Math 725, Spring 2006
Solution Set #6

#1. [West 4.3.15] Let G be a weighted graph with weight function $\mathrm{wt} : E \to \mathbb{R}_{\geq 0}$. For each spanning tree T of G, define $a(T) = \min\{\mathrm{wt}(e) : e \in T\}$ and for each edge cut $[S, \bar{S}]$, define $b([S,\bar{S}]) = \max\{\mathrm{wt}(e) : e \in [S,\bar{S}]\}$. Prove that

$$\max_{\textbf{spanning trees } T} a(T) \quad = \quad \min_{\textbf{edge cuts } [S,\bar{S}]} b([S,\bar{S}]).$$

Let T be a spanning tree T and $F = [S, \bar{S}]$ an edge cut. Then T must contain at least one edge $e \in F$, otherwise it contains no path from any member of S to any member of $\bar{S}$. Therefore

$$a(T) \;\leq\; \mathrm{wt}(e) \;\leq\; b(F) \tag{1}$$

from which it follows that

$$\max_T \; a(T) \;\leq\; \min_{[S,\bar{S}]} \; b([S,\bar{S}]).$$

Now, we want to prove that there exist T, F for which equality holds in (1). Let T be a maximum-weight spanning tree, constructed by modifying Kruskal's Algorithm (West, p. 95) so that each iteration adds the edge of *largest* weight that does not complete a cycle.

Let e be the last edge added to T. Then $T - e$ has two components, whose vertex sets partition V; call these vertex sets S and $\bar{S}$. Before the last iteration, the edges that can be added to $T - e$ that do not form a cycle are precisely the members of $F = [S, \bar{S}]$.

Then $\mathrm{wt}(e') \geq \mathrm{wt}(e)$ for every $e' \in T - e$, because the algorithm added e' to T before it added e. On the other hand, $\mathrm{wt}(e'') \leq \mathrm{wt}(e)$ for every $e'' \in F$, otherwise the algorithm would add e'' instead of e. Combining these two statements, we have

$$\mathrm{wt}(e') \geq \mathrm{wt}(e) \geq \mathrm{wt}(e'') \qquad \forall e' \in T,\; e'' \in F$$

which implies that

$$a(T) \geq b(F)$$

as desired. ■

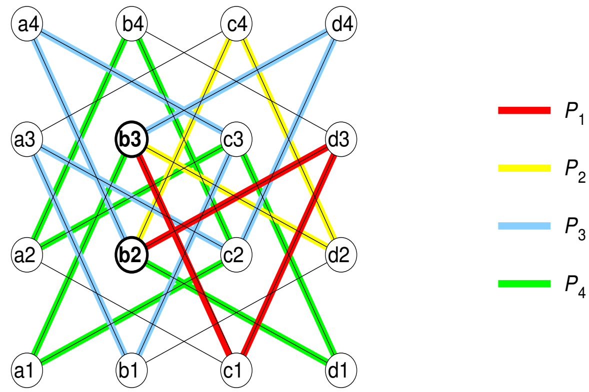
#2. Let G be the graph with 16 vertices $\{a_1, a_2, \ldots, d_4\}$ corresponding to the squares of a 4×4 chessboard, with two vertices adjacent if they are connected by a knight move (that is, if one of their row and column coordinates differs by 2, and the other by 1).

(a) Use the Max-Flow/Min-Cut algorithm to find a maximum family of PED paths joining the vertices $s = b_2$ and $t = b_3$.

To do this, we replace each undirected edge xy by a pair of directed edges $\overrightarrow{xy}$, $\overrightarrow{yx}$, each having capacity 1 (although we can omit the edges having head s or tail y). Then an acyclic s,t-flow f can be partitioned into $|f|$ PED s,t-paths. There are many ways the algorithm might proceed, but you should wind up with a family of four paths: for example,

$$\begin{array}{ll} P_1: & b_2,\ d_3,\ c_3,\ b_3 \\ P_2: & b_2,\ c_4,\ d_2,\ b_3 \\ P_3: & b_2,\ a_4,\ c_1,\ b_1,\ a_3,\ c_2,\ d_4,\ b_3 \\ P_4: & b_2,\ d_1,\ c_3,\ a_2,\ b_4,\ c_2,\ a_1,\ b_3 \end{array} \tag{2}$$

It's a little easier to illustrate this family with a picture:



(b) Certify that your answer is correct by exhibiting an s,t-edge cut of cardinality $\lambda'(s,t)$.

The set of edges incident to s, i.e., the edge cut $[\{s\}, \overline{\{s\}}]$, is an s,t-edge cut of cardinality 4.

(c) Prove that $\lambda(s,t) < \lambda'(s,t)$.

The vertex set $X = \{c4, c3, d3\}$ is an s,t-vertex cut of cardinality 3. The easiest way to verify this is to observe directly that $W = \{a_4, b_2, d_1\}$ is the vertex set of a component of $G - X$, since every other neighbor of a vertex in W belongs to X. (Notice that it is *not* enough to assert that every path P_i in (2) contains a vertex X—that in itself does not rule out the possibility that $G - X$ might have some other s,t-path.)

#3. [West 5.1.22] Given a set of lines in the plane with no three meeeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \leq 3$.

Note that $\Delta(G) \leq 4$, because each vertex lies on exactly two lines and has at most one neighbor in each direction on each line.

