Math 725, Spring 2006
Solution Set #5

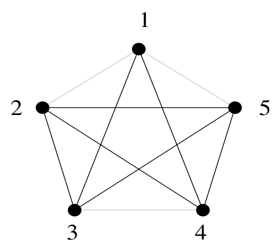
#1. [West 3.3.7] For each $k > 1$, construct a k-regular simple graph having no perfect matching.

Here is one of several possible constructions.

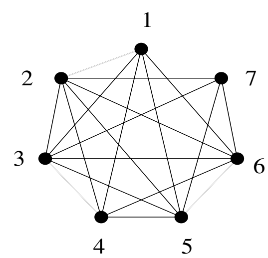
If k is even, then the complete graph K_{k+1} suffices; it is certainly k-regular, but has an odd number of vertices, hence no perfect matching.

On the other hand, suppose that $k > 1$ is odd.Construct a graph H_k by starting with a complete graph on vertices $v_1, v_2, \ldots, v_k$, and deleting the $(k+1)/2$ edges

$$v_1v_2,\ v_3v_4,\ \ldots,\ v_{k-2}v_{k-1},\ v_kv_1.$$

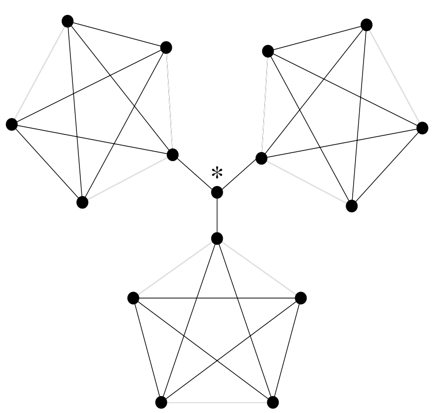


H3

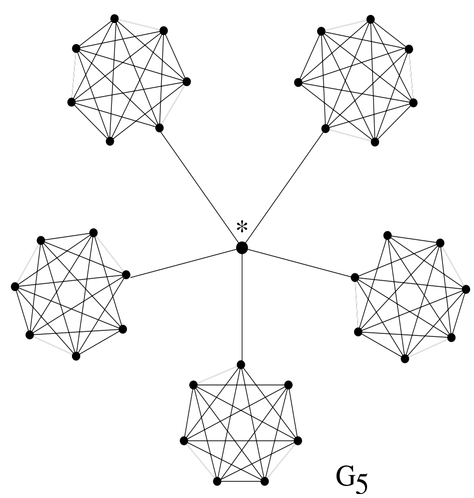


H5

Notice that $n(H_k) = k + 2$ is odd, and that H_k has $k+1$ vertices of degree k and one vertex of degree $k-1$. Now construct a graph G_k from $k \cdot H_k$ (that is, the disjoint union of k copies of H_k), inventing a new vertex $*$, and adding an edge from 0 to each vertex of degree $k-1$ in one of the copies of H_k.



G3



G5

By construction, G_k is k-regular. However, if we take $S = \{*\}$, then $|S| = 1$ and $o(G_k - S) = k$ (because $G_k - S \cong k \cdot H_k$), so G_k has no perfect matching by Tutte's theorem.

#2. [West 3.3.22] Let G be an X,Y-bigraph. Let H be the graph obtained from G by adding one vertex to Y if $n(G)$ is odd, then adding edges to make Y into a clique.

(a) Prove that G has a matching of size $|X|$ if and only if H has a perfect matching.

($\Longleftarrow$) If M is a perfect matching of H, then each vertex in X must be matched with a vertex of Y by a vertex of $E(G)$, because X is a coclique in H and $N_H(x) = N_G(x)$ for all $x \in X$. Therefore $M \cap E(G)$ is a matching of G saturating X, in particular of size $|X|$.

($\Longrightarrow$) On the other hand, if M is a matching of G of size $|X|$, then it saturates X (since G is X,Y-bipartite). If we regard M as a matching of H, then there are an even number (namely $n(H) - 2|M|$) of unsaturated vertices, all of which belong to Y, which is a clique in H. So any pairing of these vertices extends M to a perfect matching of H.

