Math 725, Spring 2006
Solution Set #7

#1. Complete the proof of Theorem 3.1 in the lecture notes on the Tutte polynomial. To do this, show that if e is a loop, then $T(G;\,x,y) \;=\; y \cdot T(G-e;\,x,y)$.

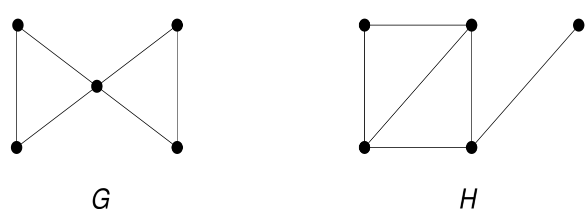
We can assume inductively that $T(G-e;\,x,y)$ is given by the closed-form expression (equation (3.3) in the lecture notes on the Tutte polynomial).

Suppose that e is a loop of G. We can rewrite $\tilde{T}(x,y)$ as a sum over all $F \subseteq E-\{e\}$, by grouping together the terms for F and $F+e$. For all such F, we have $|F+e| = |F|+1$ and $r(F+e) = r(F)$ (the latter because e is a cut-edge). Therefore:

$$\begin{aligned}
\tilde{T}(G;\,x,y) &= \sum_{\substack{F\subseteq E\\ e\notin F}} (x-1)^{r(G)-r(F)}(y-1)^{|F|-r(F)} \;+\; \sum_{\substack{F\subseteq E\\ e\in F}} (x-1)^{r(G)-r(F)}(y-1)^{|F|-r(F)}\\
&= \sum_{F\subseteq E-\{e\}} (x-1)^{r(G)-r(F)}(y-1)^{|F|-r(F)} \;+\; \sum_{F\subseteq E-\{e\}} (x-1)^{r(G)-r(F+e)}(y-1)^{|F+e|-r(F+e)}\\
&= \sum_{F\subseteq E-\{e\}} (x-1)^{r(G)-r(F)}(y-1)^{|F|-r(F)} \;+\; \sum_{F\subseteq E-\{e\}} (x-1)^{r(G)-r(F)}(y-1)^{|F|-r(F+e)+1}\\
&= \sum_{F\subseteq E-\{e\}} (x-1)^{r(G)-r(F)} \left[(y-1)^{|F|-r(F)} + (y-1)^{|F|-r(F+e)+1}\right]\\
&= \sum_{F\subseteq E(G-e)} (x-1)^{r(G)-r(F)}(y-1)^{|F|-r(F)}\,[1+(y-1)]\\
&= y \sum_{F\subseteq E(G-e)} (x-1)^{r(G)-r(F)}(y-1)^{|F|-r(F)}\\
&= y\cdot T(G-e;\,x,y)
\end{aligned}$$

as desired. ■

#2. Let G and H be the graphs shown below.



Verify that G and H have the same chromatic polynomial, but not the same Tutte polynomial.

To calculate $T(G)$, I'm going to cheat slightly and use the fact that the Tutte polynomial of a graph is the product of the Tutte polynomials of its blocks. In this case, the blocks of G are two copies of K_3, so by Example 2.3 of the lecture notes, we have

$$T(G) = T(K_3)^2 = (x^2+x+y)^2.$$

Now the graph H has a cut-edge, so $T(H) = x \cdot T(W)$, where W is K_4 with an edge removed. This observation already tells us that $T(H) \neq T(G)$, since the former polynomial is divisible by x and the latter is not. We have already calculated $T(W) = x(x^2 + x + y) + (x^2 + xy + y^2 + x + y)$ (see Example 2.6 in the lecture notes), so

$$T(H) = x(x^3 + 2x^2 + 2y^2 + xy + x + y).$$

We can calculate the chromatic polynomials of G and H using Theorem 4.2 of the notes:

$$\chi(G;\, k) \;=\; (-1)^{n(G)-c(G)} k^{c(G)} T(G;\, 1-k, 0)$$

and similarly for H. We find that

$$\chi(G;\, k) \;=\; \chi(H;\, k) \;=\; k^5 - 6k^4 + 13k^3 - 12k^2 + 4k \;=\; k(k-1)^2(k-2)^2. \qquad \blacksquare$$

#3. [West 6.1.21] Let G be a connected plane graph and let $F \subseteq E(G)$. Prove that F is a spanning tree of G if and only if $E(G^*) - F^*$ is a spanning tree of G^*.

Let $F \subseteq E(G)$.

Claim 1: $|F| = n(G) - 1$ if and only if $|E(G^*) - F^*| = n(G^*) - 1$.