Draw a new line ℓ that is not perpendicular to any line in the arrangement, and rotate the coordinates so that ℓ becomes the x-axis. Now no line of the arrangement is vertical, so in particular every vertex has a different x-coordinate.

Now, apply the greedy coloring algorithm, ordering the vertices by their x-coordinates. For each vertex v with collinear neighbors u, w, we have either $u < v < w$ or $u > v > w$. In particular, at most one of u, w can precede v in the ordering, and the same is true for the other line on which v lies. Therefore, at most two neighbors of v have been colored before it is, and the coloring uses at most three colors. ■

Alternately, for any subgraph $H \subseteq G$, let v be the vertex of H with the smallest x-coordinate. By the same reasoning as above, $d_H(x) \leq 2$, so $\delta(H) \leq 2$. So $\chi(G) \leq 3$ by Theorem 5.1.19 (which we did not discuss in class).

#4. [West 5.1.38] Prove that $\chi(G) = \omega(G)$ if $\bar{G}$ is bipartite.

Let $G = (V, E)$. If $G \cong K_n$, then there is nothing to prove: $\chi(G) = \omega(G) = n$. Otherwise, let U be the set of politicians in G, that is,

$$U = \big\{v \in V \mid N(v) = V - \{v\}\big\}.$$

Then every maximal clique contains U, and each member of U forms a singleton color class with respect to every proper coloring. Therefore $\chi(G-U) = \chi(G) - |U|$ and $\omega(G-U) = \omega(G) - |U|$. Hence we may replace G with $G - U$, which is a politician-free graph whose complement is bipartite. Then

$$\begin{aligned} \omega(G) &= \alpha(\bar{G}) && \text{(by definition)} \\ &= \beta'(\bar{G}) && \text{(by Cor. 3.1.24)} \\ &= n(G) - \alpha'(\bar{G}) && \text{(by Thm. 3.1.22).} \end{aligned}$$

Let M be a maximum matching on $\bar{G}$, i.e., $|M| = \alpha('\bar{G})$. Construct a coloring of G as follows: for each $e = vw \in M$, the vertices $\{v, w\}$ form a color class, and if a vertex x is M-unsaturated, then it is a singleton color class. This coloring is proper, and uses $n(G) - \alpha'(\bar{G})$ colors. Therefore $\chi(G) \leq n(G) - \alpha'(\bar{G}) = \omega(G)$. Since $\chi(G) \geq \omega(G)$ in general (Prop. 5.1.7), we have $\chi(G) = \omega(G)$. ■

#5. [West 5.3.4] (a) Prove that the chromatic polynomial of the n-cycle is $\chi(C_n;\, k) = (k-1)^n + (-1)^n(k-1)$.

We proceed by induction on n.

Base case: If $n = 2$, then C_2 is the digon, and its chromatic polynomial is $k(k-1)$. Indeed, the formula gives

$$(k-1)^2 + (-1)^2(k-1) \;=\; (k^2 - 2k + 1) + (k-1) \;=\; k^2 - k \;=\; k(k-1).$$

Inductive step: Suppose that $n \geq 2$ and $\chi(C_n;\, k) = (k-1)^n + (-1)^n(k-1)$. We will prove the analogous formula for C_{n+1}. Let $e \in E(C_{n+1})$. Then $G - e \cong P_{n+1}$ (a tree with $n+1$ vertices) and $G/e \cong C_n$, so

$$\begin{aligned} \chi(C_{n+1};\, k) &= \chi(P_{n+1};\, k) - \chi(C_n;\, k) \\ &= k(k-1)^n - ((k-1)^n + (-1)^n(k-1)) \\ &= (k-1)^{n+1} + (-1)^{n+1}(k-1). \end{aligned}$$

■

(b) For $H = G \vee K_1$, prove that $\chi(H;\, k) = k \cdot \chi(G;\, k-1)$.

Let v be the vertex of K_1 in H. Since $N_H(v) = V(G)$, the vertex v must form its own color class in any proper coloring. Therefore, a k-coloring f of H consists of a choice for $f(v)$ together with a proper coloring of G using the remaining $k-1$ colors, from which the desired formula follows.

(c) Use (a) and (b) to find the chromatic polynomial of the wheel $W_n = C_n \vee K_1$.

$\chi(W_n;\, k) = \chi(C_n \vee K_1;\, k) = k \cdot \chi(C_n;\, k-1) = k\big[(k-2)^n + (-1)^n(k-2)\big]$.

Bonus problem: We proved in class that a tree T on n vertices has chromatic polynomial $\chi(T;\, k) = k(k-1)^{n-1}$ (see Proposition 5.3.3). Conversely, suppose that G is a graph with chromatic polynomial $\chi(G;\, k) = k(k-1)^{n-1}$ for some n. Must G be a tree?

Almost. I didn't say anything about G being simple! Indeed, if G is simple, then since $\chi(G;\, k) = k(k-1)^{n-1} = k^n - (n-1)k^{n-1} + \cdots$, we know by Theorem 5.3.10 that G has n vertices and $n-1$ edges. Moreover, G cannot be disconnected, because if it has c components then $\chi(G;\, k)$ is divisible by k^c, which is true only for $c = 1$. Therefore G must be a tree.

Therefore, in general, $\chi(G;\, k) = k(k-1)^{n-1}$ if and only if G is a loopless graph on n vertices whose underlying simple graph is a tree.