Math 725, Spring 2006
Problem Set #4
Due Wednesday, March 15, in class

#1. [West 3.1.24] A *permutation matrix* P is a square matrix all of whose entries are 0 or 1, with exactly one 1 in each row and in each column. For k a positive integer, prove that a square matrix of nonnegative integers can be written as the sum of k permutation matrices if and only if every row and every column has sum k.

#2. [West 3.1.31] Use the König-Egerváry Theorem to prove Hall's Marriage Theorem. That is, assume that $\alpha'(G) = \beta(G)$ for G bipartite, and use that fact to come up with an alternate proof of Hall's Theorem. You should not assume from the start that Hall's Theorem holds (else this exercise would be trivial), and your proof should not resemble West's proof of either of these results.

#3. [West 3.1.19, Schrijver] Let Y be a finite set and $\mathbf{A} = \{A_1, \ldots, A_m\}$ a family of subsets of Y (not necessarily disjoint). A *system of distinct representatives* (or SDR) for $\mathbf{A}$ is a set of distinct elements $y_1, \ldots, y_m \in Y$ such that $y_i \in A_i$ for all i.

(a) Prove that $\mathbf{A}$ has an SDR if and only if $\left|\bigcup_{i \in S} A_i\right| \geq |S|$ for all $S \subseteq [m]$.

(b) Let $\mathbf{B} = \{B_1, \ldots, B_m\}$ be another family of subsets of Y. Prove that $\mathbf{A}$ and $\mathbf{B}$ have a *common* SDR if and only if for each $I \subseteq [n]$, the set $\bigcup_{i \in I} A_i$ meets at least $|I|$ of the sets B_j.

#4. [Schrijver] Let $G = (V, E)$ be a simple graph with $n = n(G)$ and $\delta(G) \geq 2$. Define a *bimatching* to be an edge set $B \subseteq E$ such that no vertex belongs to more than <u>two</u> edges in B, and define a *bicover* to be an edge set $C \subseteq E$ if every vertex belongs to at least two edges in C. Let

$$\begin{aligned} \tilde{\alpha} &= \tilde{\alpha}(G) &= \max\{|B| : B \text{ is a bimatching}\}, \\ \tilde{\beta} &= \tilde{\beta}(G) &= \min\{|C| : C \text{ is a bicover}\}. \end{aligned}$$

Prove that $\tilde{\alpha} \leq \tilde{\beta}$ and that $\tilde{\alpha} + \tilde{\beta} = 2n$. (Hint: As in the proof of Gallai's theorem, show separately that $\tilde{\alpha} + \tilde{\beta} \leq 2n$ and that $\tilde{\alpha} + \tilde{\beta} \geq 2n$. However, unlike in the proof of Gallai's theorem, you will probably not be able to say anything useful about the structure of a minimum bicover.)