Math 796 Problem Set #1
Due Friday, February 1

Problem #1 Let D_n be the set of all divisors of n (including n itself), partially ordered by divisibility.

(#1a) Prove that D_n is a ranked poset, and describe the rank function.

(#1b) When is D_n a chain? When is D_n a Boolean algebra? For which integers n, m is it the case that $D_n \cong D_m$?

(#1c) Prove that D_n is a distributive lattice. Describe its meet and join operations and its join-irreducible elements.

(#1d) Prove that D_n is *self-dual*, i.e., there is a bijection $f : D_n \to D_n$ such that $f(x) \leq f(y)$ if and only if $x \geq y$.

Problem #2 Let G be a graph with connected components $G_1, \dots, G_r$. Describe the clique poset of G in terms of the clique posets of $G_1, \dots, G_r$.

Problem #3 Prove that if L is a lattice, then

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x, y, z \in L$$

if and only if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \forall x, y, z \in L.$$

(A consequence is that L is distributive if and only if L^* is; that is, distributivity is a self-dual condition.)

Problem #4

(#4a) Describe the join-irreducible elements of Young's lattice Y.

(#4b) Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition, and let $\lambda = \mu_1 \vee \mu_2 \vee \cdots \vee \mu_k$ be the unique minimal decomposition of λ into join-irreducibles. Explain how to find k from the Ferrers diagram of λ.

Problem #5

(#5a) Count the maximal chains in $L_n(q)$. (Recall that this is the lattice of vector subspaces of the finite field with q elements).

(#5b) Count the maximal chains in the interval $[\emptyset, \lambda] \subset Y$ if the Ferrers diagram of λ is a $2 \times n$ rectangle.

(#5c) Ditto if λ is a hook shape (i.e., $\lambda = (n+1, 1, 1, \dots, 1)$, with a total of m copies of 1).

Problem #6 Fill in the details in the proof of Birkhoff's theorem by showing the following facts.

(#6a) For a finite distributive lattice L, show that the map $\phi : L \to J(\mathrm{Irr}(L))$ given by

$$\phi(x) \;=\; \langle p \mid p \in \mathrm{Irr}(L),\; p \leq x \rangle$$

is indeed a lattice isomorphism.

(#6b) For a finite poset P, show that an order ideal in P is join-irreducible in $J(P)$ if and only if it is principal (i.e., generated by a single element).