Math 821, Spring 2014
Solution Set #1
Due date: Friday, January 31

Problem #1 Let X be a path-connected topological space. Prove that X is connected. (Recall that the converse is not true — the topologists' sine curve is a counterexample.)

Solution: We need to show that X has no subset U that is "clopen" (i.e., both closed and open) other than $\emptyset$ or X. Suppose U is such a clopen subset, so that its complement $V = X \setminus U$ is also clopen. Suppose that $p \in U$ and that $q \in X \setminus U$. Let P be a p,q-path in X parametrized by $f : I = [0,1] \to X$. Then $P = (P \cap U) \cup\!\!\!\cdot\, (P \cap V)$. Let $A = f^{-1}(P \cap U)$ and $B = f^{-1}(P \cap V)$. Then

- $A \cup B = I$ (because $f(I) = P$);
- $A \cap B = \emptyset$ (because $f(A) \cap f(B) = \emptyset$);
- A, B are open in I (by continuity);
- $\emptyset \subsetneq A, B \subsetneq I$ (because $0 \in A \setminus B$ and $1 \in B \setminus A$).

But this is a contradiction because I is connected. Therefore, no such pair p, q can exist, which says that one of $U, X \setminus U$ is empty. It follows that X is connected. (In principle, this argument uses the continuity of f to reduce the "path-connected implies connected" statement about an arbitrary topological space to the same statement about the familiar topological space I.) □

Alternate Solution: Fix $x \in X$. For every $y \in X$, choose a path $f_y : I \to X$ with $f_y(0) = x$ and $f_y(1) = y$, and let $P_y = f_y(I) \subseteq X$. Then each P_y is connected (because it is the continuous image of the connected space I) and $\bigcap_{y \in X} P_y \neq \emptyset$ (because it contains x), so $\bigcup_{y \in X} P_y = X$ is connected.

Problem #2 Let Γ be a finite graph. Prove that if Γ is connected, then it is path-connected.

Solution: I'll prove something more general: every space X that is both connected and locally path-connected is path-connected. ("Locally path-connected" means that every point x has a connected open neighborhood U_x.) A graph is locally path-connected because every vertex has a neighborhood that looks like the vertex itself plus d rays sticking out (where d is the degree of the vertex — the number of edges attached to it, possibly infinite, counting a loop as two edges) and every point on the interior of an edge has a neighborhood that looks like an open interval.

Let $x \in X$ and let Y be the set of all points that are joined to x by a path. For every $y \in Y$ and $q \in U_y$, we have an x,y-path and a y,q-path; concatenating them produces an x,q-path. Therefore $Y = \bigcup_{y \in Y} U_y$ is open.

On the other hand, let $Z = X \setminus Y$. For every $z \in Z$ and $q \in U_z$, we cannot have an q,x-path, since we certainly have a z,q-path and concatenating the two would produce an z,x-path, which cannot exist. Therefore, $Z = \bigcup_{z \in Z} U_z$ is open.

We have constructed a clopen decomposition $X = Y \cup\!\!\!\cdot\, Z$ with $Y \neq \emptyset$ (because $x \in Y$). Since X is connected, we must have $Y = X$ and $Z = \emptyset$. This is precisely the statement that X is path-connected.

In fact, we don't even need the assumption of finiteness — any cell complex will work. What this tells us, among other things, is that the topologists' sine curve cannot be realized as a cell complex.

Problem #3 Let X and Y be topological spaces and let $f : X \twoheadrightarrow Y$ be a continuous function that is onto. Prove that if X is compact, then so is Y.

