Math 821, Spring 2014
Solution Set #2
Due date: Friday, February 14

Problem #1 Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y. Prove also the corresponding statements with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent to X.

Solution: In general, if p is a point in a topological space, let's write $[p]$ for the component of that space containing p, and $\langle p \rangle$ for the path-component of that space containing p.

First, we want to prove that

$$\langle x \rangle = \langle x' \rangle \iff \langle f(x) \rangle = \langle f(x') \rangle.$$

The $\implies$ direction is easy: if ϕ is an x, x'-path in X, then $f \circ \phi$ is an $f(x), f(x')$-path in Y.

For the reverse direction, if ψ is an $f(x), f(x')$-path in Y, then $g \circ \psi$ is a $g(f(x)), g(f(x'))$-path in X. On the other hand, $\langle x \rangle = \langle g(f(x)) \rangle$ because, by definition of homotopy equivalence, there is a homotopy $h_t : X \to X$ with $h_0 = \mathbb{1}$ and $h_1 = g \circ f$; the function $\gamma : I \to X$ given by $\gamma(t) = h_t(x)$ therefore defines a path from x to $g(f(x))$. Similarly, we can construct a path from $g(f(x'))$ to x'. Concatenating these paths with $g \circ \psi$ gives an x, x'-path in X and establishes the $\impliedby$ direction.

Second, we want to prove that $[x] = [x'] \iff [f(x)] = [f(x')]$. The $\implies$ direction follows from the fact that the continuous image of a connected space is connected.

We now want to show that if $[f(x)] = [f(x')]$, then $[x] = [x']$. By the $\implies$ direction, the hypothesis implies that $[g(f(x))] = [g(f(x'))]$. Again, consider the homotopy $H : X \times I \to X$ with $H(z, 0) = z$ and $H(z, 1) = g(f(z))$. Let $U = [H(x, 1)]$ and $V = X \setminus U$; then the clopen decomposition $X = U \cup V$ pulls back to a clopen decomposition

$$X \times I = H^{-1}(U) \cup H^{-1}(V)$$

but both $(x, 1)$ and $(x', 1)$ lie in the same piece of this decomposition because H maps them into the same component of X. That piece must $H^{-1}(U)$, On the other hand, path-components are contained in connected components, and X has paths from $(x, 0)$ to $(x, 1)$ and from $(x', 0)$ to $(x', 1)$, namely

$$\phi(t) = H(x, t), \qquad \phi'(t) = H(x', t),$$

so $(x, 0)$ and $(x', 0)$ belong to $H^{-1}(U)$ as well, which is to say that $x, x' \in U$ as desired.

Third, once we know that f maps (path-)components to (path-)components, we know that for every component C of X, the composition $g \circ f$ maps C to some (path-)component C' of X. On the other hand, if $\{h_t\}$ is a homotopy between $g \circ f$ and $\mathbb{1}_X$, then for every $p \in C$ there is a path $t \mapsto h_t(p)$ from p to $g(f(p))$. This is only possible if $C' = C$. Now $\{h_t|_C\}$ is a homotopy between $(g \circ f)|_C = (g|_D) \circ (f|_C)$ (where D is the (path-)component of Y containing $f(C)$) and $\mathbb{1}_C$.

The "deduce that…" part is immediate. This problem is another justification that homotopy equivalence is a sensible thing to consider.

Some students (at least in 2011) argued that since the continuous image of a (path-)connected space is (path-)connected, the maps f and g induce surjections $P_X \to P_Y$ and $P_Y \to P_X$ respectively, where P_X means the set of (path-)connected components of X; therefore, $|P_X| \geq |P_Y| \geq |P(X)|$ and equality holds throughout. However, these numbers may be infinite, when the statement "$a \leq b \leq a \implies a = b$" is the (highly nontrivial) Bernstein-Schröder-Cantor theorem of set theory. In the context of topology, I think it's more natural to show directly that f and g induce bijections.

