Math 725, Spring 2006
Problem Set #2
Due Friday, February 10, in class

#1. [West 1.2.40] Let P and Q be paths of maximum length in a connected graph G. Prove that P and Q have a common vertex.

#2. [West 1.3.24] Prove that $K_{3,2}$ is not a subgraph of any hypercube Q_n.

#3. Does every connected graph G with $\delta(G) \geq 2$ have a connected Eulerian spanning subgraph? (Either prove that it does, or give a counterexample.)

#4. [West 1.4.10] Prove that a digraph D is strongly connected if and only if for each partition of its vertex set $V(D) = S \sqcup T$, there is an edge whose tail is in S and whose head is in T.

#5. [West 2.1.29]
(a) Prove that every tree is bipartite.
(b) Let X, Y be a bipartition of a tree T, and suppose that $|X| \geq |Y|$. Prove that X contains a leaf of T.

#6. Let T be a tree with ℓ leaves. Prove that T is a *caterpillar* (that is, there is some path in T that either contains or is incident to every edge) if and only if its diameter is $\ell - 2$.

#7. [West 2.1.37] Let T, T' be two spanning trees of a connected graph G. For every $e \in E(T) - E(T')$, prove that there exists an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ and $T - e + e'$ are both spanning trees of G.

Bonus problem: Recall that an *orientation* of a graph G is a digraph whose underlying graph is G. Let G be connected. Prove that G has a strong orientation if and only if it has no cut-edge.