**#1. Prove that a bipartite Eulerian graph must have an even number of edges.**

Let $G$ be Eulerian and bipartite with partite sets $X, Y$. Since $G$ is Eulerian, $d_G(x)$ is even for every $x \in X$. Also, each edge has one endpoint in $x$, so

$$e(G) = \sum_{x \in X} d(x)$$

is even as well.

Alternately, since $G$ is Eulerian it can be decomposed into cycles; that is, there are cycles $C_1, \dots, C_r$ such that $E(G) = E(C_1) \sqcup \cdots \sqcup E(C_r)$. Since $G$ is bipartite, each of these cycles has even length, hence $e(G) = \sum_{i=1}^{r} e(C_i)$ is even.

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**#2. (a) Let $G$ and $H$ be connected graphs such that $V(G) \cap V(H) = \emptyset$, and let $v \in V(G)$ and $w \in V(H)$. Let $J$ be the graph formed from $G$ and $H$ by identifying the vertices $v$ and $w$. Prove that $\tau(J) = \tau(G)\tau(H)$.**

We have $E(J) = E(G) \sqcup E(H)$ and $n(J) = n(G) + n(H) - 1$.

Suppose that $T \subseteq E(G)$ and $U \subseteq E(H)$ are spanning trees of $G$ and $H$ respectively. Then $|T \cup U| = (n(G)-1) + (n(H)-1) = n(J) - 1$. Also, $T \cup U$ is connected, because if $x, y \in V(G)$, then either $x, y \in V(G)$ (when they are joined by a path in $T$), or $x, y \in V(H)$ (when they are joined by a path in $U$), or $x \in V(G)$, $y \in V(H)$ (when there is an $x, v$-path $P \subseteq G$ and a $w, y$-path $Q \subseteq H$, and $P \cup Q$ is an $x, y$-path in $J$). So $T \cup U$ is a spanning tree of $J$.

Now suppose that $S \subseteq J$ is a spanning tree, and let $T = S \cap E(G)$ and $U = S \cap E(H)$. Then $T$ and $U$ are acyclic (since they are subsets of the acyclic edge set $S$). In particular

$$|T| \leq n(G) - 1 \qquad \text{and} \qquad |U| \leq n(H) - 1. \tag{1}$$

But $|S| = |T| + |U| = (n(G) - 1) + (n(H) - 1)$. So equality must hold in both parts of (1). Hence $T$ and $U$ are spanning trees of $G$ and $H$ respectively.

It follows that there is a bijection

$$\{\text{spanning trees of } G\} \times \{\text{spanning trees of } H\} \to \{\text{spanning trees of } J\},$$

and in particular $\tau(J) = \tau(G)\tau(H)$.

**(b) Let $a, b \geq 2$ be integers, and let $G_{a,b}$ be the graph formed by identifying an edge of the cycle $C_a$ with an edge of the cycle $C_b$. Use the deletion-contraction recurrence and the result of #2a to find a closed-form formula for $\tau(G_{a,b})$ in terms of $a$ and $b$.**

Let $e$ be the "glued" edge of $G_{a,b}$. Then $G_{a,b} - e \cong C_{a+b-2}$, and $G_{a,b}/e$ is the graph $J$ built from $G = C_{a-1}$ and $H = C_{b-1}$ as in (a). So

$$\tau(G_{a,b}) = \tau(C_{a+b-2}) + \tau(J) = (a + b - 2) + (a - 1)(b - 1) = ab - 1.$$

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**#3. Let $G$ be a connected simple graph with girth 4. What are the possible values for the girth of its complement $\overline{G}$?**

$G$ must have at least four vertices. If it has four vertices exactly then it must be $C_4$; its complement consists of two disconnected edges and has infinite girth (because it is acyclic). On the other hand, if $n(G) > 5$ then let $w, x, y, z$ be a 4-cycle and $v$ another vertex. Then $v$ cannot have both a neighbor in $\{w, y\}$ and a neighbor in $\{x, z\}$, else $G$ would have a 3-clique, i.e., a 3-cycle. So either $\{v, w, y\}$ or $\{v, x, z\}$ is a coclique in $G$, hence a 3-cycle in $\bar{G}$. So $\bar{G}$ has girth 3.

In summary, $\overline{G}$ has girth either 3 or $\infty$.

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**#4. Prove or disprove the statement that every tree has at most one perfect matching.**

This is true. Let $T$ be a tree and let $M, M'$ be perfect matchings. We know that every component of $M \triangle M'$ is either a path or an even cycle. The latter is impossible because $T$ is acyclic. On the other hand, for each component that is a nontrivial path, each endpoint of the path is saturated by exactly one of the matchings $M, M'$. Since both are perfect matchings, no component of $M \triangle M'$ can be a nontrivial path. We conclude that $M \triangle M' = \emptyset$, that is, $M = M'$. So $T$ has at most one perfect matching.

It is also possible to proceed by induction on $n(T)$. If $n(T) = 1$ (indeed, if $n(T)$ is any odd number) then $T$ has no perfect matching, while if $n(T) = 2$ then $T \cong K_2$ has one perfect matching consisting of its single edge. Otherwise, let $v$ be a leaf of $T$ and $w$ its unique neighbor. If $M$ is a perfect matching of $T$, then in particular $v \in V(M)$, so $vw \in M$. So $M$ consists of the edge $vw$ together with a perfect matching $M'$ on the graph $T' = T - v - w$; in turn $M'$ is the union of perfect matchings on each component of $T'$. On the other hand, $T'$ is a forest (it is certainly acyclic but need not be connected), so each of its components is a tree, and by induction has at most one perfect matching. Therefore $T'$ has at most one perfect matching, and so does $T$. (While perhaps a little less elegant, this proof does give an algorithm for computing a perfect matching of $T$ if one exists.)

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**#5. (a) Prove that if $G$ is bipartite, then $\alpha'(G) \geq e(G)/\Delta(G)$.**

The König-Egerváry Theorem says that $\alpha'(G) = \beta(G)$. Let $Q$ be a minimum vertex cover, so that $|Q| = \beta(G) = \alpha'(G)$. Each vertex of $Q$ has degree $\leq \Delta(G)$, hence covers at most $\Delta(G)$ edges. So $Q$ must have at least $e(G)/\Delta(G)$ vertices if it is to have any hope of being a vertex cover. That is, $\alpha'(G) \geq e(G)/\Delta(G)$, as desired.

**(b) Use the result of #5a to prove that every regular bipartite graph has a perfect matching.**

Let $G$ be a $k$-regular $X, Y$-bigraph. By the bipartite version of handshaking, we have $e = k|X| = k|Y|$. Also, $\Delta(G) = k$. By problem #5a, we have $\alpha'(G) \geq e(G)/\Delta(G) = k|X|/k$. So a maximum matching $M$ has $|X| = |Y|$ edges, hence saturates $2|X| = n(G)$ vertices; that is, $M$ is a perfect matching.