Math 410, Spring 2009
Homework Problems (updated 2/19/09)

In the 6th century, the Indian mathematician Aryabhata wrote that "*Half the circumference multiplied by half the diameter is the area of a circle.*"

Problem #1 Is Aryabhata's statement correct? Why or why not?

The quotation above doesn't say anything about the actual numerical value of π. However, Aryabhata also gave the following rule.

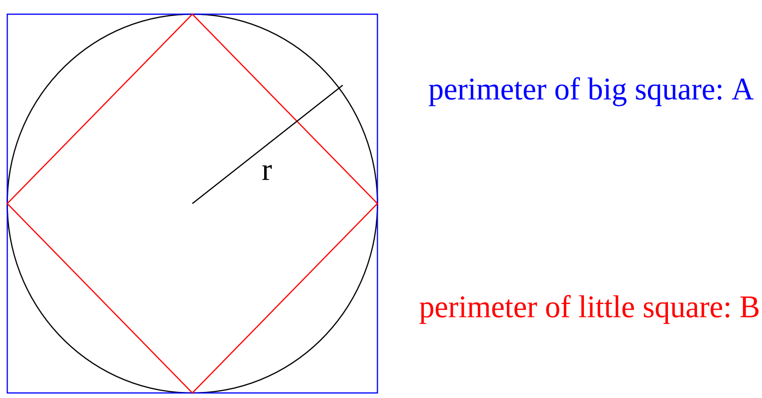
> *Add four to one hundred, multiply by eight and then add sixty-two thousand. The result is approximately the circumference of a circle of diameter twenty thousand. By this rule the relation of the circumference to diameter is given.*

Problem #2 What numerical value of π is implicit in Aryabhata's formula?

In class, we (briefly) discussed Archimedes' method for estimating the value of π. What his method boils down to is this: if you inscribe a regular polygon inside a circle, then its perimeter — let's call it B — will be a lower bound for the circumference of the circle. Likewise, if you circumscribe a regular polygon around a circle, then its perimeter A will be an upper bound for the circumference. That is,

$$A < \text{circumference} < B.$$

The more sides that P has, the closer these estimates will be (and, of course, the harder to compute).

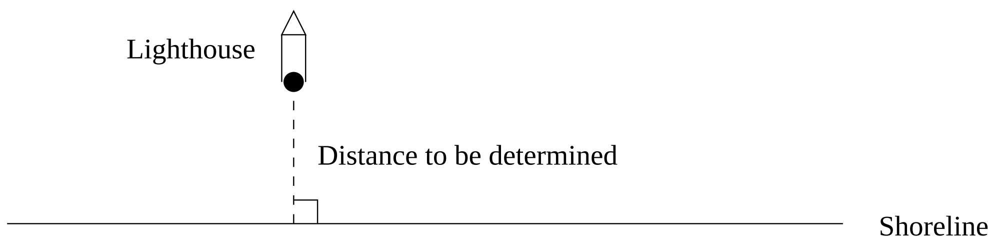


Problem #3 Figure out the formulas for A and B, in terms of r, when the regular polygon is a square (as in the above figure).

For extra credit, figure out the values of A and B for other polygons (or even, if you want, for a general polygon).

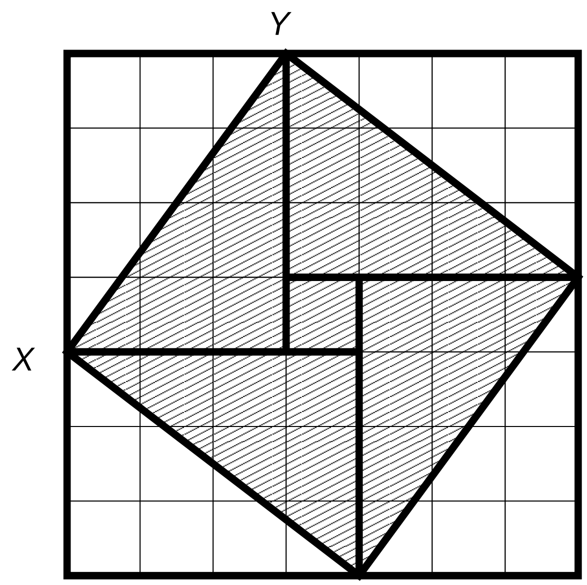
Problem #4 Give a proof that $\sqrt{3}$ is irrational. (If you understand the Pythagorean proof that of the irrationality of $\sqrt{2}$, you'll be able to modify it to deal with the case of $\sqrt{3}$.)

