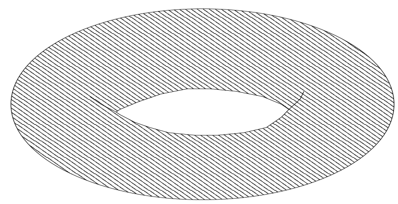
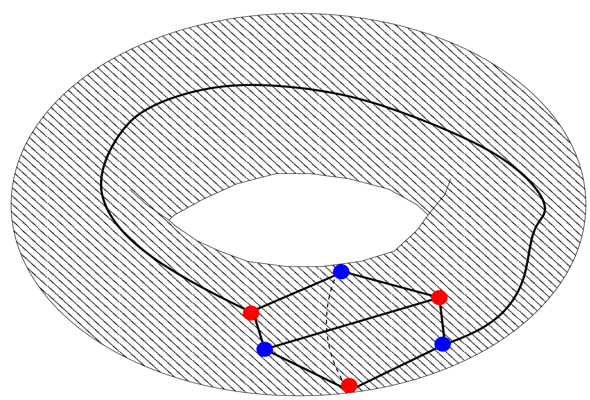
Embedding Graphs on Surfaces of Higher Genus

If we can't embed $K_{3,3}$ in the plane (or, equivalently, the sphere), what if we build a bridge to avoid crossings?

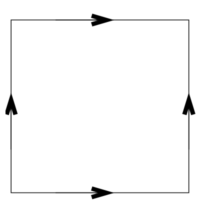
Essentially, we are adding a "handle" to the sphere. What we get is, topologically speaking, a torus.

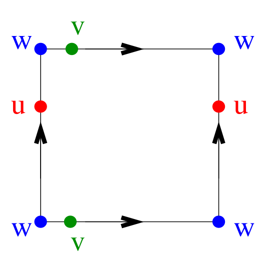


If $\nu(G) = 1$, then G is embeddable on a torus. For example, let $G = K_{3,3}$:



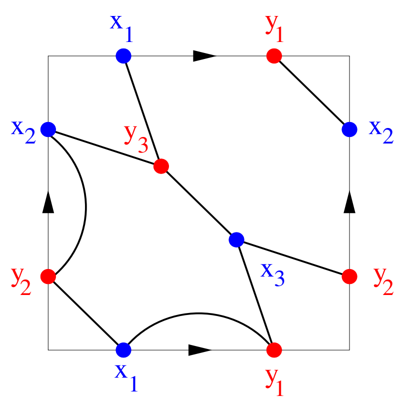
This picture is awkward, but there's a nicer way to draw pictures on the torus. To construct a torus, we could take a sheet of paper, glue the top and bottom edges together, and glue the left and right sides together. The topological diagram for this gluing is as shown on the left:



