Math 796 Problem Set #3
Due Friday, February 29

Problem #1 Let X and Y be disjoint sets of vertices, and let B be an X,Y-bipartite graph: that is, every edge of B has one endpoint in each of X and Y. For $V = \{x_1, \dots, x_n\} \subset X$, a *transversal* of V is a set $W = \{y_1, \dots, y_n\} \subset Y$ such that $x_i y_i$ is an edge of B. Let $\mathscr{I}$ be the family of all subsets of X that have a transversal. Prove that $\mathscr{I}$ is a matroid independence system. (A matroid that arises in this way is called a *transversal matroid*.)

Problem #2 Construct two sets of vectors $S, S' \subset \mathbb{R}^3$ such that $T(M(S)) = T(M(S'))$ but $M(S) \not\cong M(S')$. (Instead of giving coordinates, describe each matroid by an affine point arrangement in $\mathbb{R}^2$.)

Problem #3 Let $G = (V, E)$ be a connected graph. An orientation of G is called *strong* if every edge is part of a (directed cycle); equivalently, there is a path from every vertex to every other vertex by following directed edges. For instance, the orientation on the left is strong, but the one on the right is not.

Let $s(G)$ be the number of strong orientations of G. Prove that $s(G) = T_G(0, 2)$. (Hint: Find a deletion-contraction recurrence for $s(G)$.)

Problem #4 Use the corank-nullity form of the Tutte polynomial, together with the definition of matroid duality, to prove that $T(M, x, y) = T(M^*, x, y)$ for every matroid M.

Problem #5 Let P be a chain-finite poset. The *kappa function* of P is the element of the incidence algebra $I(P)$ defined by $\kappa(x, y) = 1$ if $x \lessdot y$, $\kappa(x, y) = 0$ otherwise.

(#5a) Give a condition on κ that is equivalent to P being ranked.

(#5b) Give combinatorial interpretations of $\kappa * \zeta$ and $\zeta * \kappa$.

Problem #6 Let Π_n be the lattice of set partitions of $[n]$. Recall that the order relation on Π_n is given as follows: if $\pi, \sigma \in \Pi_n$, then $\pi \leq \sigma$ if every block of π is contained in some block of σ (for short, "π refines σ"). In this problem, you're going to calculate the number $\mu_n := \mu_{\Pi_n}(\hat{0}, \hat{1})$.

(#6a) Calculate μ_n by brute force for $n = 1, 2, 3, 4$. Make a conjecture about the value of μ_n in general.

(#6b) Define a function $f : \Pi_n \to \mathbb{Q}[x]$ as follows: if X is a finite set of cardinality x, then

$$f(\pi) = \#\{h : [n] \to X \quad \big| \quad h(s) = h(s') \iff s, s' \text{ belong to the same block of } \pi\}.$$

For example, if $\pi = \hat{1} = \{\{1, 2, \dots, n\}\}$ is the one-block partition, then $f(\pi)$ counts the constant functions from $[n]$ to X, so $f(\pi) = x$. Find a formula for $f(\pi)$ in general.

(#6c) Let $g(\pi) = \sum_{\sigma \geq \pi} f(\sigma)$. Prove that $g(\pi) = x^{|\pi|}$ for all $\pi \in \Pi_n$. (Hint: What kinds of functions are counted by the sum?)

(#6d) Apply Möbius inversion and an appropriate substitution for x to calculate μ_n.