Math 796 Problem Set #5
Due Friday, April 11

Problem #1 Let D be a loopless directed graph with vertices V and edges $A = \{a_1, \dots, a_n\}$. That is, V is a finite set, and E is a set of ordered pairs (v, w), where v, w are distinct vertices. We do not allow parallel copies of the same edge, but we do allow antiparallel edges (i.e., (v, w) and (w, v) can both be edges). Replacing each directed edge $(v, w) \in A$ with an undirected edge vw yields a graph G, called the *underlying graph* of D.

Let $(a_{i_1}, \dots, a_{i_m})$ be distinct edges that form a cycle in G of length $m \geq 2$. That is, there are distinct vertices $v_1, \dots, v_m$ such that for every $j \in [m]$, either $a_{i_j} = (v_j, v_{j+1})$ or $a_{i_j} = (v_{j+1}, v_j)$ (where for convenience $v_{m+1} = v_m$). Define an n-tuple $c = (c_1, \dots, c_n) \in \{+, -, 0\}^n$ as follows:

- If $a_{i_j} = (v_j, v_{j+1})$, then $c_{i_j} = +$.
- If $a_{i_j} = (v_{j+1}, v_j)$, then $c_{i_j} = -$.
- $c_i = 0$ if $i \notin \{i_1, \dots, i_m\}$.

Prove that the set $\mathscr{C}$ of all such c forms a circuit system for an oriented matroid.

Problem #2 Let $G = (V, E)$ be a connected graph. A *matching* in G is a set of edges $M \subset E$ such that no two share an endpoint; that is, each vertex of G is incident to at most one member of E. A *vertex cover* is a set $K \subset V$ such that every edge has at least one endpoint in K. Let $m(G)$ be the maximum size of a matching in G, and let $c(G)$ be the minimum size of a vertex cover.

(#2a) Prove that $m(G) \leq c(G)$.

(#2b) Use the Max-Flow/Min-Cut Theorem to prove that $m(G) = c(G)$ when G is bipartite. (Hint: Build a network N from G by adjoining a source s adjacent to all white vertices of G and a sink t adjacent to all black vertices. Choose a capacity function appropriately so that flows and cuts in N correspond to matchings and vertex covers in G, respectively.)

Problem #3 Let P be a finite poset, and let λ and μ be the partitions described in the Greene-Kleitman Theorem.

(#3a) Construct a poset P such that for every antichain A of size μ_1, there does *not* exist any antichain A' disjoint from A such that $|A \cup A'| = \mu_2$.

(#3b) Verify the Greene-Kleitman Theorem for the poset you have constructed.

Problem #4 Prove that supersolvable graphs are perfect. Use the chraacteriazation of supersolvability given in Theorem 5 from the class notes on 3/12/08. Do not use either the fact that "supersolvable" and "chordal" are equivalent (unless you prove it), or the Strong Perfect Graph Theorem (unless you prove it).

Problem #5 Let a and b be relatively prime integers, both greater than 1. Find the number of equivalence classes of necklaces with a red beads and b blue beads in terms of a and b. (Hint: Consider two cases—either a, b are both odd, or else one is odd and one is even.)