Math 724, Fall 2017
Homework #5
Deadline: Monday, October 30, 5:00pm

Instructions: Typeset your solutions in LaTeX. Email your solutions to Jeremy (jlmartin@ku.edu) as a PDF file named with your last name and the problem set number (e.g., `Macaulay5.pdf`). Collaboration is encouraged, but each student must write up his or her solutions independently and acknowledge all collaborators.

(#1) Bogart, Problem #168. (Parts (b), (c) and (d) don't require an explanation. For the other parts, a couple of sentences should be enough.)

(#2) Bogart, Problem #170.

(#3) Bogart, Problem #197.

(#4) Bogart, Problem #199, parts (a)–(f). (The typesetting of this problem is horrible. In the preamble, the last sentence should read, "Then one possible picture of the tree T is the product

$$P(T) = \prod_{\{i,j\}\colon\ i \text{ and } j \text{ are adjacent}} x_i x_j.''$$

Likewise, part (e) would be clearer with a displayed equation: "...the picture enumerator

$$\sum T:\ T \text{ is a tree on [n]} P(T).''$$

Oh, and the second sentence in part (d) should end with a question mark.)

(#5) Bogart, Problems #205–207 (they're really a package deal).

(#6) Let $R = \mathbb{C}[x_1, \dots, x_n]$ be the ring[1] of polynomials in n variables with complex coefficients. As a vector space, R decomposes as a direct sum

$$R = \bigoplus_{k=0}^{\infty} R_k = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

where R_k means the vector subspace of polynomials that are homogeneous of degree k. (This "direct sum" statement is just a fancy way of saying that every polynomial can be written in exactly one way as a constant plus a homogeneous linear form plus a homogeneous quadratic form plus yada yada yada.) The *Hilbert series* of R is defined as the generating function for the vector space dimensions of the graded pieces R_k:

$$H_R(q) = \sum_{k \geq 0} (\dim_{\mathbb{C}} R_k) q^k.$$

Find a nice closed-form expression for $H_R(q)$. (Hint: You have already done this. The problem really is about understanding the algebra well enough to recognize the appropriate combinatorics.)

(Note: The Hilbert series can be defined for every graded ring, i.e., every ring R with a decomposition $R = \bigoplus_k R_k$ such that $R_k R_\ell \subseteq R_{k+\ell}$. Graded rings are ubiquitous in algebraic combinatorics, and the Hilbert series of a graded ring is its most fundamental invariant. For example, the quotient of $\mathbb{C}[x_1, \dots, x_n]$ by any homogeneous ideal (i.e., any ideal I generated by homogeneous polynomials) is a graded ring; one remarkable fact is that the Hilbert series of $\mathbb{C}[x_1, \dots, x_n]/I$ is always a rational function of q.)

[1] Quick definition of "ring": a set in which addition, subtraction and multiplication are well-defined operations that behave the way you think they ought to.