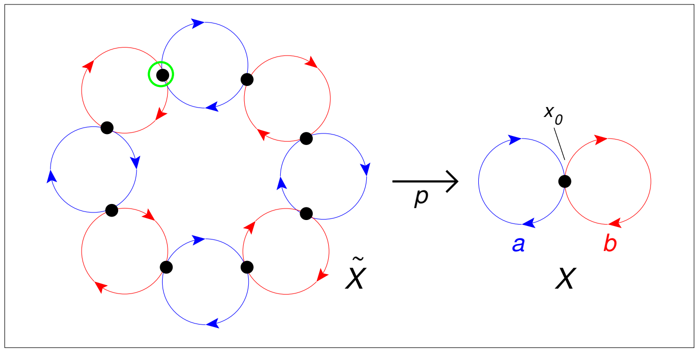
Another argument uses Van Kampen's theorem. Let e be the n-cell that gets attached (so $e \cong D^n$) and let $f : \partial e = S^{n-1} \to X$ be the attaching map. Write $Y = X \cup Z$ where Z is obtained by fattening ∂e slightly into an open set that contains, and deformation-retracts onto, it. (This is a mapping cylinder neighborhood in the sense of Example 0.15.) Then Z is contractible, hence simply-connected (since it deformation-retracts onto an n-ball) and $X \cap Z$ is simply-connected (because it deformation-retracts onto the simply-connected $(n-1)$-sphere ∂e). Now applying Van Kampen's theorem to the decomposition $Y = X \cup Z$ gives a surjection $\pi_1(X) \to \pi_1(Y)$ whose kernel is zero.

Problem #6 (Hatcher p.79, #2) Show that if $p_1 : \tilde{X}_1 \to X_1$ and $p_2 : \tilde{X}_2 \to X_2$ are covering spaces, then so is their product $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$.

Solution: For $i = 1, 2$, let $\{U^i_\alpha\}$ be an open cover of X_i that is "good", i.e., every component of $p_i^{-1}(U^i_\alpha)$ is mapped homeomorphically to U^i_α by p_i. The components V of $(p_1 \times p_2)^{-1}(U^1_\alpha \times U^2_\alpha)$ are just the products $V_1 \times V_2$, where V_i is a component of $p_i^{-1}(U^i_\alpha)$, and since $p_i|_{U_i}$ are homeomorphisms, so is $(p_1 \times p_2)|_V$. So $\{U^1_\alpha \times U^2_\alpha\}$ is a good cover of $X_1 \times X_2$. By the way, this argument implies that the number of sheets of a covering space is multiplicative under direct product.

Problem #7 (Hatcher p.80, #12) Let a and b be the generators of $\pi_1(S^1 \vee S^1, x_0)$ corresponding to the two copies of S^1, with x_0 their common point. Draw a picture of the covering space $\tilde{X}$ of $S^1 \vee S^1$ corresponding to the normal subgroup of $\pi_1(S^1 \vee S^1)$ generated by a^2, b^2, and $(ab)^4$, and prove that this covering space is indeed the correct one. (I.e., this group should be $p_*\pi_1(\tilde{X}, \tilde{x}_0)$.)

Solution: The space $\tilde{X}$ is a necklace of eight circles:



The number of sheets of the covering is 8 (the cardinality of the preimage of any point in X, for example x_0). If we take Take $\tilde{x}_0 \in \tilde{X}_0$ to be the highlighted point. Then the colored loops in $\tilde{X}_0$ given by

(1) the blue circle containing $\tilde{x}_0$,
(2) the red circle containing $\tilde{x}_0$, and
(3) walking all the way around the "outer arcs" of the necklace

map via p_* to the loops a^2, b^2 and $(ab)^4$ in $G = \pi_1(X)$.

Claim 1: $p_*\pi_1(\tilde{X})$ is normal in $\pi_1(X)$. This is true by Prop. 1.39 in Hatcher, but we didn't get to this theorem until Friday 3/14, so here's a more elementary proof. By the symmetry of $\tilde{X}$, the same colored loops are available at every basepoint, so the group $p_*\pi_1(\tilde{X})$ is independent of the choice of basepoint. Given $g \in \pi_1(X)$ and $h \in p_*\pi_1(\tilde{X})$, lift g to a path $\tilde{g}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$ and lift h to a loop $\tilde{h}$ at $\tilde{x}_1$. Then $\tilde{g} \cdot \tilde{h} \cdot \overline{\tilde{h}}$ is a loop at $\tilde{x}_0$ that maps via p_* to $gh\bar{g}$. It follows that $p_*\pi_1(\tilde{X})$ is closed under conjugation in G, proving Claim 1. Moreover,

$$H := \langle\!\langle a^2, b^2, (ab)^4 \rangle\!\rangle \quad \subseteq \quad p_*\pi_1(\tilde{X}) \quad \trianglelefteq \quad G$$

(recall that the notation $\langle\!\langle \dots \rangle\!\rangle$ means "normal subgroup generated by") and therefore

$$G/H = \langle a, b \mid a^2, b^2, (ab)^4 \rangle.$$

Claim 2: The set

$$C := \{e,\ a,\ b,\ ab,\ ba,\ aba,\ bab,\ abab\} \ \subset\ G$$

contains a set of coset representatives for G/H. Indeed, any word with two consecutive instances of the same letter can be replaced with a shorter word that is equivalent modulo H (i.e., in the same coset of H), so we can pick a set of coset representatives consisting of words alternating between a's and b's (cf. the discussion of $\mathbb{Z}_2 * \mathbb{Z}_2$ on p.42 of Hatcher). The defining relations of G/H say that $abab$ is its own inverse; on the other hand $(abab)(baba)$ reduces to the empty word, so $(abab)^{-1} = abab = baba$. The relations also say that $ababa = bab$ and $babab = aba$, so any alternating word of length 5 or more can be reduced modulo H to an element of C. This proves Claim 2.

We have shown that

$$\begin{aligned} [G : p_*\pi_1(\tilde{X})] &\leq [G : H] && \text{(because } H \subseteq p_*\pi_1(\tilde{X})\text{)} \\ &\leq 8 && \text{(by the construction of coset representatives)} \\ &= [G : p_*\pi_1(\tilde{X})] && \text{(since } \tilde{X} \text{ is an 8-sheeted covering).} \end{aligned}$$

Therefore, equality must hold throughout, and it follows that $p_*(\pi_1\tilde{X}) = H$ as desired, By the way, G/H is isomorphic to the dihedral group D_4 — which is easily seen to be the set of deck transformations, i.e., the symmetries of the colored octagon.