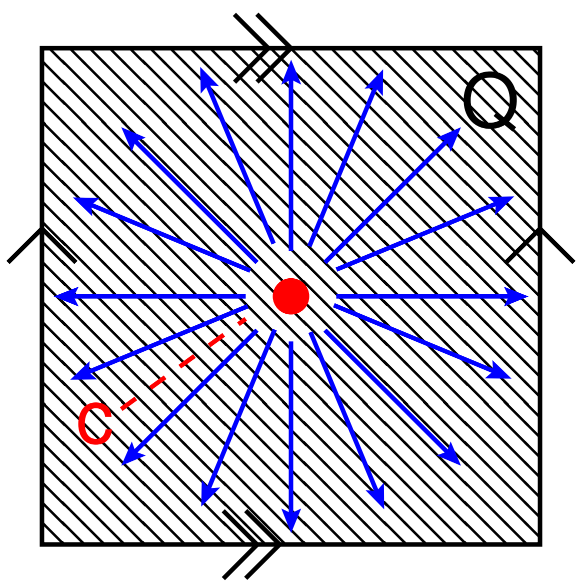
Solution: Let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of Y. Then $\{f^{-1}(U_\alpha) \mid \alpha \in A\}$ is an open cover of X (because every point in X gets mapped to a point in at least one U_α, hence belongs to $f^{-1}(U_\alpha)$). By compactness, it has a finite subcover: $\{f^{-1}(U_\alpha) \mid \alpha \in A'\}$, where $A' \subseteq A$ is finite. I.e., $X = \bigcup_{\alpha \in A'} f^{-1}(U_\alpha)$. This implies set-theoretically that $Y = \bigcup_{\alpha \in A'} U_\alpha$, so there we have the desired finite subcover.

Problem #4 [Hatcher p.18 #1] Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point.

Solution: Draw a square Q from which the center point c has been deleted. For each point $p \in Q \setminus c$, draw the ray $\overrightarrow{cp}$ and let $f(p)$ be the point where that ray hits the boundary ∂Q. Define $F : Q \times I \to Q$ by

$$F(p, t) = (1 - t)p + tf(p)$$

(where the arithmetic is vector arithmetic in $\mathbb{R}^2$). Note that $F(0, p) = p$ and $F(1, p) = f(p)$. Moreover, if $p \in \partial Q$ then $F(p, t) = p$ for all t. So F is a deformation retraction.



If we pass from Q to the torus T by identifying opposite sides, the map F is still well-defined and is a deformation retraction. Note that ∂Q maps onto the union of two circles that meet in a point — one longitudinal and one meridional circle.

Note: Many of you found the explicit formula for f. Specifically, if $p = (x, y)$ in Cartesian coordinates, then

$$f(p) = \frac{1}{\max(|x|, |y|}p$$

which is well-defined precisely because $p \neq (0, 0)$. However, the geometric description above is sufficient.

One solver (Billy) found another way to say this: build the torus by starting with the closed unit **disk**, partitioning its boundary circle into four 90° arcs (say, the intersections with the four quadrants in $\mathbb{R}^2$), and identifying them. Then the deformation retraction can be expressed very naturally in polar coordinates: $F((r, \theta), t) = ((1-t)r + t, \theta)$.

Problem #5 [Hatcher p.18 #3, more or less] (a) Show that homotopy equivalence of spaces is an equivalence relation.

For reflexivity, the identity map is a homotopy equivalence, and symmetry is immediate from the definition. For transitivity, suppose we have maps as shown that are all homotopy equivalences.

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \underset{k}{\overset{h}{\rightleftarrows}} Z$$

Let $p = g \circ k \circ h \circ f$. We need to construct a homotopy $\alpha \simeq \mathbb{1}_X$. By hypothesis, suppose we have a homotopy

$$q_t : Y \to Y, \qquad q_0 = \mathbb{1}_Y, \quad q_1 = k \circ h.$$

Then $g \circ q_t \circ f$ is a homotopy with $g \circ q_1 \circ f = p$ and $g \circ q_t \circ f = g \circ f$. Therefore

$$p = g \circ k \circ h \circ f \simeq g \circ f \simeq \mathbb{1}_X.$$

(b) Fix spaces X, Y and let f, g be maps $X \to Y$. Show that the relation "f is homotopic to g" is an equivalence relation.

Reflexivity: $f \simeq f$ by the homotopy $F(x,t) = f(x)$.

Symmetry: If $F(x,t)$ is a homotopy between f and g then $F(x, 1-t)$ is a homotopy between g and f.