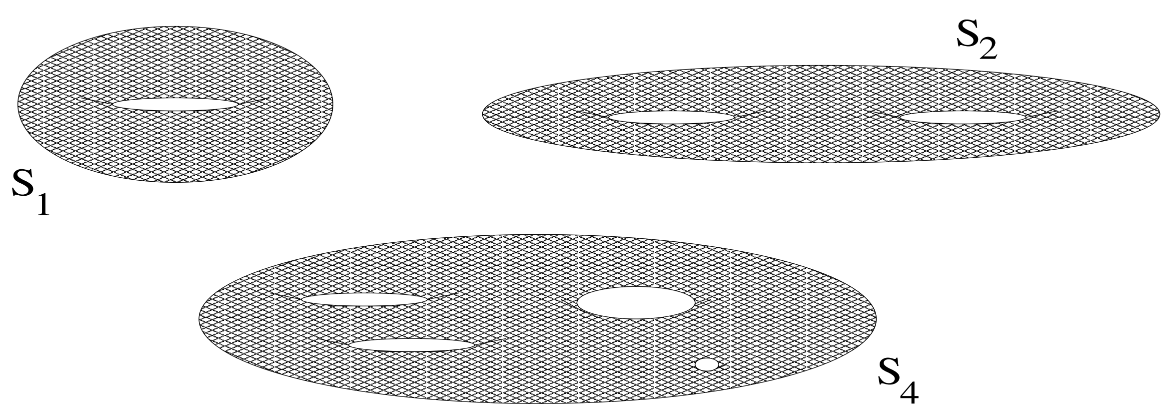


The arrows indicate the orientations of the edges when they are glued together. (Reversing one of the arrows would produce a Klein bottle instead of a torus.) Thus the two red dots represent the same point v. So do the two green dots (v) and the four blue ones (w).

That means that we can represent toroidal embeddings of graphs by drawing them on a square. For example, here's an embedding of $K_{3,3}$:



Definition: The ***g*-holed torus** S_1 is the surface obtained by adding g handles to the sphere.



The **genus** of a graph G is

$$\gamma(G) = \min\{g \mid G \text{ embeds on } S_g\}.$$

Notice that $\gamma(G) \leq \nu(G)$, because we can eliminate a crossing by adding a bridge. However, the inequality can be quite sharp. In fact, not only does

K_5 have genus 1, but so does K_7 (whose crossing number is 9)! (See p. 267 for the figure.)

Euler's formula for tori If G has a 2-cell embedding on a surface of genus g, then

$$n - e + f \ = \ 2 - 2g.$$

Corollary: $e \leq 3(n - 2 + 2g)$.

Theorem (Heawood 1890) If G is embeddable on a surface of genus $g > 0$, then

$$\chi(G) \leq \lfloor c \rfloor,$$

where

$$c = \frac{7 + \sqrt{1 + 48g}}{2}.$$

Proof. It suffices to prove that G has a vertex of degree $\leq c - 1$; the desired result will then follow from induction on n. There is no problem if $n \leq c$, so we assume that $n > c$.

The quantity c is a positive root of the polynomial

$$c^2 - 7c + (12 - 12g) = 0$$

which is equivalent to

$$c - 1 \ = \ 6 - \frac{12 - 12g}{c}$$

so

$$\begin{aligned} \frac{2e}{n} &\leq \frac{6(n - 2 + 2g)}{n} \quad \text{by the Corollary} \\ &= 6 + \frac{12g - 12)}{n} \\ &\leq 6 + \frac{12g - 12)}{c} \quad \text{this is the key step!} \\ &= c - 1. \end{aligned} \qquad (*)$$

Notice that the key inequality requires that $12g - 12 \geq 0$, that is, $g \geq 1$. So the average degree of G is less than c, which means that at least one vertex must have degree $\leq c - 1$ as desired. ∎

Note that $c = 4$ for $g = 0$. However, Heawood's argument does not suffice to prove the Four-Color Theorem, because the second inequality in $(*)$ is valid only if $g \geq 0$.

For $g = 1$, we have $c = 7$. So every toroidal graph is 7-colorable.

In fact, Heawood's bound is sharp for $g > 0$; this is quite nontrivial but can be proven more easily than the Four-Color Theorem. So, strangely, the problem of determining the maximum chromatic number of genus-g graphs is most difficult when $g = 0$.

Wagner's Theorem can be generalized for the torus, in the following sense:

Theorem: For every $n \geq 0$, there is some **finite** set Φ_n of (isomorphism types of) graphs such that

$$\gamma(G) \leq n \qquad \Longleftrightarrow \qquad G \text{ has no minor in } \Phi_n.$$

For $n = 0$, we have $\Phi_0 = \{K_5, K_{3,3}\}$.

Lots and lots of elements of Φ_1 are known, but not the complete list.

How do we know that the set is finite? Well, there is an amazing result called the Graph Minor Theorem (GMT), due to Robertson and Seymour:

In every infinite list of graphs, some graph is a minor of another.

It follows from the GMT that every list of minimal obstructions must be finite, since no two elements of it are comparable.