Math 725, Spring 2006
Solution Set #4

#1. [West 3.1.24] A *permutation matrix* P is a square matrix all of whose entries are 0 or 1, with exactly one 1 in each row and in each column. For k a positive integer, prove that a square matrix of nonnegative integers can be written as the sum of k permutation matrices if and only if every row and every column has sum k.

Let A be an $n \times n$ square matrix with nonnegative integer entries a_{ij} Construct a bipartite graph G with partite sets $X = \{x_1 \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ in which a_{ij} is the number of parallel edges joining x_i and y_j. (In essence, A is to be regarded as the adjacency matrix of G; although it does not quite match Definition 1.1.17, it contains enough information to specify G up to isomorphism.) This construction is reversible, so that we have a bijection between X, Y-bipartite graphs and $n \times n$ nonnegative integer matrices.

The sum of the entries in the i^{th} row (respectively, j^{th} column) of A gives $d_G(x_i)$ (resp. $d_G(y_j)$). Hence, the condition that all these sums equal k is equivalent to the condition that G be k-regular. Then G has a perfect matching M by Corollary 3.1.13. The graph $G - M$ is $(k-1)$-regular, so by induction on k we can partition the edges of G into k perfect matchings. Each such matching corresponds to a permutation matrix, and the sum of these permutation matrices is precisely the matrix A, as desired. ■

#2. [West 3.1.31] Use the König-Egerváry Theorem to prove Hall's Marriage Theorem.

Let G be an X, Y-bigraph, and assume the König-Egerváry Theorem; that is, $\alpha'(G) = \beta(G)$. We want to prove Hall's Theorem, i.e., that G has a matching saturating X if and only if Hall's condition holds for X, that is, $|N(S)| \geq |S|$ for every $S \subseteq X$. As we know, the "only if" direction is easy, so the gist of the problem is to use the König-Egerváry condition to prove the "if" direction.

Let $x = |X|$ and $y = |Y|$. We lose nothing by deleting all isolated vertices from Y. Then $N(X) = Y$, so $x \leq y$ by Hall's condition.

Let Q be a vertex cover of G. Let $S = Q \cap X$, $T = Q \cap Y$, $s = |S|$ and $t = |T|$. By definition of vertex cover, we must have $N(X - S) \subseteq T$, so $t \geq |N(X - S)| \geq |X - S| = x - s$, the last inequality following from Hall's condition. From this it follows that $|Q| = s + t \geq x$. On the other hand, X itself is a vertex cover, so it must be a minimum vertex cover; that is, $\beta(G) = x$. By the König-Egerváry Theorem, $\alpha'(G) = x$. Hence any maximum matching of G must saturate X, as desired. ■

#3. [West 3.1.19, Schrijver] Let Y be a finite set and $\mathbf{A} = \{A_1, \ldots, A_m\}$ a family of subsets of Y (not necessarily disjoint). A *system of distinct representatives* (or SDR) for A is a set of distinct elements $y_1, \ldots, y_m \in Y$ such that $y_i \in A_i$ for all i.

(a) Prove that A has an SDR if and only if $\left|\bigcup_{i \in S} A_i\right| \geq |S|$ for all $S \subseteq [m]$.

Construct a simple bipartite graph G with partite sets $\{A_1, \ldots, A_m\}$ and $\{y_1, \ldots, y_m\}$, with an edge from A_i to y_j if and only if $y_j \in A_i$. Then an SDR for $\mathbf{A}$ is just a perfect matching of G. Moreover, by the construction of G we have

$$\bigcup_{i \in S} A_i = \bigcup_{i \in S} N(A_i) = N\left(\cup_{i \in S} A_i\right),$$

so by Hall's theorem the stated condition is equivalent to the existence of a perfect matching for G.

(b) Let $\mathbf{B} = \{B_1, \ldots, B_m\}$ be another family of subsets of Y. Prove that A and B have a *common* SDR if and only if for each $I \subseteq [n]$, the set $\bigcup_{i \in I} A_i$ meets at least $|I|$ of the sets B_j.

Oops! I forgot to specify the crucial condition that $\mathbf{A}$ and $\mathbf{B}$ are not supposed to be just any old set families—they are supposed to be *set partitions* of Y. That is, each element of Y belongs to exactly one A_i and exactly one B_j.