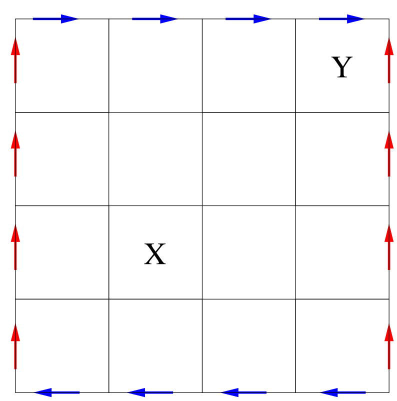
Problem #2 Let p, q be distinct points on S^2, and let X be the space obtained by gluing them together. Determine the homotopy type of X.

Solution: $X \simeq S^1 \vee S^2$. This is Example 0.11 in Hatcher (p.13), so I only assigned 5 points for it. But it is helpful to be able to see the homotopy equivalence for yourself. Here is my description of it.

To see this, start with $S^1 \vee S^2$, which looks like a sphere glued to a circle at a point p. Let's put the circle inside the sphere (figure; left) Think of the circle as a closed loop from p to itself. Move the other one endpoint of the loop to another point $q \neq p$ (figure; center); this move is a homotopy equivalence by Proposition 0.18. Then collapse the resulting non-closed path to a point (figure; right); this is a homotopy equivalence by Prop. 0.17, and the resulting space is X.

Problem #3 For $k \geq 1$, let T_n denote the n-holed torus. Construct a cell complex structure on T_n.

Solution: Again, this is an example in Hatcher (p.5), so only 5 points. There are lots of ways to impose a cell complex structure. Here's one, which stems from the observation that $T_n = T_1 \# T_{n-1}$, where $\#$ is the operation of *connected sum*: cut out a small disk from each of the operands, then glue their boundaries together.[1] This basically reduces the problem to finding a fine enough cell structure on the torus that is compatible with this operation. For example, start with the cell structure on the 1-hole torus shown below. Make n photocopies, delete the cells $X_1, Y_2, X_2, \ldots, X_{n-1}, Y_n$ (here, e.g., Y_2 means "the cell labeled Y in the 2nd photocopy") and identify the boundaries: $\partial X_1 = \partial Y_2$, ..., $\partial X_{n-1} = \partial Y_n$.



Problem #4 Let X be a finite graph lying in a half-plane $P \subset \mathbb{R}^3$ and intersecting the edge e of P in a subset of its vertices. Describe the homotopy type of the "surface of revolution" obtained by rotating X about e.

Solution: Let $R(X)$ be the surface of revolution thus obtained. First, we can assume that X is connected — if it has multiple connected components X_α, then $R(X)$ is the disjoint union of the $R(X_\alpha)$.

[1]This is not a purely topological operation, in the sense that you need to know the dimensions of two spaces in order to build their connected sum.

Second, observe that if a is an edge of X that has at least one endpoint not in e, then $R(X) \simeq R(X/a)$; this corresponds to deformation-retracting an annulus (specifically, the piece of $R(X)$ corresponding to a) onto a circle. (We just need to make sure that no other point hits the boundary during the contraction process. For example, if such a contraction produces an edge between two vertices on e, then that edge needs to be some kind of an arc rather than a line segment.)

Let

$$\begin{aligned} n &= \text{total number of vertices of } X, \\ b &= \text{number of vertices of } X \text{ lying on } e, \\ m &= \text{number of edges of } X. \end{aligned}$$

If $b = 0$, then we can eventually replace X with a graph Y with one vertex and $m - n + 1$ loops, while preserving the homotopy type of $R(X)$. We then have

$$R(X) \simeq R(Y) = S^1 \times X = S^1 \times (S^1)^{(m-n+1)}$$

(which is the infamous "not-a-torus" space).

If $b > 1$, then we will eventually end up with a graph Y all of whose vertices lie on e, and such that $R(Y) \simeq R(X)$. Then Y has b vertices and $m - (n - b)$ edges.

