Math 821 Problem Set #3
Posted: Friday 2/25/11
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Problem #1 **Recall that for a space X and base point $p \in X$, we have defined $\pi_1(X,p)$ to be the set of homotopy classes of p,p-paths on X — or equivalently of continuous functions $S^1 \to X$. Recall also that S^0 consists of two points (let's call them a and b) with the discrete topology. Accordingly, we could define $\pi_0(X,p)$ to be the set of homotopy classes of continuous functions $f : S^0 \to X$ such that $f(a) = p$.**

Describe the set $\pi_0(X,p)$ intrinsically in terms of X. Is there a natural way to endow it with a group structure?

Solution: The homotopy type of such a thing is determined by the path-connected component of X containing $f(b)$. Therefore, a reasonable interpretation for $\pi_0(X,x)$ is as the set of connected components. This set cannot naturally be made into a group.

Problem #2 **(Hatcher, p.38, #2) Show that the change-of-basepoint homomorphism β_h (see p.28) depends only on the homotopy class of the path h.**

Solution: Recall the setup: $x_0, x_1 \in X$; h is a x_0,x_1-path in X; and β_h is the map $\pi_1(X,x_1) \to \pi_1(X,x_0)$ given by $[f] \mapsto [h \cdot f \cdot \bar{h}]$.

Suppose that h_t is a homotopy of x_0,x_1-paths. Then $h_t \cdot f \cdot \overline{h_t}$ is a homotopy of x_0-loops. In particular, if $h \simeq h'$, then $\beta_h[f] \simeq \beta_{h'}[f]$.

Problem #3 **(Hatcher, p.38, #7) Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta,s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on *one* of the boundary circles, but not by any homotopy f_t that is stationary on *both* boundary circles.**

Solution: Visualize $S^1 \times I$ as a cylinder made of rubber, and f as a full twist of the cylinder. (Imagine opening a jar full of extremely old rubber cement.)

(i) Define $f_t : S^1 \times I \to S^1 \times I$ by
$$f_t(\theta,s) = (\theta + 2\pi ts, s).$$
This is evidently a homotopy (it is continuous in each of θ, s, t); f_0 is the identity map; and f_1 is the given map f. Moreover, f_t is stationary on the circle $S^1 \times \{0\}$, i.e., $f_t(\theta,0) = (\theta_0)$.

(ii) Suppose that f_t is a homotopy that is stationary on both boundary circles. That is, $f_t : S^1 \times I \to S^1 \times I$ with
$$f_0(\theta,s) = (\theta,s), \qquad f_t(\theta,0) = (\theta,0),$$
$$f_1(\theta,s) = (\theta + 2\pi s, s), \qquad f_t(\theta,1) = (\theta,1).$$

We want to derive a contradiction. The idea is to draw a line down the side of the cylinder, so that twisting by f wraps the line around the outside in a spiral. Projecting these two paths from $S^1 \times I$ to S^1 will give two closed loops in S^1, one trivial and one that winds once around the circle — so they cannot be homotopic.

Here is a precise argument. Fix some basepoint $\theta_0 \in S^1$. Let g_t be the loop at θ_0 obtained from f_t by restricting its domain to $\{\theta_0\} \times I$, then projecting onto the S^1 factor. That is,

$$g_t(s) = p(f_t(\theta_0, s))$$

where p is the projection map $S^1 \times I \to S^1$. I claim that $\{g_t\}$ is a path homotopy. It certainly is a continuously varying family of functions $I \to S^1$, and

$$g_t(0) = p(f_t(\theta_0, 0)) = (\theta_0, 0) = \theta_0,$$
$$g_t(1) = p(f_t(\theta_0, 1)) = (\theta_0, 1) = \theta_0,$$

which says that each g_t defines a closed loop with basepoint θ_0.

We then have

$$\begin{aligned} g_0(s) &= p(f_0(\theta_0, s)) \\ &= p(\theta_0, s) \\ &= \theta_0, \\ g_1(s) &= p(f_1(\theta_0, s)) \\ &= p(\theta_0 + 2\pi s, s)) \\ &= \theta_0 + 2\pi s. \end{aligned}$$

But these paths have winding numbers 0 and 1 respectively (since they lift to $\widetilde{g_0}(s) = 0$ and $\widetilde{g_1}(s) = 2\pi s$ respectively; recall that the winding number of a loop can be computed from any lift). Therefore, by what we know about $\pi_1(S^1)$, they cannot be homotopic. This is a contradiction and says that no such homotopy f_t can exist.

Problem #4 [Hatcher p.38 #8] Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$? Why or why not?

No. If we parametrize the torus $S^1 \times S^1$ as $\{f(s,t) = (e^{is}, e^{it}) : \ s, t \in [0, 2\pi]\}$, then the antipode of $f(s,t)$ is the point $f(s+\pi, t+\pi)$. We can naturally embed the torus in $\mathbb{R}^3$ as a donut by, e.g.,

$$(e^{is}, e^{it}) \mapsto (5\cos s + \cos t \cos s, \ 5 \sin s + \cos t \sin s, \ \sin t).$$

Then the map $P : S^1 \times S^1 \to \mathbb{R}^2$ given by projection onto the xy-plane satisfies $P(q) = -P(-q) \neq (0,0)$ for all points q on the torus.

Problem #5 [Hatcher p.39 #9] Use the 2-dimensional case of the Borsuk-Ulam theorem (Hatcher, Thm. 1.10, p.32) to prove the "Ham and Cheese Sandwich Theorem": if A_1, A_2, A_3 are compact measurable sets in $\mathbb{R}^3$, then there is a plane in $\mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.

