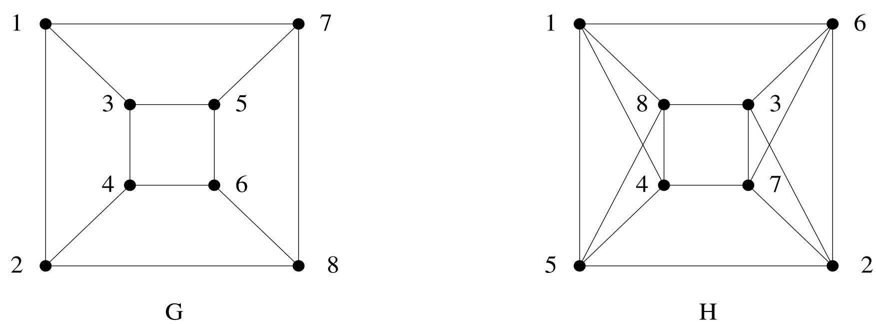
Math 725, Spring 2006
Solution Set #1

#1. For any integers $a, b \geq 3$, the graph $C_a + C_b$ is 2-regular and simple, but is not a cycle (since it is not connected). However, it is true that every *connected* 2-regular simple graph must be a cycle. (For example, such a graph must be Eulerian, and the constructive proof of Theorem 1.2.26 can be used to show that the Eulerian circuit contains one cycle only.)

#2. Consider the following labelings of the vertices by $[8] = \{1, 2, \ldots, 8\}$:



It is now routine to check that $ij \in E(G)$ if and only if $ij \notin E(H)$, which is precisely the statement that $\bar{G} \cong H$. (Technically, the isomorphism is the bijection $V(G) \to V(\bar{H})$ given by the identity map on [8].)

#3. (a) A spanning subgraph of G is determined by its edge set, which can be any subset of $E(G)$. Therefore, the number of spanning subgraphs is just $2^{e(G)}$.

(b) If G is connected, then it is a connected spanning subgraph of itself. On the other hand, if it has a connected spanning subgraph H, then any two vertices in $V(G) = V(H)$ are connected by a path in H, which is perforce a path in G, so G is connected.

(c) Suppose that G is disconnected. Let H be a component of G, let $A = V(H)$, and let $B = V(G) - V(H)$. Both of these sets are nonempty, and G does not contain any edge with one endpoint in A and the other in B. So every such edge belongs to $\bar{G}$; that is, $\bar{G}$ contains a subgraph isomorphic to $K_{a,b}$, where $a = |A|$ and $b = |B|$. This subgraph is spanning (since $a + b = n(G)$) and connected. So $\bar{G}$ is connected by (b).

#4. Let a_{ij} be the entry of the adjacency matrix in position (i, j); that is, a_{ij} is the number of edges joining vertices i and j. Assume G is loopless, so $a_{ii} = 0$ for all i. The (i, j) entry of A^2 is

$$\sum_{k=1}^{n} a_{ik} a_{kj}.$$

This counts *the number of i, j-walks of length two.* Indeed, each such walk must go through one other vertex $k \in [n] - \{i, j\}$. There are a_{ik} choices for the edge joining i and k and $a_{kj} = a_{jk}$ choices for the edge joining k and j.

#5. There are many ways to prove that the Petersen graph (call it G for now) doesn't have a 7-cycle.

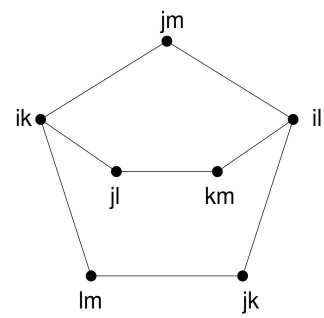
Proof #1: It is enough to show that if we delete any three vertices from G, then the resulting graph $G - X$ does not have a 7-cycle. Let $X \subseteq V(G)$ have cardinality 3. What is $e(G - X)$? Recall that G is 3-regular, so each vertex in X is incident to exactly three edges. Deleting X will kill all of these edges; however, we don't want to double-count the edges with both endpoints in G, so we conclude that

$$e(G - X) = e(G) - 9 + e(G[X]) = 6 + e(G[X]).$$

On the one hand, this had better be at least 7 if $G - X$ is going to have a 7-cycle. So X cannot be a coclique. On the other hand, $e(G[X])$ can be at most 2 (since we know that G has no 3-cycles).

CASE 1: $e(G[X]) = 1$. Let ij and $k\ell$ be the two adjacent vertices in X. The other vertex in X cannot be adjacent to either of these; without loss of generality it is ik. In particular, $jm \in V(G - X)$. But $N_G(jm) = \{ik, i\ell, k\ell\}$, so jm has only one neighbor in $G - X$, hence cannot be part of a cycle.

CASE 2: $e(G[X]) = 2$. That is, $G[X] \cong P_3$. Without loss of generality, we may assume that $X = \{ij, k\ell, im\}$. Then $G - X$ is the following graph:



Notice that $n(G - X) = 7$ and $e(G - X) = 8$. In particular, if $G - X$ contains a 7-cycle then it can be obtained by deleting one edge. But deleting any single edge will cause some vertex to have degree 1, so $G - X$ cannot contain a 7-cycle.