(b) Prove that if G satisfies Hall's condition (that is, $|N(S)| \geq |S|$ for all $S \subseteq X$), then H satisfies Tutte's condition (that is, $o(H - T) \leq |T|$ for all $T \subseteq V(H)$).

Suppose that G satisfies Hall's condition. Let $T \subset V(H)$ and $S = V(H) - T = V(H - T)$. Define $T_X = T \cap X$, $T_Y = T \cap Y$, $S_X = S \cap X$, $S_Y = S \cap Y$. Also define

$$Z = \{x \in S_X \mid N(x) \subseteq T_Y\}.$$

The vertices of S_Y form a clique in $H - T$, so in particular they all belong to the same component J of $H - T$. For each $x \in S_X$, if x has a neighbor in S_Y (that is, if $x \notin Z$) then $x \in V(J)$, while if $x \in Z$ then x is an isolated (hence odd) component of $H - T$. Thus we have

$$V(J) = V(H) - Z - T_Y \tag{1}$$

and

$$o(H - T) = \begin{cases} |Z| & \text{if } n(J) \text{ is even,} \\ |Z| + 1 & \text{if } n(J) \text{ is odd.} \end{cases} \tag{2}$$

Since $N(Z) \subseteq T_Y$, Hall's condition implies that

$$|Z| \leq |T_Y|. \tag{3}$$

<u>Case 1: $T_X \neq \emptyset$.</u> Then $|T| \geq |T_Y| + 1 \geq |Z| + 1$ (by (3)) $\geq o(H - T)$ (by (2)).

<u>Case 2: $T_X = \emptyset$.</u> If $|Z| < |T_Y|$ then $|T| = |T_Y| \geq |Z| + 1 \geq o(H - T)$. On the other hand, if $|Z| = |T_Y|$ then by (1) it follows that $n(J) = n(H) - |Z| - |T_Y| = n(H) - 2|Z|$ is even, so $o(H - T) = |Z| \leq |T_Y| = |T|$.

In all cases H satisfies Tutte's condition.

(c) Use parts (a) and (b) to conclude that Tutte's 1-Factor Theorem 3.3.3 implies Hall's Theorem 3.1.11.

We have shown that if Hall's condition holds, then the graph H satisfies Tutte's condition (part (b)), hence has a perfect matching by Tutte's theorem, so that G has a matching saturating X (part (a)). This is the difficult direction of Hall's theorem.

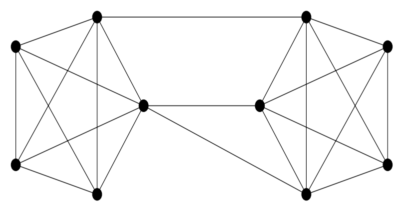
#3. [West 4.1.9] For each choice of integers k, ℓ, m with $0 < k \le \ell \le m$, construct a simple graph G such that $\kappa(G) = k$, $\kappa'(G) = \ell$, and $\delta(G) = m$.

If $k = \ell = m$ then we may take $H = K_{m+1}$. Otherwise, if $k < m$, we proceed as follows.

Let H be the disjoint union of two copies of K_{m+1}, with vertex sets $X = \{x_1, x_2, \ldots, x_{m+1}$ and $Y = \{y_1, y_2, \ldots, y_{m+1}$ respectively. We then construct G by adding to H the edges

$$\{x_i y_i \mid 1 \le i \le k\} \ \cup \ \{x_k y_j \mid k+1 \le j \le \ell\}.$$

For example, if $k = 2$, $\ell = 4$ and $m = 6$, then G is the graph shown below.



Then:

- H is m-regular and $d_G(x_{m+1}) = d_H(x_{m+1})$. So $\delta(G) = m$.

- The edge cut $[X, Y]$ has cardinality ℓ, so $\kappa'(G) \le \ell$. Every other edge cut F has the form $[X' \cup Y', X'' \cup Y'']$, where $X = X' \sqcup X''$, $Y = Y' \sqcup Y''$, and either X', X'' are both nonempty. That is, F must contain an edge cut of at least one of the cliques X, Y, so $|F| \ge \kappa'(K_{m+1}) = m$. Hence $[X, Y]$ is a minimum edge cut and $\kappa'(G) = \ell$.