Problem #5 The year is 500 BCE. You are a Greek mathematician, currently working for the Egyptian government as a consultant. You've been asked to determine the distance from a lighthouse, which stands on a rock in the sea, to the mainland (see figure).



Explain to your Egyptian employers how you're going to carry out the project, and why your method works. You can use your compass, and you can measure as many line segments as you need, as long as they are all between points on land (the slaves who are going to carry out the measurements can't swim). Hint: You actually need to measure only one line segment, as long as you choose the points carefully.

What we call the Pythagorean theorem was known (as *Gougu*) to ancient Chinese mathematicians. One of the earliest and most elegant proofs of the Pythagorean theorem comes from the *Zhoubi suanjing (Zhou Shadow Gauge Manual)*, which was compiled sometime between 100 BCE and 100 CE. The text contains the following diagram[1]



Problem #6 Using this diagram, explain why the line segment $\overline{XY}$ has length 5 (making it the hypotenuse of a 3-4-5 right triangle). Don't use the Pythagorean theorem in your answer (since that is ultimately what you're trying to explain). Hint: Find the area of the shaded square.

The *Zhoubi suanjing* contains the following statement:[2]

> A person gains knowledge by analogy, that is, after understanding a particular line of argument they can infer various kinds of similar reasoning.... Whoever can draw inferences about other cases from one instance can generalize.... To be able to deduce and then generalize... is the mark of an intelligent person.

Problem #7 In that spirit, take your explanation in the previous problem and use it to explain why the Pythagorean theorem works in general: that is, for any right triangle with sides of lengths a and b and hypotenuse of length c, the equation $a^2 + b^2 = c^2$ holds.

[1]More or less. The actual diagram in the text contains Chinese characters that I have not reproduced here, and the middle square is not shaded.

[2]Taken from the MacTutor website, which has a lot more information on ancient Chinese mathematics: http://www-history.mcs.st-andrews.ac.uk/HistTopics/Chinese_overview.html.

Problem #8 This problem concerns Euclid's proof of the theorem that there are infinitely many prime numbers (see the class notes for February 11).

Suppose that Euclid had defined $x = p_1 p_2 \dots p_n + 2$ instead of $x = p_1 p_2 \dots p_n + 1$. Would the argument still be valid?

(Important note: I am not asking you to say whether Euclid's theorem is still true, but whether this alternate proof would be logically correct. If you try to prove a theorem and your proof is incorrect, it doesn't automatically mean that the theorem is false.)

It was pointed out in class that you can test whether a number n is divisible by 3 by adding up its digits. For example, if you add up the digits of $n = 72465702$, you get $7+2+4+6+5+7+0+2 = 33$. Since 33 is a multiple of 3, so is 72465702. The same test works for divisibility by 9 — since 33 is not a multiple of 9, neither is 72465702.

Here's something else you might try doing to a number n: look at the *alternating* sum of the digits. That is, add up the digits, but put a minus sign on every other digit. For example,

$$n = 72465702 \quad \text{becomes} \quad 7-2+4-6+5-7+0-2 \;=\; 5.$$

(Of course, it's possible to get zero or a negative number in this way, but so what?)

Miracle of miracles! This procedure really is a test for divisibility by some number d. Your job is to figure out what d is.

Problem #9 (i) Suppose that you apply this test to all positive numbers n from 1 to 100. (Don't actually do it — just think about what would happen.) What are all the values of n for which the alternating sum of digits is 0?

(ii) Based on your answer to (i), make a conjecture about the value of d.

(iii) Test your conjecture on these numbers: 2009, 3124, 4567, 8481, 218702088. (That is, for each of those numbers, form the alternating sum and say what your conjecture implies about whether it is divisible by d or not. Then check your answer by actually dividing the number by d — it is OK to use a calculator for this step.)

Problem #10 Verify that Archimedes' trisection of the angle (described in the class notes for 2/18/09) is correct.

Problem #11 Here is a Euclidean construction that comes close to being a trisection, but doesn't quite work. (Many people have discovered this construction and thought that it worked. In fact, lots and lots of people throughout history have insisted wrongly that they were able to solve the classical trisection problem, by this or other Euclidean constructions.)

1. Let $\angle AOB$ be the angle to be trisected.
2. Construct circles of radii 2, 3, and 4 centered at O. Call them C_2, C_3 and C_4 respectively.
3. Bisect $\angle BOA$ with a line ℓ. Let C be the point where line ℓ meets C_2.
4. Bisect $\angle COB$ with a line m. Let D be the point where line m meets C_4.
5. Draw the segment $\overline{CD}$. Let T be the point where $\overline{CD}$ meets C_3.
6. Construct $\angle TOB$. (This is supposed to be the trisection of $\angle AOB$.)

