Math 796 Problem Set #4
Due Friday, March 14

Problem #1 (Stanley, EC1, 3.45) Prove the *q-binomial theorem*:

$$\prod_{k=0}^{n-1}(x-q^k) \;=\; \sum_{k=0}^{n}\binom{\mathbf{n}}{\mathbf{k}}(-1)^k q^{\binom{k}{2}} x^{n-k}.$$

Here $\binom{\mathbf{n}}{\mathbf{k}}$ denotes the *q-binomial coefficient*:

$$\binom{\mathbf{n}}{\mathbf{k}} \;=\; \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})};$$

see Stanley, EC1, pp. 26–28, or Aigner [who uses the notation $\binom{n}{k}_q$ and calls them "Gaussian coefficients"], pp. 69, 79, 94. You may, with appropriate citation, use identities such as (17b) on p. 26 of Stanley.

(Hint: Let $V=\mathbb{F}_q^n$ and let X be a vector space over $\mathbb{F}_q$ with x elements. Count the number of one-to-one linear transformations $V\to X$ in two ways.) Derive the ordinary binomial theorem as a corollary.

Problem #2 (Stanley, EC1, Supp. 4) Let P be a finite poset, and let μ be the Möbius function of $\hat{P}=P\cup\{\hat{0},\hat{1}\}$. Suppose that P has a fixed-point-free automorphism $\sigma: P\to P$ of prime order p; that is, $\sigma(x)\neq x$ and $\sigma^p(x)=x$ for all $x\in P$. Prove that $\mu(\hat{0},\hat{1})\cong -1 \pmod{p}$. What does this say in the case that $\hat{P}=\Pi_p$?

Problem #3 Let $\mathcal{A}$ be a real hyperplane arrangement. Recall that the *dual graph* G of $\mathcal{A}$ has as its vertices the regions of $\mathcal{A}$, with two vertices joined by an edge if the corresponding regions share a common boundary. Prove that G is bipartite.

Problem #4 (Stanley, HA, 2.5) Let K be a field, let G be a graph on n vertices, and let $\mathcal{B}_G=\mathscr{B}_n\cup\mathcal{A}_G$; that is, $\mathcal{B}$ consists of the coordinate hyperplanes in K^n together with the hyperplanes $x_i=x_j$ for all edges ij of G. Calculate $\chi_{\mathcal{B}_G}(k)$ in terms of $\chi_{\mathcal{A}_G}(k)$.