Let T be a spanning tree of $R(Y)$. Then T has $b - 1$ edges, each of which gets rotated into a 2-sphere, and so $R(T) \simeq (S^2)^{\vee(b-1)}$. (This notation means "the wedge sum of $b - 1$ copies of S^2".) For each additional edge $a \notin T$, the revolution $R(a)$ is either a sphere attached to $R(T)$ at two points (if a is not a loop) or a sphere with two of its points identified, then attached to $R(T)$ (if a is a loop). In either case, the edge a contributes an additional $S^1 \vee S^2$ to the homotopy type. We conclude that

$$\begin{aligned} R(X) \simeq R(Y) &\simeq (S^2)^{\vee(b-1)} \ \vee \ (S^1 \vee S^2)^{\vee(m-(n-b)-(b-1))} \\ &= (S^2)^{\vee(b-1)} \ \vee \ (S^1 \vee S^2)^{\vee(m-n+1)} \\ &= \boxed{(S^2)^{\vee(m-n+b)} \ \vee \ (S^1)^{\vee(m-n+1)}.} \end{aligned}$$

(The graph pictured has $n = 9$, $b = 4$, $m = 15$, $m - n + b = 10$, $m - n + 1 = 7$.)

A comment: I should have required that each component of X have at least one vertex on e. (Otherwise, if for example X consists of one vertex off e and a loop attached to it, then $R(X)$ is a torus rather than a wedge of spheres.)

Problem #5 Let $0 \leq k \leq n$. Recall from class that the *Grassmannian* $G(k, n)$ is defined as the space of k-dimensional subspaces $V \subset \mathbb{R}^n$, so that in particular, $G(1, \mathbb{R}^n) = \mathbb{R}P^{n-1}$. (Fact: Everything in this problem works the same way if you change $\mathbb{R}$ to $\mathbb{C}$, except that the dimensions of all the cells get doubled.)

(#5a) Work out an explicit cell decomposition for $G(2, 4)$ as a finite CW-complex. That is, describe how to decompose the set $G(2, 4)$ into pieces, each of which is isomorphic to a $\mathbb{R}$-vector space. If you do this correctly (hint: row-reduced echelon form), then the isomorphisms should be straightforward from the construction.

Any $V \in G(2, 4)$ can be expressed as the column span of a $k \times n$ matrix M. Performing elementary column operations on the matrix doesn't change the span, and we know that we can eventually put M into a unique reduced column-echelon form, i.e., one of the following things:

$$\begin{bmatrix}1&0\\0&1\\*&*\\*&*\end{bmatrix},\quad \begin{bmatrix}1&0\\*&0\\0&1\\*&*\end{bmatrix},\quad \begin{bmatrix}1&0\\*&0\\*&0\\0&1\end{bmatrix},\quad \begin{bmatrix}0&0\\1&0\\0&1\\*&*\end{bmatrix},\quad \begin{bmatrix}0&0\\1&0\\*&0\\0&1\end{bmatrix},\quad \begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}$$

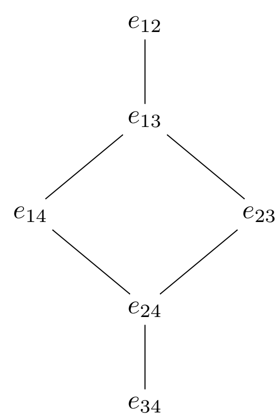
This gives a cell structure with f-polynomial

$$f(X,q) := \sum_{e_\alpha \in X} q^{\dim e_\alpha} = q^4 + q^3 + 2q^2 + q + 1.$$

Specifically, the cells classify points in $G(2,4)$ by the locations of the pivots in its reduced column-echelon form. Moving the pivots upwards gives a bigger cell; specifically, if we write e_{ij} for the cell whose pivots are in rows i, j with $i < j$, then $e_{ij} \subseteq \overline{e_{i'j'}}$ iff $i \le i'$ and $j \le j'$.

(#5b) Describe the attaching poset of $G(2,4)$. (Recall that this is the partially ordered set whose elements are the cells e_α, and whose order relation is given by $e_\alpha \ge e_\beta$ if $\overline{e_\alpha} \supseteq e_\beta$).

