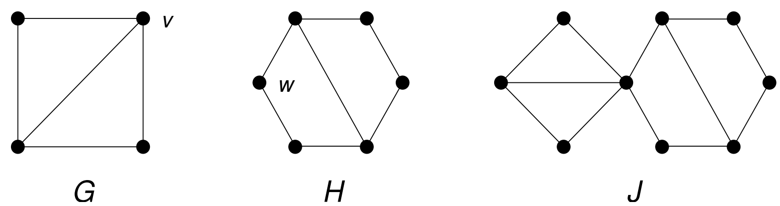
Math 725 Midterm Exam
Friday, March 10, 2006

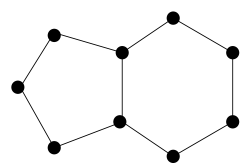
Do all five problems. No books or other notes are allowed. You can cite results proved in class provided that you indicate clearly that you are doing so.

#1. Prove that a bipartite Eulerian graph must have an even number of edges.

#2. (a) Let G and H be connected graphs such that $V(G) \cap V(H) = \emptyset$, and let $v \in V(G)$ and $w \in V(H)$. Let J be the graph formed from G and H by identifying the vertices v and w. Prove that $\tau(J) = \tau(G)\tau(H)$. (The following figure gives an example of the construction of J.)



(b) Let $a, b \geq 2$ be integers, and let $G_{a,b}$ be the graph formed by identifying an edge of the cycle C_a with an edge of the cycle C_b. For example, $G_{5,6}$ is the following graph:



Use the deletion-contraction recurrence and the result of #2a to find a closed-form formula for $\tau(G_{a,b})$ in terms of a and b.

#3. Let G be a connected simple graph with girth 4. What are the possible values for the girth of its complement $\overline{G}$?

#4. Prove or disprove the statement that every tree has at most one perfect matching.

#5. (a) Prove that if G is bipartite, then

$$\alpha'(G) \geq e(G)/\Delta(G).$$

(As a reminder, $\alpha'(G)$ is the size of a maximum matching in G, and $\Delta(G)$ is the maximum degree of a vertex.)

(b) Use the result of #5a to prove that every regular bipartite graph has a perfect matching.