Carry out this construction in Sketchpad. (You don't have to send me the sketch.)

a. By moving A around, tell me the value of $m\angle TOB$ for each of these values of $\angle AOB$: $30°$, $90°$, $144°$, $6°$, $180°$.

b. Tell me how the value of $\angle AOB$ (i.e., small, medium or large) affects the accuracy of the construction.

Problem #12 **(Extra credit)** Prove that $\sqrt{n}$ is constructible for every odd positive integer n.

Problem #13 The Babylonian method of approximating $\sqrt{r}$ was to start with an estimate s_0, then to compute successively more accurate approximations $s_1, s_2, s_3 \ldots$ using the formula

$$s_{n+1} = \frac{s_n + r/s_n}{2}.$$

a. Use the Babylonian method to approximate $\sqrt{3}$, using a starting estimate of $s_0 = 1$. How many iterations do you need for s_n to be accurate to 7 decimal places?

b. What happens if you start with a less accurate estimate, such as $s_0 = 5$?

c. What happens if you start with $s_0 = -1$? (The Babylonians probably wouldn't have tried this because, so far as we know, they didn't know about negative numbers.)

d. What happens if you try to approximate $\sqrt{-3}$ by using a starting estimate of $s_0 = -1$? (The Babylonians *certainly* wouldn't have tried this.)

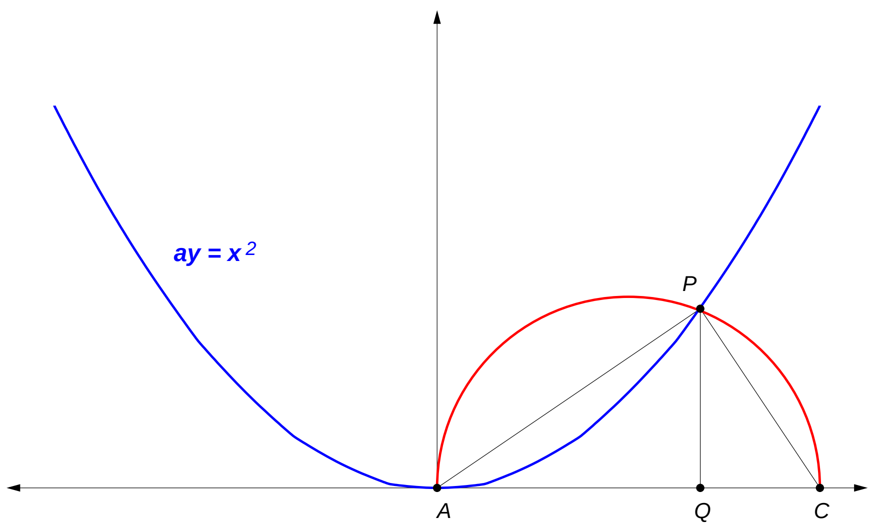
Problem #14 (Adapted from Burton, pp. 300–301) The Persian Omar Khayyam (c. 1050–1130 CE), best known as a poet, was also an outstanding mathematician. Several centuries before del Ferro (and/or Tartaglia and/or Cardano) solved the cubic equation algebraically, Khayyam came up with a geometric solution. His solution is non-Euclidean because it involves a parabola, but it's not hard to see that it works.

Khayyam considered the equation

$$z^3 + a^2 z = b$$

where a and b are positive real numbers. His solution is as follows (in modern coordinate notation):

1. Construct the parabola with equation $x^2 = ay$ (shown in blue below).
2. Construct a semicircle with diameter $AC = b/a^2$ on the x-axis (shown in red below).
3. Let P be the point where the parabola meets the semicircle. Drop a perpendicular from P to the x-axis to find the point Q.
4. Let z be the length of segment AQ.



Verify that Khayyam's construction of z is correct, by the following steps.

a. Prove that
$$z^2 = a \cdot PQ.$$

b. Prove that
$$\frac{z}{PQ} = \frac{PQ}{QC}.$$
(Hint: Use similar triangles and a theorem or two from Euclidean geometry.)

c. Use (b) to write $(PQ)^2$ in terms of a, b and z.

d. (Before going on, take a step back and remind yourself of what Khayyam was trying to do!) Combine the equations from parts (a) and (c) to complete your verification that Khayyam's construction is correct.