Proof #2: (This is shorter than the first proof, but it is less elementary in that it requires the use of Proposition 1.1.38, which states that each two nonadjacent vertices in G have exactly one common neighbor.) Let $C \subset G$ be a 7-cycle with vertices $v_1, v_2, \ldots, v_7$. Then v_1 and v_4 have a unique common neighbor in G, which is evidently not a vertex of C; call it x. Since Pete is 3-regular, we see that $N(v_1) = \{v_2, v_7, x\}$. One of these must be the common neighbor of v_1 and v_5.
— If $N(v_1) \cap N(v_5) = \{v_2\}$, then v_2, v_3, v_4, v_5 is a 4-cycle in G.
— If $N(v_1) \cap N(v_5) = \{v_7\}$, then v_5, v_6, v_7 is a 3-cycle in G.
— If $N(v_1) \cap N(v_5) = \{x\}$, then v_4, v_5, x is a 3-cycle in G.
In each case we have shown that Pete has girth ≤ 4, which contradicts Corollary 1.1.40.

Proof #3: Classify all the closed paths of length 7. To do this, we use the labeling of the vertices by two-element subsetse of [5], with adjacency given by disjointness. For each path v_1, v_2, v_3, we can write $v_1 = ij$, $v_2 = k\ell$, $v_3 = mi$, where $\{i, j, k, \ell, m\} = [5]$. We can continue the path by repeatedly choosing $v_{k+1} \in N(v_k) - \{v_1, \ldots, v_{k-1}\}$ for $k = 4, 5, \ldots$; for example, v_4 can be either jk or $j\ell$, and then the choice of v_5 depends on the choice of v_4. I won't draw out the whole search tree, but if you do it, you will see that it is impossible to have $v_8 = v_1$.

#6. I'll prove the contrapositive. Suppose that G is bipartite with bipartition X, Y. That is, each edge of G has one endpoint in each of X and Y. It is then immediate that for any subgraph H the pair $X \cap V(H)$, $Y \cap V(H)$ is a bipartition of H.

We have seen that odd cycles are not bipartite. Therefore, no graph containing an odd cycle can be bipartite.

#8. Let H be a component of $G - v$. There must be an edge $e \in E(G)$ with one endpoint in H and one endpoint in $V(G) - V(H)$; otherwise, H would be a maximal connected subgraph of G and G wouldn't be connected. On the other hand, the condition that H is a component of $G - v$ says that there is no edge with one endpoint in H and the other endpoint in $V(G - v) - V(H)$. So the other endpoint of e must be v.

If v is a cut-vertex of G, then $G - v$ is disconnected, hence has at least two components. By the previous observation, this implies that $d_G(v) \geq 2$.

#9. Recall that every walk in a bipartite graph must alternate between the two partite sets. Therefore, every walk of even (odd) length must have its endpoints in the same (different) partite sets. Now let G be a connected X, Y-bigraph and let X', Y' be another bipartition. Suppose that there exist vertices $x \in X \cap X'$ and $y \in X \cap Y'$. Since G is connected, it contains an x, y-walk; but by the second observation, its length must be both even and odd, a contradiction. Therefore, either $X = X'$ and $Y = Y'$, or else $X = Y'$ and $Y = X'$.

Now suppose that G is disconnected and bipartite. Let H be one of its components and $J = G - H$. Let X, Y be a bipartition of H and let X', Y' be a bipartition of $J - H$. Then $(X \cup X'), (Y \cup Y')$ and $(X \cup Y'), (Y \cup X)$ are genuinely different bipartitions of G. Alternately, given a proper 2-coloring of G, it is possible to swap the colors in a single component. (Indeed, up to swapping, the number of proper colorings of a bipartite graph G is $2^{c(G)-1}$.)

#10. Suppose that G is disconnected. Let X be the vertex set of one of its components, and let $Y = V(G) - X$; then there is no edge with one endpoint in each of X and Y.

Now suppose that G is connected and that $V(G) = X \sqcup Y$. Let P be a minimum-length X, Y-path (that is, a path with one endpoint in each of X and Y). Say that the vertices of P in order are $v_0, v_1, \ldots, v_n$, with $v_0 \in X$ and $v_n \in Y$. Suppose that $n \geq 2$. If $v_1 \in X$, then deleting v_0 produces a shorter X, Y-path, which contradicts the minimality of P. On the other hand, if $v_1 \in Y$, then deleting v_n produces a shorter X, Y-path. We conclude that $n = 1$ and P is a single edge, as desired. (This argument looks a bit artificial, but it does at least provide an algorithm for reducing a given X, Y-path to an X, Y-edge, deleting one endpoint at a time.)

Bonus problem: Fix n, and let X (resp. Y) be the set of bit strings with an even (resp. odd) number of 1's. Thus $V(Q_n) = V(R_n) = X \sqcup Y$.

First, suppose that n is odd. If two bit strings v, w are adjacent in R_n, then they agree in exactly one bit, hence differ in the other $n-1$ bits. Since $n-1$ is even, the parity of v is the same as that of w. That is, there is no edge with one endpoint in each of X and Y, so R_n is disconnected (by #10 above). But Q_n is connected, so $Q_n \not\cong R_n$.

Now, suppose that n is even. For a bit string v, write $\bar{v}$ for the string obtained by flipping every bit in v; for instance, $\overline{01001} = 10110$. Define a function $f : V(Q_n) \to V(R_n)$ by

$$f(v) = \begin{cases} v & \text{if } v \in X, \\ \bar{v} & \text{if } v \in Y. \end{cases}$$

Since n is even, v and $\bar{v}$ have the same parity. Hence the function f is a bijection (indeed, it is its own inverse).

To check that f is an isomorphism, note that if $vw \in E(Q_n)$, then they differ in exactly one bit, so in particular exactly one of them, say v, is even. Thus $f(v) = v$ and $f(w) = \bar{w}$ agree in exactly one bit, and hence $f(v)f(w) \in E(R_n)$. By essentially the same argument, $f(v)f(w) \in E(R_n)$ only if $vw \in E(Q_n)$.