The 5-Color Theorem

Theorem: Let $G = (V, E)$ be a planar graph. Then $\chi(G) \leq 5$.

Proof. We induct on $n = n(G)$. For the base case(s), if $n \leq 5$ then G is certainly 5-colorable. So assume inductively that every planar graph on fewer than n vertices is 5-colorable.

Recall that Euler's formula implies $e(G) \leq 3n - 6$. Therefore $\delta(G) \leq 5$ (otherwise $e(G) = (\sum_{v \in V} d(v))/2 \geq 3n$). Let v be a vertex of minimal degree.

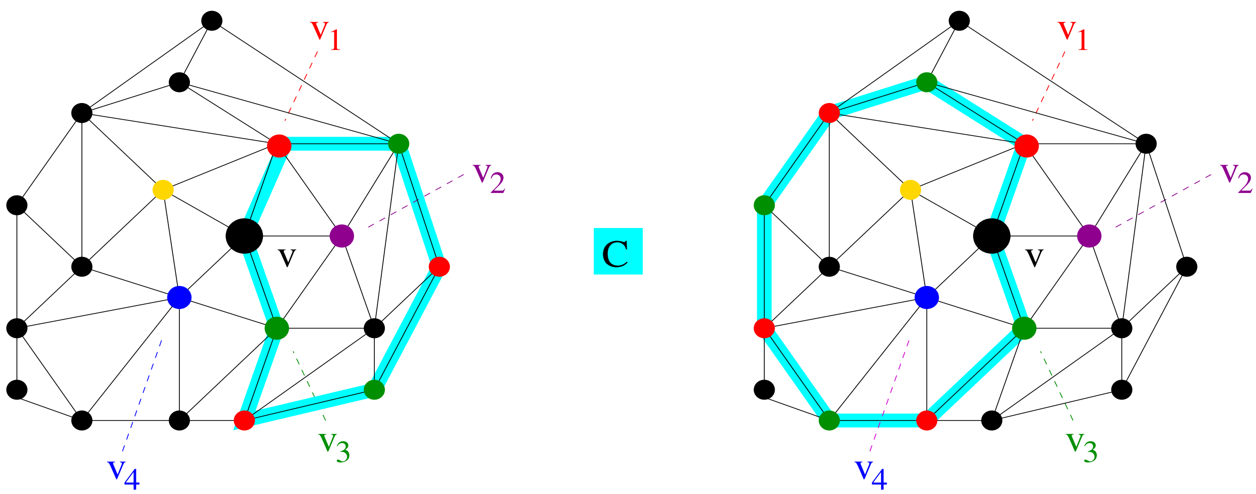
The graph $G - v$ is planar, and by induction has a proper 5-coloring f. If $\delta < 5$, or if $\delta = 5$ and two neighbors of v have the same color, then we can extend f to a 5-coloring of G.

Otherwise, fix a planar embedding of G. Let v_1, v_2, v_3, v_4, v_5 be the neighbors of v, listed in cyclic order around v. Assume WLOG that $f(v_i) = i$. Let G_{ij} be the induced subgraph of $G - v$ on the vertices of colors i and j.

Claim: There is a pair of colors i, j such that v_i and v_j lie in different components of G_{ij}.

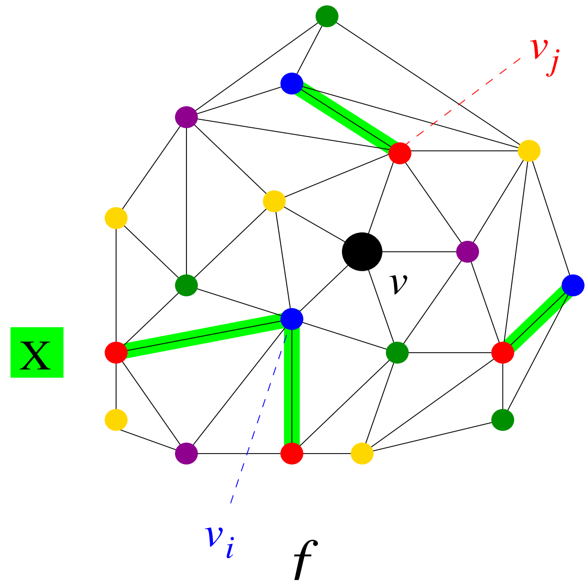
Proof of claim: Either v_1, v_3 is such a pair or it isn't. If it is, we're done. If not, then v_1, v_3 lie in the same component Y of $G - v$. So Y contains a v_1, v_3-path that alternates between colors 1 and 3.

Adding the edges vv_1 and vv_3 to this path forms a cycle $C \subset G$ separating v_2 and v_4. That is, either v_2 is inside C and v_4 lies outside C, or vice versa. Here are the two possibilities:

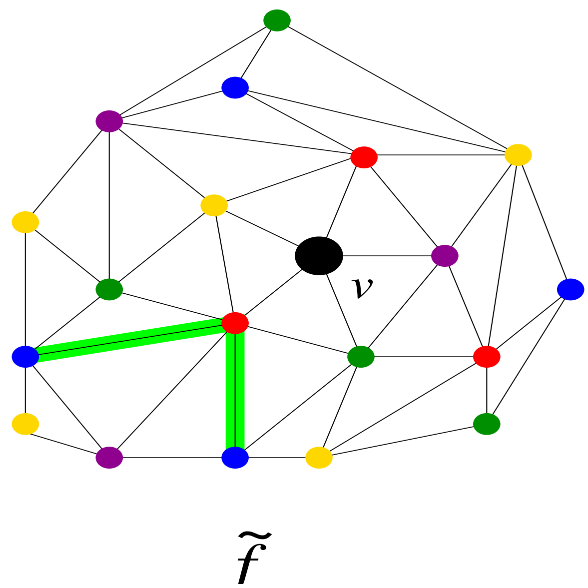


But then $V(C)$ is a v_2, v_4-cut, and none of its vertices belongs to G_{24}. Hence v_2 and v_4 lie in different components of G_{24}, proving the claim.

Let i, j be as in the claim, and let X be the component of G_{ij} that contains v_i. In the example below, color i is indicated in blue and color j in red. The edges of G_{ij} are highlighted in green.



Now form a new coloring $\tilde{f}$ by swapping the colors i, j on X, leaving the color of every other vertex unchanged.



It is easy to check that $\tilde{f}$ is a proper 5-coloring. Moreover, $\tilde{f}(w) \neq i$ for $w \in N(v)$. So we can set $\tilde{f}(v) = i$ to produce a proper 5-coloring of G, as desired.

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