It looks like this:

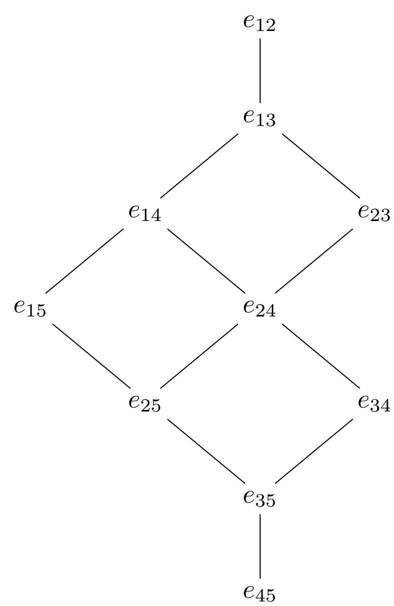


(#5c) Describe the attaching poset of $G(2,5)$.

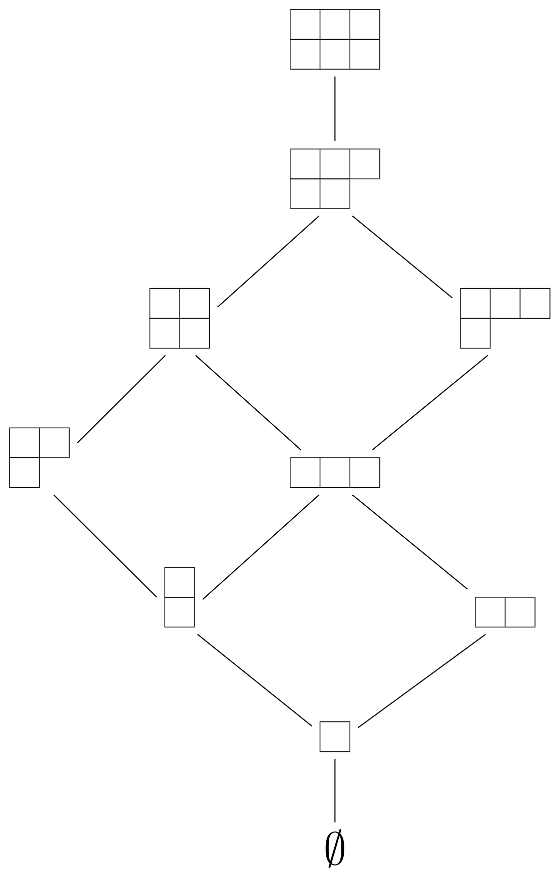
The cells can be labeled $\{e_{ij} \mid 1 \le i < j \le 5\}$. The order relation is given combinatorially by

$$\overline{e_{ij}} \supset e_{i'j'} \qquad \Longleftrightarrow \qquad i \le i' \quad \text{and} \quad j \le j'.$$

Here is the whole partially ordered set:



(#5d) Write out the poset $P(2,3)$ of all Ferrers diagrams with at most two rows and at most three columns, ordered by containment (as sets of squares). Compare it to your previous answer.



The two posets are isomorphic. This is true in general — the attaching poset of the Schubert cell decomposition of $G(k,n)$ is isomorphic to the lattice of partitions that fit inside a $(n-k)\times k$ rectangle. The number of squares in a partition equals the dimension of the corresponding Schubert cell. By the way, the general formula for the number of Schubert cells of each dimension is very nice:

$$\sum_{e_\alpha \in G(k,n)} q^{\dim e_\alpha} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}.$$

This is called a *q-binomial coefficient.* It isn't even obvious that this expression is a polynomial — but in fact it is.

Problem #6 [Extra credit; Hatcher p.19, #20] Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure on p.19 of Hatcher, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

Solution: I'm not going to try to draw the picture in LaTeX, but here's the idea – try to see it in your head. Look at the disk where the cylinder intersects itself. Squash that to a point — this is a homotopy equivalence. What is left looks like a sphere in which three points have been identified. By an argument just like that of Problem #2 above, this space is homotopy-equivalent to $S^2 \vee S^1 \vee S^1$.