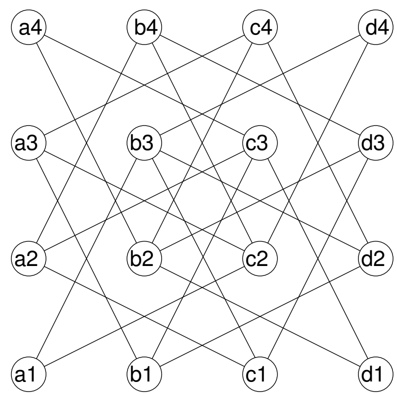
Math 725, Spring 2006
Problem Set #6
Due Friday, April 21, in class

#1. [West 4.3.15] Let G be a weighted graph with weight function $\mathrm{wt} : E \to \mathbb{R}_{\geq 0}$. For each spanning tree T of G, define $a(T) = \min\{\mathrm{wt}(e) : e \in T\}$ and for each edge cut $[S, \bar{S}]$, define $b([S, \bar{S}]) = \max\{\mathrm{wt}(e) : e \in [S, \bar{S}]\}$. Prove that

$$\max_{\text{spanning trees } T} a(T) \quad = \quad \min_{\text{edge cuts } [S,\bar{S}]} b([S, \bar{S}]).$$

#2. Let G be the graph with 16 vertices $\{a_1, a_2, \ldots, d_4\}$ corresponding to the squares of a 4×4 chessboard, with two vertices adjacent if they are connected by a knight move (that is, if one of their row and column coordinates differs by 2, and the other by 1). The graph G can be pictured as follows:



(a) Use the Max-Flow/Min-Cut algorithm to find a maximum family of PED paths joining the vertices $s = b_2$ and $t = b_3$.

(b) Certify that your answer is correct by exhibiting an s,t-edge cut of cardinality $\lambda'(s,t)$.

(c) Prove that $\lambda(s,t) < \lambda'(s,t)$.

#3. [West 5.1.22] Given a set of lines in the plane with no three meeeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \leq 3$.

#4. [West 5.1.38] Prove that $\chi(G) = \omega(G)$ if $\bar{G}$ is bipartite.

#5. [West 5.3.4] (a) Prove that the chromatic polynomial of the n-cycle is $\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$.

(b) For $H = G \vee K_1$, prove that $\chi(H; k) = k \cdot \chi(G; k-1)$. (Here $\vee$ denotes join; see Defn. 3.3.6 on p. 138.)

(c) Use (a) and (b) to find the chromatic polynomial of the wheel $W_n = C_n \vee K_1$.

Bonus problem: We proved in class that a tree T on n vertices has chromatic polynomial $\chi(T; k) = k(k-1)^{n-1}$ (see Proposition 5.3.3). Conversely, suppose that G is a graph with chromatic polynomial $\chi(T; k) = k(k-1)^{n-1}$ for some n. Must G be a tree?