- The set $\{x_1, \ldots, x_k\}$ is a vertex cut of cardinality k, so $\kappa(G) \le k$. For every $Q \subset V(G)$, the graph $G - Q$ has at most two components, since $X \cap V(G-Q)$ and $Y \cap V(G-Q)$ are both cliques in G, and in fact it will have two components only if Q contains an endpoint of every edge joining X to Y—that is, only if either $x_i \in Q$ or $y_i \in Q$ for every $i \in [k]$. Hence $\kappa(G) \ge k$.

#4. [West 4.1.14] Let G be a connected graph such that for every edge e, there are cycles C_1, C_2 such that $E(C_1) \cap E(C_2) = \{e\}$. Prove that G is 3-edge-connected.

First, the given condition implies that every edge of G belongs to a cycle. That is, G has no cut-edge (by Theorem 1.2.14), hence is 2-edge-connected.

Suppose now that G has an edge cut of cardinality 2, say $F = [S, \bar{S}] = \{e, f\}$. We may assume without loss of generality that F is a bond (a minimal edge cut); then $G[S]$ and $G[\bar{S}]$ are connected by Prop. 4.1.15. Note that f is a cut-edge of $G - e$, and the endpoints of e lie in different components (namely $G[S]$ and $G[\bar{S}]$) of $(G - e) - f = G - F$. Therefore, for every cycle $C \subseteq G$ containing e, the path $C - e$ between the endpoints of e must contain the edge f. Hence there is no pair of cycles whose intersection is exactly $\{e\}$, a contradiction. We conclude that G has no edge cut of cardinality 2, hence is 3-edge-connected.

#5. [West 4.2.23] Let G be an X, Y-bigraph. Let H be the graph obtained from G by adding two new vertices s, t, an edge sx for every $x \in X$, and an edge ty for every $y \in Y$.

(a) Prove that $\alpha'(G) = \lambda_H(s, t)$.

For every perfect matching $\{x_1y_1, \dots, x_ry_r\} \subseteq E(G)$, the set

$$\{(s, x_1, y_1, t), \ \dots, \ (s, x_r, y_r, t)\}$$

is a p.i.d. family of s, t-paths in H. Hence $\alpha'(G) \leq \lambda_H(s, t)$. On the other hand, if $\mathbf{P}$ is a family of p.i.d. s, t-paths, then each $P_i \in \mathbf{P}$ contains an edge $e_i \in [X, Y]$, and no two of the edges e_i share an endpoint (by p.i.d.-ness), hence the set of all e_i form a matching of cardinality equal to that of $\mathbf{P}$. So $\alpha'(G) \geq \lambda_H(s, t)$.

(b) Prove that $\beta(G) = \kappa_H(s, t)$.

Let $Q \subseteq V(G) = V(H) - \{s, t\}$. Then Q is a vertex cover of G if and only if $G - Q$ has no edges, i.e., $H - Q$ has no edges from X to Y, which is equivalent to the condition that s, t belong to different components of $H - Q$, i.e., that Q is an s, t-cut in H. Hence the maximum size of a set Q satisfying these equivalent conditions is $\beta(G) = \kappa_H(s, t)$.

#6. [West 4.2.12] Use Menger's Theorem to give a proof that $\kappa(G) = \kappa'(G)$ when G is 3-regular.

Let $x, y \in V(G)$, and let $\mathbf{P} = \{P_1, P_2, \dots$ be a family of pairwise edge-disjoint x, y-paths. I claim that the family $\mathbf{P}$ is in fact pairwise *internally* disjoint. Indeed, suppose that $P_1, P_2 \in \mathbf{P}$ share some internal vertex z. Then each P_i contains two edges e_{i1}, e_{i2} incident to z, and the p.e.d. condition implies that $e_{11}, e_{12}, e_{21}, e_{22}$ are all distinct. But then $d(z) \geq 4$, which contradicts the assumption of 3-regularity.

It follows that $\lambda(x, y) = \lambda'(x, y)$ for every $x, y \in V(G)$. By the two versions of Menger's Theorem, we have

$$\kappa(G) \ = \ \min_{x,y \in V(G)} \lambda(x, y) \ = \ \min_{x,y \in V(G)} \lambda'(x, y) \ = \ \kappa'(G).$$