Math 796 Problem Set #6
Due Wednesday, May 7

Problem #1 Consider the permutation action of the symmetric group $\mathfrak{S}_4$ on the vertices of the complete graph K_4, whose corresponding representation is the defining representation ρ_{def} (let's say over $\mathbb{C}$). Let σ be the 3-dimensional representation corresponding to the action of $\mathfrak{S}_4$ on pairs of opposite edges of K_4.

(#1a) Compute the character of σ.

(#1b) Explicitly describe all G-equivariant linear transformations $\phi : \rho_{\text{def}} \to \sigma$. (Hint: Schur's lemma should be useful.)

Problem #2 Recall that the *alternating group* $\mathfrak{A}_n$ consists of the $n!/2$ even permutations in $\mathfrak{S}_n$, that is, those with an even number of even-length cycles.

(#2a) Show that the conjugacy classes in $\mathfrak{A}_4$ are not simply the conjugacy classes in $\mathfrak{S}_4$. (Hint: Consider the possibilities for the dimensions of the irreducible characters of $\mathfrak{A}_4$.)

(#2b) Determine the conjugacy classes in $\mathfrak{A}_4$, and the complete list of irreducible characters.

(#2c) Use this information to determine $[\mathfrak{A}_4, \mathfrak{A}_4]$ without actually computing any commutators.

Problem #3 Let n be a positive integer, and let D_n be the dihedral group $\langle x, y \mid x^n = y^2 = 1,\ yxy = x^{-1}\rangle$. Let C_n be the cyclic subgroup generated by x, and let ρ be the irreducible representation of C_n mapping x to $\zeta = e^{2\pi i/n} \in \mathbb{C}$. Show that $\operatorname{Ind}_{C_n}^{D_n} \rho$ is isomorphic to the defining representation of D_n (that is, its two-dimensional representation as the group of symmetries of $\mathbb{R}^2$ fixing a regular n-gon).

Problem #4 For each $\mu \vdash 4$, let ρ_μ be the permutation representation of the symmetric group $\mathfrak{S}_4$ on tabloids of shape μ [see class notes 4/18/08].

(#4a) Compute the characters $\chi_\mu = \chi_{\rho_\mu}$. Give your answer as a 5×5 matrix $[\chi_{\lambda,\mu}]$, with rows indexed by μ and columns corresponding to the conjugacy classes $C_\lambda \subset \mathfrak{S}_4$.

(#4b) Apply the Gram-Schmidt process to the rows of this matrix to produce a list of irreducible characters $\tilde{\chi}_\nu$ of $\mathfrak{S}_4$, labeled by the partitions $\nu \vdash 4$, so that $\langle \tilde{\chi}_\nu,\, \chi_{\rho_\nu} \rangle_G \neq 0$ and $\left\langle \tilde{\chi}_\nu,\, \chi_{\rho_\mu} \right\rangle_G = 0$ if $\nu < \mu$.

(#4c) Express the characters χ_{ρ_μ} as linear combinations of the irreducible characters $\tilde{\chi}_\nu$.

Problem #5 Recall that for $\lambda, \mu \vdash n$, the Kostka number $K_{\lambda\mu}$ is defined as the number of column-strict tableaux of shape λ and content μ (that is, having μ_1 1's, μ_2 2's, etc.) Prove that $K_{\lambda\mu} = 0$ unless $\lambda \trianglerighteq \mu$. (Together with the fact that $K_\lambda = 1$ for all λ, this implies that the Schur symmetric functions are a graded $\mathbb{Z}$-basis for Λ.)