With this additional condition in hand, we can construct a bipartite graph G with $\mathbf{A}$ and $\mathbf{B}$ the partite sets, and edges $\{A_iB_j \ : \ A_i \cap B_j \neq \emptyset\}$. Then G has a perfect matching if and only if $\mathbf{A}$ and $\mathbf{B}$ have a common SDR; notice that the condition that $\mathbf{A}$ is a partition—that $A_i \cap A_{i'} = \emptyset$ for $i \neq i'$—is precisely what we need to conclude that a common system of representatives for $\mathbf{A}$ and $\mathbf{B}$ is in fact a common system of *distinct* representatives. The condition stated in the problem is then just Hall's condition for G.

#4. [Schrijver] Let $G = (V, E)$ be a simple graph with $n = n(G)$ and $\delta(G) \geq 2$. Define a *bimatching* to be an edge set $B \subseteq E$ such that no vertex belongs to more than <u>two</u> edges in B, and define a *bicover* to be an edge set $C \subseteq E$ if every vertex belongs to at least two edges in C. Let

$$\tilde{\alpha} = \tilde{\alpha}(G) = \max\{|B| : B \text{ is a bimatching}\},$$
$$\tilde{\beta} = \tilde{\beta}(G) = \min\{|C| : C \text{ is a bicover}\}.$$

Prove that $\tilde{\alpha} \leq \tilde{\beta}$ and that $\tilde{\alpha} + \tilde{\beta} = 2n$.

For $v \in V$ and $F \subseteq E$, write $d_F(v)$ for the number of edges of F having v as an endpoint; equivalently, the degree of v in the spanning subgraph (V, F). Let B be a bimatching and C a bicover. By the handshaking formula (Proposition 1.3.3 in West), we have $2|B| = \sum_{v \in V} d_B(v)$ and $2|C| = \sum_{v \in V} d_C(v)$. But $d_B(v) \leq 2 \leq d_C(v)$ for all v, so it follows that $|B| \leq |C|$. Therefore $\tilde{\alpha} \leq \tilde{\beta}$ (indeed, $\tilde{\alpha} \leq n \leq \tilde{\beta}$).

First, let B be a maximum bimatching on G, so $|B| = \tilde{\alpha}$. For $k = 0, 1, 2$, let s_k be the number of vertices with degree k in B; that is, in the spanning subgraph (V, B). The handshaking formula (Proposition 1.3.3 in West) gives

$$\tilde{\alpha} = |B| = \frac{s_1 + 2s_2}{2}. \tag{1}$$

Now form a bicover C by adding to B two edges incident to each vertex with degree 0 in B, and one edge incident to each vertex with degree 1 in B. The desired edges exist because $\delta(G) \geq 2$; on the other hand, some of the newly chosen edges may coincide. Also, we have $|C| \geq \tilde{\beta}$ by definition of $\tilde{\beta}$. So

$$\tilde{\beta} \leq |C| \leq |B| + 2s_0 + s_1 = \frac{4s_0 + 3s_1 + 2s_2}{2} \tag{2}$$

and adding (1) and (2) gives

$$\tilde{\alpha} + \tilde{\beta} \quad \leq \quad \frac{s_1 + 2s_2}{2} + \frac{4s_0 + 3s_1 + 2s_2}{2} \quad = \quad 2(s_0 + s_1 + s_2) \ = \ 2n. \tag{3}$$

Now let C be a minimum bicover on G, so $|C| = \tilde{\beta}$. The handshaking formula gives

$$\tilde{\beta} = |C| = \sum_{v \in V} \frac{d_C(v)}{2}. \tag{4}$$

Notice that if $e = xy \in C$, then $d_C(x)$ and $d_C(y)$ cannot both be ≥ 3, for in that case $C - e$ is a bicover. To put it another way, the vertices v for which $d_C(v) > 2$ form a coclique in C. Construct a bimatching B from C by removing exactly $d_C(v) - 2$ edges incident to each such v. By the previous observation, no two of these edges coincide. So in fact we are removing $d_C(v) - 2$ edges incident to *every* $v \in V$, since $d_C(v) - 2 \geq 0$ for all v. So we can calculate $|B|$ exactly:

$$|B| = |C| - \sum_{v \in V} (d_C(v) - 2). \tag{5}$$

By definition we have $|B| \leq \tilde{\alpha}$. Putting this together with (4) and (5), we obtain

$$\tilde{\alpha} + \tilde{\beta} \quad \geq \quad 2 \sum_{v \in V} \frac{d_C(v)}{2} - \sum_{v \in V} (d_C(v) - 2) \quad = \quad \sum_{v \in V} 2 \ = \quad 2n. \tag{6}$$

The desired result now follows from (3) and (6).