Transitivity: If $F(x,t), G(x,t)$ realize homotopies $f \simeq g$ and $g \simeq h$, then define $H(x,t) = F(x,2t)$ for $0 \leq t \leq 1/2$ and $H(x,t) = G(x, 2t-1)$ for $1/2 \leq t \leq 1$.

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Lemma: Let f_0, f_1 be homotopic maps $X \to Y$. Let $a : W \to X$ and $b : Y \to Z$ be maps. Then $f_0 \circ a \simeq f_1 \circ a$ and $b \circ f_0 \simeq b \circ f_1$.

Proof: If $F : X \to I$ is a homotopy between f_0 and f_1, then $F \circ (a \times \mathbb{1})$ is a homotopy between $f_0 \circ a$ and $f_1 \circ a$ and $b \circ F$ is a homotopy between $b \circ f_0$ and $b \circ f_1$.

Corollary: If $f_0 \simeq f_1$ and g is a homotopy inverse for f_0, then by the lemma we have $f_1 \circ g \simeq f_0 \circ g \simeq \mathbb{1}_Y$ and $g \circ f_1 \simeq g \circ f_1 \simeq \mathbb{1}_X$. So something even stronger is true: if g is a homotopy inverse for f then it is a homotopy inverse for any map to which F is homotopic.

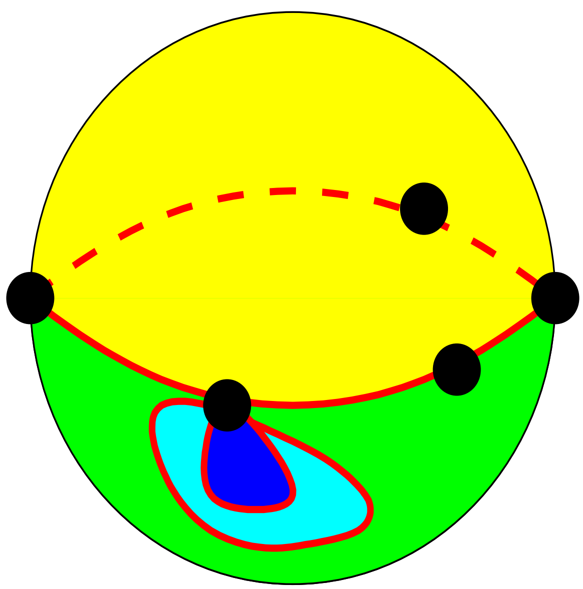
Problem #6 [Hatcher p.19 #14] Given nonnegative integers v, e, f with $v - e + f = 2$ (and $v, f > 0$), construct a cell structure on S^2 having v 0-cells, e 1-cells, and f 2-cells. (Do not use any facts about spanning trees or Euler characteristic.)

Solution: First, a terminological note. The following phrases are all synonyms:

- "cell structure on S^2";
- "cell complex homeomorphic to S^2";
- "cellular 2-sphere";
- "cellular S^2";
- "cellulation of S^2".

There is only one cellular S^2 with $(v, e, f) = (1, 0, 1)$: take a 2-cell and squash its boundary to a point. Equivalently, this is the one-point compactification of $\mathbb{R}^2$.

For the case $e > 0$, there are several constructions; here is one. Draw a sphere with an equator, and put v vertices on the equator, making v edges between them. (If $e = 1$ this means that the equator is a loop; that's okay.) Then pick one of the vertices and draw $f - 2$ nested loops at it, all reaching into the southern hemisphere. We end up with a cell structure with v vertices and $v + f - 2 = e$ edges. The number of faces is f because the equator separates the globe into two 2-cells (the northern and southern hemispheres), and each loop adds one more to the count of 2-cells. For example, here is a picture with $(v, e, f) = (5, 7, 4)$:



Note 1: In order to have a *regular* cellular 2-sphere, I believe it is necessary and sufficient to have at least two cells of each dimension.

Note 2: Many of you broke the problem into several cases depending on the value of e. This is OK as a means of solving the problem, but when you write it up, you should see if you can find a simpler solution that does not require case analysis.