Indeed, $|F| = n(G) - 1$ iff $|E(G^*) - F^*| = e(G) - (n(G) - 1) = e(G) - n(G) + 1$. On the other hand, the definition of planar duality says that $n(G^*) = f(G)$, the number of faces of G, which by Euler's formula is $e(G) - n(G) + 2$.

Claim 2: F is acyclic if and only if $E(G^*) - F^*$ is connected (as a spanning subgraph of G^*).

By Theorem 6.1.14, F^* is acyclic iff the dual edge set $F^* \subseteq E(G^*)$ does not contain a bond, i.e., is not a disconnecting set of G^*. This is equivalent to the statement that $E(G^*) - F^*$ is connected.

Now the desired conculsion follows from Claims 1 and 2, together with Theorem 2.1.4. $\blacksquare$

#4. Let G be a connected planar graph. Prove that $T(G;\, x, y) = T(G^*;\, y, x)$.

Proof #1 (closed formula). Let $E = E(G)$ and $E^* = E(G^*)$. As in #3, an edge set $F \subseteq E$ is acyclic if and only if $E^* - F^*$ is connected, i.e., contains a spanning tree of G^* Therefore

$$\begin{aligned} r(F) &= \max\{|A| : \; A \subseteq F \text{ acyclic}\} \\ &= \max\{|A^*| : \; A^* \subseteq F^*, \; B^* = E^* - A^* \text{ spanning}\} \\ &= |E^*| - \min\{|B^*| : \; B^* \supseteq E^* - F^* \text{ spanning}\} \\ &= |E^*| - r(G^*) + r(E^* - F^*) - |E^* - F^*| \\ &= |F^*| - r(G^*) + r(E^* - F^*). \end{aligned} \tag{1}$$

If G has n vertices and f faces, then G^* has f vertices. Since both graphs are connected, we have $r(G) = n-1$ and $r(G^*) = f - 1$. By Euler's formula, $r(G) + r(G^*) = n + f - 2 = |E| = |E^*|$, and solving for $r(G)$ gives

$$r(G) \;=\; |E^*| - r(G^*). \tag{2}$$

Applying (1) and (2) to the closed form of the Tutte polynomial gives

$$\begin{aligned} T(G;\,x,y) &= \sum_{F\subseteq E} (x-1)^{(|E^*|-r(G^*))-(|F^*|-r(G^*)+r(E^*-F^*))}\,(y-1)^{|F^*|-(|F^*|-r(G^*)+r(E^*-F^*))} \\ &= \sum_{F\subseteq E} (x-1)^{|E^*|-|F^*|-r(E^*-F^*)}\,(y-1)^{r(G^*)-r(E^*-F^*)} \\ &= \sum_{D^*\subseteq E^*} (x-1)^{|D^*|-r(D^*)}\,(y-1)^{r(G^*)-r(D^*)} \end{aligned}$$

where $D^* = E^* - F^*$. This is precisely $T(G^*;\,y,x)$. ■

Proof #2 (recursive). Induct on $e = e(G) = e(G^*)$ If $e = 1$, then $G \cong G^* \cong K_1$ and $T(G;\,x,y) = 1 = T(G^*;\,y,x)$.

Suppose $e > 1$. Let $a \in E(G)$. We observe that:

— If a is not a cut-edge, then it borders two distinct faces F, F', and the planar dual of $G - a$ can be constructed by removing the dual edge a^* and identifying the vertices of G^* corresponding to F and F'. That is,

$$(G-a)^* \cong G^*/a^*.$$

— If a is not a loop, then the faces of G/a are the same as the faces of G; however, the dual edge $a^* \in E(G^*)$ (whose endpoints are the two faces of G separated by a) does not occur in $(G/a)^*$. That is,

$$(G/a)^* \cong G^* - a^*.$$

Now, we make the inductive hypothesis that $T(H;\,x,y) = T(H^*;\,y,x)$ for all connected planar graphs H with $e(H) < e(G)$. In particular, this equality holds for G/a (if a is not a loop) and for $G - e$ (if e is not a cut-edge). Moreover, a is a loop (resp. cut-edge) if and only if a^* is a cut-edge. Therefore

$$\begin{aligned} T(G;\,x,y) &= \begin{cases} y\cdot T(G-a;\,x,y) & \text{if } a \text{ is a loop} \\ x\cdot T(G/a;\,x,y) & \text{if } a \text{ is a cut-edge} \\ T(G-a;\,x,y)+T(G/a;\,x,y) & \text{otherwise} \end{cases} \\ &= \begin{cases} y\cdot T((G-a)^*;\,y,x) & \text{if } a \text{ is a loop} \\ x\cdot T((G/a)^*;\,y,x) & \text{if } a \text{ is a cut-edge} \\ T((G-a)^*;\,y,x)+T((G/a)^*;\,y,x) & \text{otherwise} \end{cases} \\ &= \begin{cases} y\cdot T(G^*/a^*;\,y,x) & \text{if } a^* \text{ is a loop} \\ x\cdot T(G^*-a^*;\,y,x) & \text{if } a^* \text{ is a cut-edge} \\ T(G^*/a^*;\,y,x)+T(G^*-a^*;\,y,x) & \text{otherwise} \end{cases} \\ &= T(G^*;\,y,x). \end{aligned}$$

