Axioms, definitions and theorems for plane geometry

Math 409, Spring 2009

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The building blocks for a coherent mathematical system come in several kinds:

- **Undefined terms.** These are typically extremely simple and basic objects (like "point" and "line"), so simple that they resist being described in terms of simpler objects. Every system has to have some undefined terms—you've got to start somewhere. (But in general, the fewer the better.)
- **Postulates/Axioms**. These are basic facts about undefined terms. The simpler and more fundamental they are, the better. For example, "every pair of points determines a line", or "if $x = y$, then $y = x$".
- **Definitions**. We can define new terms using things that we already know.
- **Theorems**. These are the statements that makes mathematics what it is—they are facts that we prove using axioms, definitions, and theorems that we've proved earlier. (Propositions, Lemmas, Corollaries are all species of theorems.)

1 Undefined terms

We will start with the following undefined terms:[1]

- Point
- Line (= infinite straight line)
- Angle
- Distance between two points
- Measure of an angle

We'll write AB for the distance between two points A and B, and we'll write $m\angle ABC$ for the measure of angle $\angle ABC$.

We'll be careful to distinguish between an angle (which is a thing) and its measure (which is a number). So, for instance, the two statements "$\angle ABC = \angle XYZ$" and "$m\angle ABC = m\angle XYZ$" don't mean the same thing—the first says these two angles are actually the same angle, while the second just says that they have the same measure.

[1]Euclid included definitions of these terms in the *Elements*, but to a modern reader, his definitions are really intuitive explanations rather than precise mathematical definitions. For example, he defined a point as "that which has no part", a line as "length without breadth", and an angle as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line." (The intuition here is that an angle records the difference between the directions of two lines meeting in a point.)

2 Some definitions

When writing definitions, it is good practice to emphasize the word you are defining—that makes it easier on the reader (and for that matter the writer as well).

Definition 1. A collection of three or more points is collinear if there is some line containing all those points.

Definition 2. Two lines are parallel if they never meet.

Definition 3. When two lines meet in such a way that the adjacent angles are equal, the equal angles are called right angles, and the lines are called perpendicular to each other.

Definition 4. A circle is the set of all points equally distant from a given point. That point is called the center of the circle.

What about the term "line segment"? We all know what that is—it's the portion of a line between two points. But what does "between" mean?

With a little thought, we can define "between" using two concepts we already have: the undefined term distance and the definition of collinear. Once we've done that, we can define what a line segment is. It's important to get these two definitions in the proper order.

Definition 5. Given three distinct collinear points A, B, C, we say that B is between A and C if $AC > AB$ and $AC > BC$.

Definition 6. The line segment $\overline{AB}$ between two points A and B consists of A and B themselves, together with the set of all points between them.

3 Axioms

Axiom 1. *If A, B are distinct points, then there is exactly one line containing both A and B, which we denote $\overleftrightarrow{AB}$ (or $\overleftrightarrow{BA}$).*

This is Euclid's first axiom. Notice that it includes a definition of notation. Also, it's false in spherical geometry—if points A and B are polar opposites, then every one of the infinitely many great circles containing A contains B as well (and vice versa).

The next group of axioms concern distance.

Axiom 2. $AB = BA$.

Axiom 3. $AB = 0$ *iff* $A = B$.

The word "iff" is mathematician's jargon for "if and only if". That is, the axiom says that two different things are true. First, if $A = B$, then $AB = 0$. Second, if $AB = 0$, then $A = B$. Logically, these are two separate statements.

Axiom 4. *If point C is between points A and B, then $AC + BC = AB$.*

Axiom 5. *(The triangle inequality) If C is not between A and B, then $AC + BC > AB$.*

Now, some axioms about angle measure.

Axiom 6. ***New version.*** *(a.)* $m(\angle BAC) = 0°$ *iff* B, A, C *are collinear and* A *is not between* B *and* C.

(b.) $m(\angle BAC) = 180°$ *iff* B, A, C *are collinear and* A *is between* B *and* C.

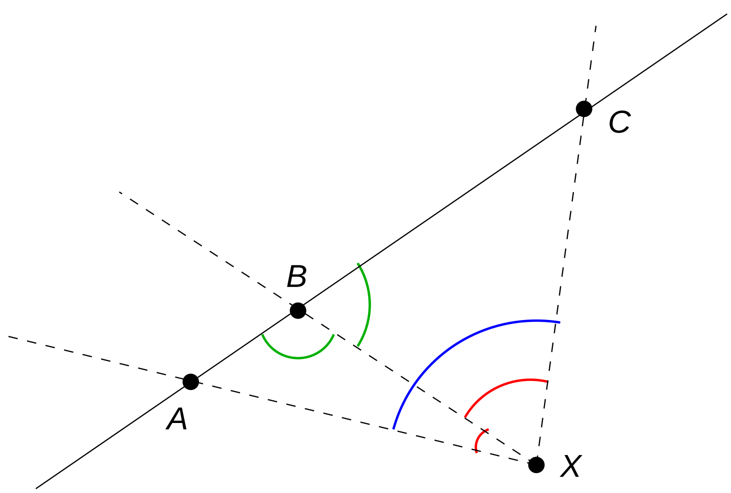
(Note: In an earlier version of these notes, I said "if" instead of "iff", but in fact we want both directions of both assertions as axioms.)

It might seem odd to start by talking about angles that "aren't really angles" (because they are defined by three collinear points). On the other hand, it's always a good idea in mathematics to look at extreme cases. These axioms make sense if you think about what happens if the points move around a little bit. To understand the first part of Axiom 6, imagine nudging B so that it is just off the segment $\overline{AC}$; then $m(\angle ABC)$ should be very close to 180°, and the less you've nudged B, the closer $m(\angle ABC)$ gets to 180°.