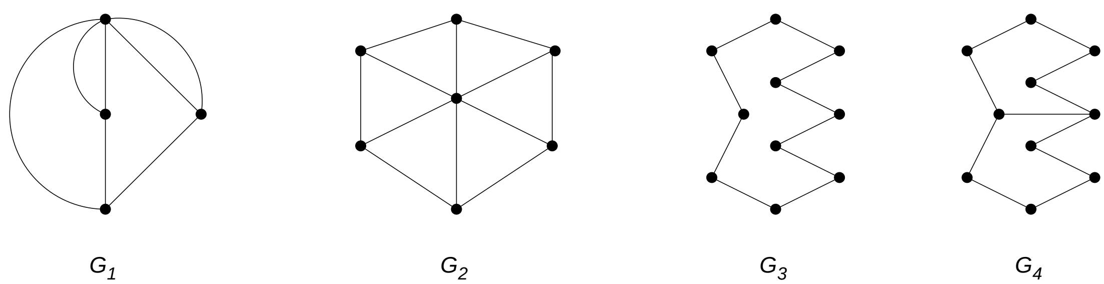
Problem #15 Recall the theorem about when a graph G has an Euler circuit or or Euler path:

- If G has no vertices of odd degree, then it has an Euler circuit.
- If G has two vertices of odd degree, then it has an Euler path but no Euler circuit.
- If G has four or more vertices of odd degree, then it has neither an Euler circuit nor an Euler path.

Obviously there are some missing cases — what if G has one or three vertices of odd degree? In this problem, you'll convince yourself that those cases can't actually happen.

(a) For each of the four graphs G_1, G_2, G_3, G_4 below:
(i) find the degree of each vertex;
(ii) add up the degrees of all the vertices;
(iii) count the edges.

(Suggestion: Keep track of the answers to (ii) and (iii) in a table.)



(b) What do you notice? Based on the numbers you've written down, make a conjecture about what happens in an arbitrary graph (and, for extra credit, explain why your conjecture is true).

(c) What does your conjecture imply about the number of odd-degree vertices in any graph?

Problem #16 Is it possible to draw a map with ten countries — let's call them Algeria, Australia, Belarus, Bulgaria, Congo, Cyprus, Namibia, Nepal, Pakistan, Portugal — so that each country is contiguous, and every two countries without the same first letter share a common border? (For example, Belarus borders Pakistan, but Congo does not border Cyprus.) Either draw such a map, or explain why it is impossible to do so.

Problem #17 Let $f(x) = \dfrac{\arctan x}{\pi} + \dfrac{1}{2}$ for $x \in \mathbb{R}$.

(a) What is the range of f? (Your answer should be a subset of $\mathbb{R}$.)

(b) Explain why f is a bijection between its domain and range. (You might have to use a little calculus.)

(c) What can you conclude about the cardinalities (i.e., sizes) of the domain and the range of f?

Problem #18 You are the manager of a hotel with infinitely many rooms. One night, a guest wants to check into the hotel, but unfortunately all the rooms are already full. How can you accommodate the new guest? (Hint: You may need to ask other guests to move, but of course you can't actually kick anyone out of the hotel.)

Problem #19 Let $I = [0, 1]$ (i.e., the set of real numbers x such that $0 \leq x \leq 1$). For each $x \in I$, let x_i be the i^{th} digit in the decimal expansion of x. For example, if $x = \pi/10 = 0.31415926\ldots$, then $x_1 = 3$, $x_2 = 1$, $x_3 = 4$, $x_4 = 1$, etc. So, in general, the decimal expansion of x is $x_1x_2x_3\ldots$.

(By the way, how do you express 1 in this way? Not a problem — in fact $1 = 0.99999999\ldots$.)

Now, for $x, y \in I$, let $f(x, y)$ be the number you get by "interweaving" the decimal expansions of x and y to get $f(x, y) = 0.x_1y_1x_2y_2x_3y_3\cdots$ For example, if

$$x = \pi/10 = 0.31415926\ldots \quad \text{and} \quad y = 8/9 = 0.88888888\ldots,$$

then

$$f(x, y) = 0.3818481858982868\ldots$$

(a) What is the range of f?

(b) Explain why the function f is one-to-one.

(c) Explain why the function f is onto.

(d) Using the conclusions of the previous parts, explain why there are as many points on a single side of a square as there are in the entire square.

References

[1] David M. Burton, *Burton's History of Mathematics: An Introduction*, 3rd edn., Wm. C. Brown Publishers, 1995.

[2] Underwood Dudley, *Mathematical Cranks*, Mathematical Association of America, 1992.

[3] John J. O'Connor and Edmund F. Robertson, The MacTutor History of Mathematics Archive. Homepage: http://www-groups.dcs.st-and.ac.uk/ history/index.html.

[4] John Stillwell, *Mathematics and its History*, Springer, New York, 1989.