■

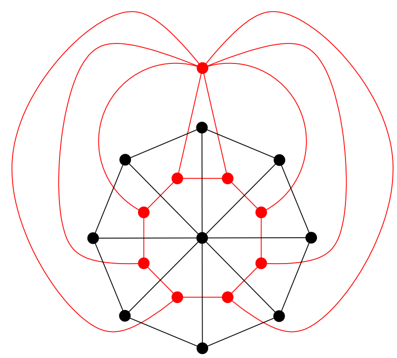
#5. [West 6.1.25] A planar graph G is called *self-dual* if $G \cong G^*$.

(a.) Prove that if G is self-dual, then $e(G) = 2n(G) - 2$.

A self-dual graph must have $n(G) = f(G)$, so Euler's formula becomes $n(G) - e(G) + n(G) = 2$, or $e(G) = 2n(G) - 2$.

(b.) For all $n \geq 4$, construct a self-dual simple graph of order n.

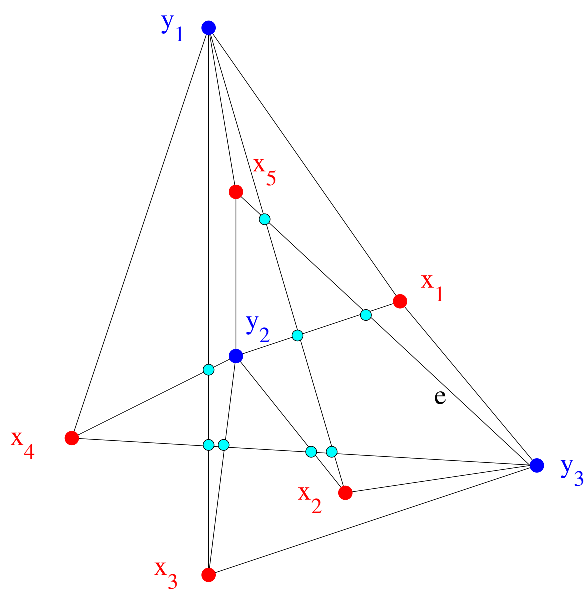
The $(n-1)$-wheel $W_n = C_{n-1} \vee K_1$ is self-dual for all $n \geq 4$ (and in fact even for $n = 2$ and $n = 3$, although those graphs are not simple.) Note that W_n is planar and that $e(W_n) = 2n - 2$. The $n - 1$ triangular faces form a cycle in $(W_n)^*$, and the outer face is adjacent to every triangular face. This accounts for all the edges of $(W_n)^*$, and exactly matches the definition of the join $C_{n-1} \vee K_1$ (see Defn. 3.3.6, p. 138).



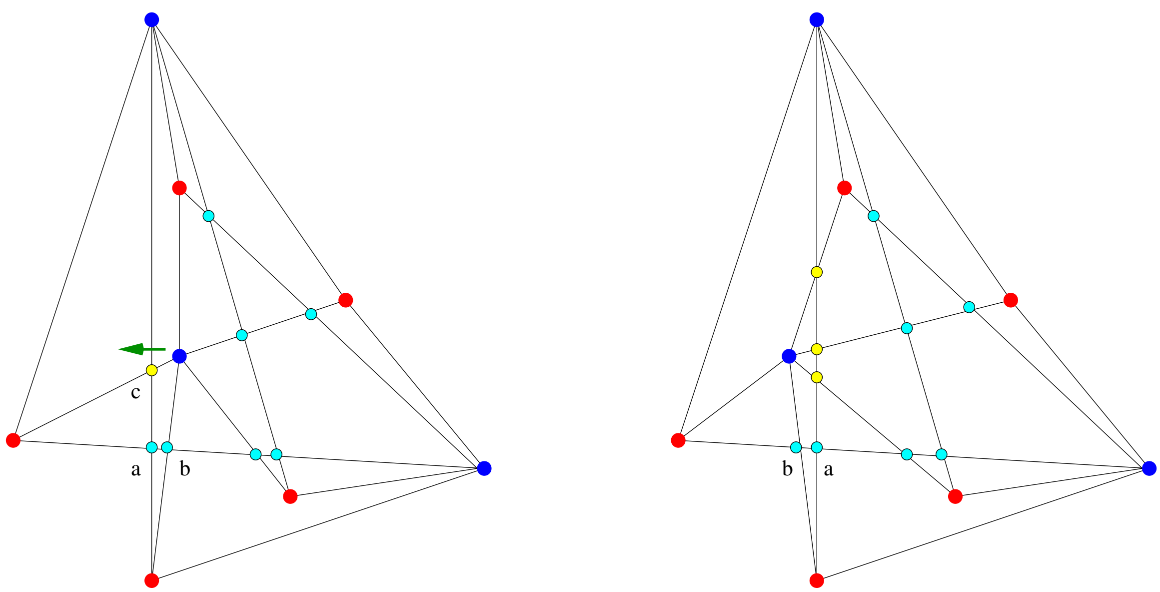
Bonus problem [West 6.3.28] Let m and n be odd. Prove that in all drawings of $K_{m,n}$, the parity of the number of pairs of edges that cross is the same. (Consider only drawings in which edges cross at most once, and edges sharing an endpoint do not cross.) Conclude that $\nu(K_{m,n})$ is odd if and only $m - 3$ and $n - 3$ are both divisible by 4.