Solution: WLOG (scaling if necessary), assume that $A_1, A_2, A_3 \subset D^3$.

For $\mathbf{v} \in S^2$, let $L_\mathbf{v}$ be the line through $\mathbf{v}$ and $-v$. For $t \in [-1, 1]$, let $P(t, \mathbf{v})$ be the plane parallel to $L_\mathbf{v}$ that meets it at the point $t\mathbf{v}$. Let B_3 be the part of A_3 that is on the same Let $f_\mathbf{v}(t)$ be the fraction of the volume of A_3 that is on the same side of $P(t, \mathbf{v})$ as $-2\mathbf{v}$ is. Thus $f_\mathbf{v}(t)$ increases continuously and monotonically from 0 to 1 as t increases from -1 to 1. Therefore $f_\mathbf{v}^{-1}(1/2)$ is some nonempty closed connected set, i.e., an interval of the form $[a_\mathbf{v}, b_\mathbf{v}]$ (where $a_\mathbf{v}, b_\mathbf{v}$ also depend continuously on $\mathbf{v}$). Let $Q(\mathbf{v}) = P((a_\mathbf{v} + b_\mathbf{v})/2, t)$. Thus $Q_\mathbf{v}$ is a plane parallel to $L_\mathbf{v}$ that depends continuously on $\mathbf{v}$ and, for every $\mathbf{v}$, splits A_3 into two equal-volume pieces. Note also that $Q(\mathbf{v}) = Q(-\mathbf{v})$.

Now, for $\mathbf{v} \in S^2$ and $i = 1, 2$, let $f_i(\mathbf{v})$ be the fraction of the volume of A_i that is on the same side of $Q(v)$ as $\mathbf{v}$ itself is. By the Borsuk-Ulam theorem, there is some pair of antipodal points $\pm\mathbf{v}$ such that $f_i(\mathbf{v}) = f_i(-\mathbf{v})$ for $i = 1, 2$. Since $f_i(-\mathbf{v}) = 1 - f_i(\mathbf{v})$, we have $f_i(\mathbf{v}) = 1/2$, so the plane $Q(\mathbf{v})$ splits each of A_1 and A_2 into two equal pieces as well.

Problem #6 [Hatcher p.39 #12] Fix $p \in S^1$. Show that every homomorphism $\pi_1(S^1, p) \to \pi(S^1, p)$ can be realized as the induced homomorphism ϕ_* for some $\phi : S^1 \to S^1$.

Regard S^1 as the unit circle in $\mathbb{C}$ and let $p = 1$. The path $f : I \to S^1$ given by $s \mapsto \exp(2\pi i s)$ generates the infinite cyclic group $\pi_1(S^1, p) \cong \mathbb{Z}$. Every homomorphism $\alpha : \mathbb{Z} \to \mathbb{Z}$ is specified by the number $n = \alpha(1)$.

Meanwhile, for any $n \in \mathbb{Z}$, the complex function $\phi(z) = z^n$ maps S^1 to S^1, and the path $\phi_* f$ has winding number n because $\phi \circ f(s) = f(s)^n = \exp(2\pi i n s)$ lifts to the map $I \to \mathbb{R}$ given by $s \mapsto ns$.

Problem #7 [Hatcher, p.52, #1] Recall that the center of a group G is defined as $Z(G) = \{g \in G : \; gh = hg \; \forall h \in G\}$.

(#7a) Show that the free product $G * H$ of nontrivial groups G and H has trivial center.

Solution: Any non-identity element $w \in G * H$ can be written uniquely as a product $w = w_1 \cdots w_n$ of non-identity elements of G and H, with letters w_i alternating between G and H (p.42). If $w_1 \in H$ then w does not commute with any non-identity element of G, while if $w_1 \in G$ then w does not commute with any non-identity element of H. Therefore, the only element of the center is the word of length 0, namely e.

(#7b) Show that the only elements of $G*H$ of finite order are the conjugates of finite-order elements in $G \cup H$.

Suppose that $w \in G * H$ and $w^n = e$. Write w in reduced form: $w = g_1 \cdots g_k$ where the letters alternate between G and H.

$$w^n = (g_1 \cdots g_k)(g_1 \cdots g_k) \; \cdots \; (g_1 \cdots g_k) = e.$$

We need to be able to somehow cancel this expression using only relations within G and H. The only possibility is that g_k and g_1 either both belong to G or both to H, and that $g_k = g_1^{-1}$. Note that this implies that k is odd, say $k = 2K + 1$. Canceling gives

$$w^n = (g_2 \cdots g_{k-1})(g_2 \cdots g_{k-1}) \; \cdots \; (g_2 \cdots g_k) = e.$$

Now the only possibility for cancellation is that $g_{k-1} = g_2^{-1}$. Continuing in this way, we eventually find that

$$g_k = g_1^{-1}, \quad g_{k-1} = g_2^{-1}, \quad \ldots, \quad g_{K+2} = g_K^{-1}.$$

But this says that $w = xyx^{-1}$, where $x = g_1 \cdots g_K$ and $y = g_{K+1}$. Moreover, y belongs to either G or H (because it is a single letter), and

$$y^n = (x^{-1} w x)^n = x^{-1} w^n x = x^{-1} x = e$$

so y has finite order. So we have shown that every finite-order element of $G*H$ is a conjugate of a finite-order element of one of G or H.