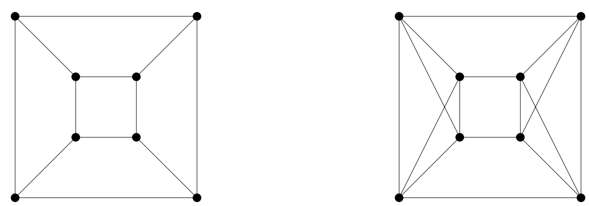
Math 725, Spring 2006
Problem Set #1
Due date Wednesday, February 1 (in class)

Note: Subsequent problem sets will probably contain fewer, and more challenging, problems than this one. Right now, the goal is to become comfortable with all the definitions, and to get some practice in the kinds of arguments necessary in graph theory.

#1. [West 1.1.5, modified] Show that the statement "If G is a 2-regular simple graph, then G is a cycle" is false, by exhibiting an explicit counterexample. What additional condition on G makes the statement true?

#2. [West 1.1.9] Prove that the graph on the right is isomorphic to the complement of the graph on the left.



#3. A subgraph $H \subseteq G$ is called a *spanning subgraph* if $V(H) = V(G)$.
(a) Let G be an arbitrary graph. How many spanning subgraphs does G have?
(b) Prove that G is connected if and only if it has a connected spanning subgraph.
(c) Use (b) to solve problem 1.1.10 in West. (Hint: If G is disconnected, what spanning subgraph must its complement have?)

#4. [West 1.1.30, shortened] Let G be a graph on vertex set $[n]$, and let $A = A(G)$ be the (n-by-n) adjacency matrix of G. What does the entry of A^2 in position (i, j) tell you about G?

#5. [West 1.1.25] Prove that the Petersen graph has no cycle of length 7.

#6. Let G be a graph and H a subgraph of G. Prove that if H is not bipartite, then neither is G. Deduce the "only if" direction of Theorem 1.2.18.

#7. Recall that the *n-dimensional cube* Q_n is the graph whose vertices are the bit strings of length n, with two vertices adjacent if and only if they differ in exactly one bit. Show that Q_n is bipartite, by exhibiting an explicit bipartition.

#8. [West 1.2.5] Let v be a vertex of a connected simple graph G. Prove that v has a neighbor in every component of $G - v$. Conclude that no graph has a cut-vertex of degree 1.

#9. [West 1.2.7] Prove that a bipartite graph has a unique bipartition (up to interchanging the two partite sets) if and only if it is connected.

#10. [West 1.2.22] Prove that a graph is connected if and only if for every partition of its vertices into two nonempty sets X, Y, there is an edge with one endpoint in each of X and Y.

Bonus problem: Let R_n be the graph on the bit strings of length n, in which two bit strings are adjacent if and only if they *agree* in exactly one bit. Show that $R_n \cong Q_n$ if and only if n is even.