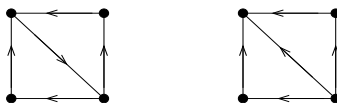


Math 796 Problem Set #3  
Due Friday, February 29

**Problem #1** Let  $X$  and  $Y$  be disjoint sets of vertices, and let  $B$  be an  $X, Y$ -bipartite graph: that is, every edge of  $B$  has one endpoint in each of  $X$  and  $Y$ . For  $V = \{x_1, \dots, x_n\} \subset X$ , a *transversal* of  $V$  is a set  $W = \{y_1, \dots, y_n\} \subset Y$  such that  $x_i y_i$  is an edge of  $B$ . Let  $\mathcal{S}$  be the family of all subsets of  $X$  that have a transversal. Prove that  $\mathcal{S}$  is a matroid independence system. (A matroid that arises in this way is called a *transversal matroid*.)

**Problem #2** Construct two sets of vectors  $S, S' \subset \mathbb{R}^3$  such that  $T(M(S)) = T(M(S'))$  but  $M(S) \not\cong M(S')$ . (Instead of giving coordinates, describe each matroid by an affine point arrangement in  $\mathbb{R}^2$ .)

**Problem #3** Let  $G = (V, E)$  be a connected graph. An orientation of  $G$  is called *strong* if every edge is part of a (directed cycle); equivalently, there is a path from every vertex to every other vertex by following directed edges. For instance, the orientation on the left is strong, but the one on the right is not.



Let  $s(G)$  be the number of strong orientations of  $G$ . Prove that  $s(G) = T_G(0, 2)$ . (Hint: Find a deletion-contraction recurrence for  $s(G)$ .)

**Problem #4** Use the corank-nullity form of the Tutte polynomial, together with the definition of matroid duality, to prove that  $T(M, x, y) = T(M^*, x, y)$  for every matroid  $M$ .

**Problem #5** Let  $P$  be a chain-finite poset. The *kappa function* of  $P$  is the element of the incidence algebra  $I(P)$  defined by  $\kappa(x, y) = 1$  if  $x < y$ ,  $\kappa(x, y) = 0$  otherwise.

(#5a) Give a condition on  $\kappa$  that is equivalent to  $P$  being ranked.

(#5b) Give combinatorial interpretations of  $\kappa * \zeta$  and  $\zeta * \kappa$ .

**Problem #6** Let  $\Pi_n$  be the lattice of set partitions of  $[n]$ . Recall that the order relation on  $\Pi_n$  is given as follows: if  $\pi, \sigma \in \Pi_n$ , then  $\pi \leq \sigma$  if every block of  $\pi$  is contained in some block of  $\sigma$  (for short, “ $\pi$  refines  $\sigma$ ”). In this problem, you’re going to calculate the number  $\mu_n := \mu_{\Pi_n}(\hat{0}, \hat{1})$ .

(#6a) Calculate  $\mu_n$  by brute force for  $n = 1, 2, 3, 4$ . Make a conjecture about the value of  $\mu_n$  in general.

(#6b) Define a function  $f : \Pi_n \rightarrow \mathbb{Q}[x]$  as follows: if  $X$  is a finite set of cardinality  $x$ , then

$$f(\pi) = \#\{h : [n] \rightarrow X \mid h(s) = h(s') \iff s, s' \text{ belong to the same block of } \pi\}.$$

For example, if  $\pi = \hat{1} = \{\{1, 2, \dots, n\}\}$  is the one-block partition, then  $f(\pi)$  counts the constant functions from  $[n]$  to  $X$ , so  $f(\pi) = x$ . Find a formula for  $f(\pi)$  in general.

(#6c) Let  $g(\pi) = \sum_{\sigma \geq \pi} f(\sigma)$ . Prove that  $g(\pi) = x^{|\pi|}$  for all  $\pi \in \Pi_n$ . (Hint: What kinds of functions are counted by the sum?)

(#6d) Apply Möbius inversion and an appropriate substitution for  $x$  to calculate  $\mu_n$ .