Let the partite sets of $K_{m,n}$ be $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. As per the hint in West (p. 514), consider what happens when we take a drawing of $K_{m,n}$ and perturb it slightly so that one of the vertices (say x_1) moves across an edge to which it is not incident (say $e = x_n y_m$). Let $A = \{x_1y_1, \ldots, x_1y_{m-1}\}$. Then each member of A that crosses e before the move does not cross it after the move, and vice versa. (Note that x_1y_m is excluded because it shares an endpoint with e, hence does not cross it either before or after the move.) Since $m - 1$ is even, the parity of the number of crossings is unaffected:

For example, consider the following drawing of $K_{5,3}$ with eight crossings. The x_i are shown in red and the y_j in blue; the pale blue points indicate crossings.



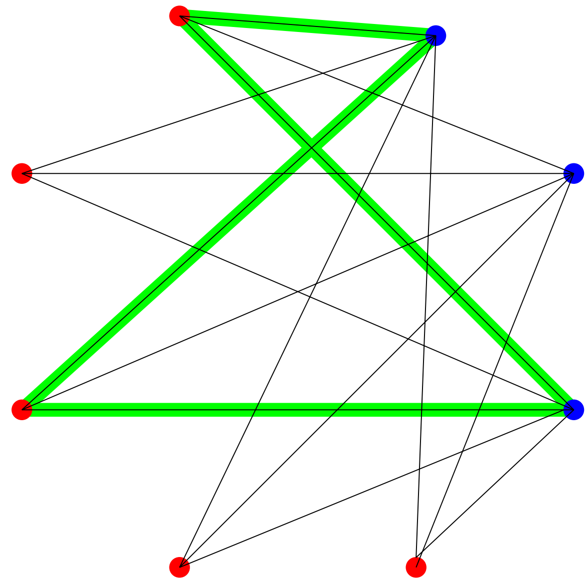
As x_1 moves southwest across e, the crossing of x_1y_2 with e is eliminated, but a crossing is created between x_1y_1 and e (the yellow point). The total number of crossings remains unchanged. Here's another possibility:



Here the crossing c (colored yellow on the left) is lost, but three new ones (colored yellow on the right) are created. So the total number of crossings increases from 8 to 10. (The crossings marked a and b swap places.)

Since we can get from any drawing of $K_{m,n}$ to any other by a sequence of moves of this sort (moving one vertex at a time across one edge at a time), it follows that the parity of the number of crossings is constant over all drawings, and therefore depends only on m and n. In particular, to determine the parity of $\nu(K_{m,n})$, it suffices to exhibit a single drawing and count the crossings.

Consider a drawing in which the vertices are placed around a circle, with each partite set consecutive. (We *don't* want to put the points at the vertices of a regular $(m + n)$-gon, because that will cause crossings between more than two lines—but we can just perturb the points a little bit to avoid this problem.) Each pair of red points and each pair of blue points will then contribute exactly one crossing.



The number of crossings is therefore

$$\binom{m}{2}\binom{n}{2} = \frac{m(m-1)n(n-1)}{4}.$$

Since m and n are odd, $m-1$ and $n-1$ are both even, say $m-1=2a$ and $n-1=2b$. So the number of crossings equals $abmn$. If both $m-3$ and $n-3$ are divisible by 4, then both a and b are odd (which would imply that $\nu(K_m,n)$ is odd), but otherwise one of a, b is even, and so is $\nu(K_m,n)$. ■