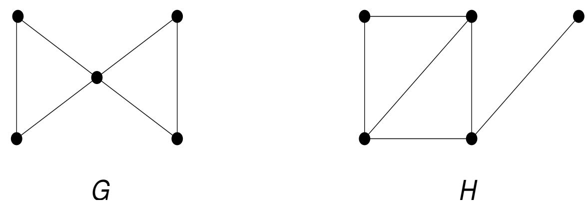
Math 725, Spring 2006
Problem Set #7
Due Friday, May 5, in class

#1. Complete the proof of Theorem 3.1 in the lecture notes on the Tutte polynomial (available from the course website). To do this, you must show that if e is a loop, then

$$T(G; x, y) \;=\; y \cdot T(G - e; x, y). \qquad (*)$$

(I've released you from having to check the case that e is a cut-edge; that calculation is in the notes. You should try to verify $(*)$ on your own, but if necessary, you can read through the calculation of the cut-edge case and mimic it to prove $(*)$.)

#2. Let G and H be the graphs shown below.



Verify that G and H have the same chromatic polynomial, but not the same Tutte polynomial.

#3. [West 6.1.21] Let G be a connected plane graph and let $F \subseteq E(G)$. Prove that F is a spanning tree of G if and only if $E(G^*) - F^*$ is a spanning tree of G^*.

#4. Let G be a connected planar graph. Prove that

$$T(G; x, y) = T(G^*; y, x).$$

(Hint: There are two ways to do this. You can use the closed formula for the Tutte polynomial and compare the rank of a subset of $E(G)$ to its dual subset of $E(G^*)$, for which results like #3 above and Theorem 6.1.14 may be helpful. Alternately, show that $(G - e)^* = G^*/e^*$ and $(G/e)^* = G^* - e^*$, and that taking the planar dual interchanges loops and cut-edges, then apply the recurrence.)

#5. [West 6.1.25] A planar graph G is called *self-dual* if $G \cong G^*$.

(a.) Prove that if G is self-dual, then $e(G) = 2n(G) - 2$.

(b.) For all $n \geq 4$, construct a self-dual simple graph of order n.

Bonus problem: [West 6.3.28] Let m and n be odd. Prove that in all drawings of $K_{m,n}$, the parity of the number of pairs of edges that cross is the same. (Consider only drawings in which edges cross at most once, and edges sharing an endpoint do not cross.) Conclude that $\nu(K_{m,n})$ is odd if and only $m - 3$ and $n - 3$ are both divisible by 4.