Math 725, Spring 2006
Problem Set #3
Due Monday, February 27, in class

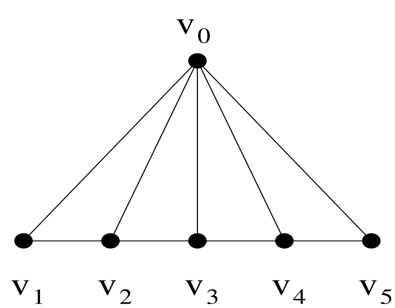
#1. [West 2.3.14] Let C be a cycle in a connected weighted graph. Let e be an edge of maximum weight in C; that is, $\mathrm{wt}(e) \geq \mathrm{wt}(e')$ for all $e' \in E(C)$. Prove that there is a minimum-weight spanning tree not containing e. Use this to prove that iteratively deleting a heaviest non-cut-edge until the graph is acyclic produces a minimum-weight spanning tree. (It may help to write out that algorithm more explicitly.)

#2. [West 2.2.1] Determine which spanning trees of K_n have Prüfer codes that (a) contain only one value; (b) contain exactly two values; or (c) contain $n-2$ distinct values.

#3. [West 2.2.16, modified] Let F_n be the "n-fan" defined by

$$\begin{aligned} V(F_n) &= \{v_0,\ v_1,\ \ldots,\ v_n\}, \\ E(F_n) &= \{v_1v_2,\ v_2v_3,\ \ldots,\ v_{n-1}v_n\} \cup \{v_0v_1,\ v_0v_2,\ \ldots,\ v_0v_n\}. \end{aligned}$$

For example, F_5 is shown below.



a. Prove that $\tau(F_n) = 3\tau(F_{n-1}) - \tau(F_{n-2})$.

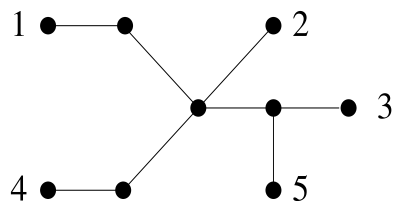
b. Based on your argument in (a), describe a sequence of graphs $G_1, G_2, G_3, \ldots$ so that $a_i = \tau(G_i)$ is the i^{th} Fibonacci number; that is, $a_1 = a_2 = 1$ and $a_i = a_{i-1} + a_{i-2}$ for $i \geq 3$. (For more on the Fibnonacci numbers, see `http://www.research.att.com/~njas/sequences/A000045`.)

#4. [West 2.2.18] Use the Matrix-Tree Theorem to compute $\tau(K_{r,s})$ for all numbers r, s. (Hint: Apply elementary row and column operations to the Laplacian matrix.)

#5. [West 2.1.58, modified] In this problem, you will prove the following theorem due to Smolenskii. Let S and T be trees with leaves $x_1, \dots, x_m$ and $y_1, \dots, y_m$ respectively, such that $\{d_S(x_i, x_j) = d_T(y_i, y_j)$ for all i, j. (Let's call the data $\{d_S(x_i, x_j) : \ 1 \le i < j \le m\}$ the *leaf-distance sequence* of T.) Then there is an isomorphism $f : S \to T$ such that $f(x_i) = y_i$ for every i.

a. Suppose that $f : G \to H$ is an isomorphism. Let $u \in V(G)$ and $v = f(u) \in V(H)$. Let G' be the graph constructed by attaching a leaf u' to G at u (that is, $V(G') = V(G) \sqcup \{u'\}$ and $E(G') = E(G) \sqcup \{uu'\}$) and let H' be the graph constructed by attaching a leaf to H at v. Show that $G \cong H$. (This is one of those statements that is intuitively true, but requires careful bookkeeping. In particular, you really have to use the definition of an isomorphism!)

b. A *stub* of S is a leaf x_i whose unique neighbor in S has degree > 2. Fix a leaf $x_i \in V(S)$ and let w be its stem (unique neighbor). Call x_i a *stub* iff $d_T(w) > 2$. For instance, in the following tree, the stubs are the leaves labeled with prime numbers.



Show that x_i is a stub if and only if for some j, k,

$$d(x_i, x_j) + d(x_i, x_k) = d(x_j, x_k) + 2.$$

Conclude that x_i is a stub of S if and only if y_i is a stub of T.

c. Suppose that x_1 is not a stub. Describe the leaves and the leaf-distance sequence of $S - x_1$ in terms of those of S,

d. Suppose that x_i is a stub. Describe the leaves and the leaf-distance sequence of $S - x_i$ in terms of those of S.

e. Prove Smolenskii's theorem by induction on the number of vertices. (Hint: Let x_1' and y_1' be the unique neighbors of x_1 and y_1 respectively. Show that there is an isomorphism $S - x_1 \to T - x_1$ mapping x_1' to y_1' and apply (a). This is easier in the case that the leaves x_1, y_1 are not stubs; if they are stubs, then you must use (b) to identify x_1' uniquely from the leaf-distance sequence of S.)

Bonus problem: [West 2.3.16] Four people must cross a canyon at night on a fragile bridge. At most two people can be on the bridge at once, and there is only one flashlight (which can cross only by being carried). Kovalevskaia can cross the bridge in 10 minutes, Legendre in 5 minutes, Macaulay in 2 minutes, and Noether in 1 minute. If two people cross together, they move at the speed of the slower person. Oh, by the way, in 18 minutes a flash flood is going to roar down the canyon and wash away the bridge (together with anyone who isn't yet safe on the other side). Can the four people get across in time?