Axiom 7. *Whenever two lines meet to make four angles, the measures of those four angles add up to* 360°.

Axiom 8. *Suppose that* A, B, C *are collinear points, with* B *between* A *and* C, *and that* X *is not collinear with* A, B *and* C. *Then* $m(\angle AXB) + m(\angle BXC) = m(\angle AXC)$. *Moreover,* $m(\angle ABX) + m(\angle XBC) = m(\angle ABC)$. *(We know that* $\angle ABC = 180°$ *by Axiom 6.)*

Axiom 8 has a bit more going on than its predecessors, so here's a picture that illustrates it. The first statement says that the measures of the two red angles add up to the measure of the blue angle. The second statement says that the two green angles add up to 180°.



An axiom about logic:

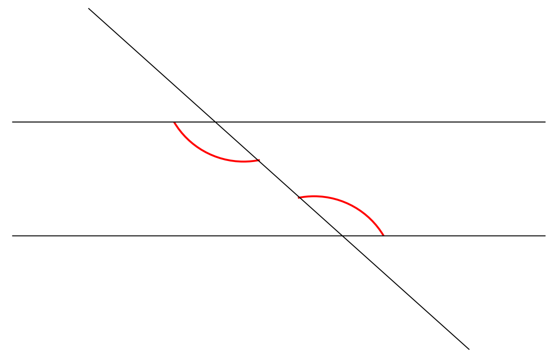
Axiom 9. *Equals can be substituted for equals.*

Two axioms about parallel lines:

Axiom 10. *Given a point* P *and a line* ℓ, *there is exactly one line through* P *parallel to* ℓ.

Axiom 11. *If* ℓ *and* ℓ' *are parallel lines and* m *is a line that meets them both, then alternate interior angles are equal.*

That is, Axiom 11 says that the two red angles are equal in the following picture (provided that the horizontal lines are parallel).



Axiom 10 (which is called Playfair's Axiom) and Axiom 11 distinguish Euclidean geometry from other geometries, such as spherical geometry (which we've talked a little about) and hyperbolic geometry (which we'll see eventually). Both axioms are intuitively correct, but it took mathematicians a long time to realize that it was possible to do geometry without them.

Now for two axioms that connect number and geometry:

Axiom 12. *For any positive whole number n, and distinct points A, B, there is some C between A, B such that $n \cdot AC = AB$.*

Axiom 13. *For any positive whole number n and angle $\angle ABC$, there is a point D between A and C such that $n \cdot m(\angle ABD) = m(\angle ABC)$.*

4 Some theorems

Now that we have a bunch of axioms and definitions in place, we can start using them to prove theorems. (We reserve the right to add more axioms and definitions later if we need to.) Many of these theorems may seem obvious, but that's the point: even seemingly obvious statements need to be proved.

The first theorem was actually one of Euclid's original five postulates (= axioms). In our axiom system, which is not the same as Euclid's, we don't need to make it an axiom—we can prove it from the axioms and definitions above.

Theorem 1. *All right angles have the same measure, namely* $90°$.

Proof. Suppose that $\angle ABX$ is a right angle. By Definition 3, this means that the segments $\overline{AB}$ and $\overline{BX}$ can be extended to perpendicular lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BX}$. Let C be a point on $\overleftrightarrow{AB}$ such that B is between A and C. Now, by Definition 3 of "perpendicular", we know that

$$m\angle ABX = m\angle XBC$$

and by Axiom 8, we know that

$$m\angle ABX + m\angle XBC = 180°.$$

Substituting the first equation into the second, we find that $2m\angle ABC = 180°$, so $m\angle ABC = 90°$. □

Notice that this proof says explicitly when it using an axiom or definition. This is an important habit to acquire when writing proofs—it's just like citing your sources.

By the way, the little box at the end is a sign that the proof is complete. (The old practice was to use the abbreviation "Q.E.D.", a Latin acronym meaning "...which was to be proven".)

The next several theorems say that certain things are unique. every line segment has exactly one midpoint, every angle has exactly one bisector, and every line has exactly one perpendicular through a point on it.

Definition 7. A midpoint of a line segment $\overline{AB}$ is a point C on $\overline{AB}$ such that $AC = BC$ and $2 \cdot AC = AB$.

Theorem 2. *Every line segment $\overline{AB}$ has exactly one midpoint.*

Proof. First, we show that $\overline{AB}$ has at least one midpoint. By Axiom 12, we can find a point C between A and B (that is, on $\overline{AB}$) such that

$$2 \cdot AC = AB. \tag{1}$$

So we just need to show that $AC = BC$. By Axiom 4, we also know that $AC + BC = AB$ so substituting for AB in equation (1) (which we can do by Axiom 9) gives us $2 \cdot AC = AC + BC$, and subtracting AC from both sides gives $AC = BC$,

The second part of the proof is to show that $\overline{AB}$ has only one midpoint. To do this, suppose that C and D are both midpoints—the goal is then to show that in fact $C = D$. By Axiom 3, we can do this by showing that $CD = 0$. We don't know anything about CD directly; what we do know is that since C and D are both midpoints of $\overline{AB}$.

$$AC = CB = AD = DB = \frac{AB}{2}. \tag{2}$$

Now C is either between A and D, or between B and D If C is between A and D, then Axiom 4 says that $AC + CD = AD$. But since $AC = AD$ (by equation (2)), this means that $CD = 0$.

If C is between B and D, then Axiom 4 says that $BC + CD = BD$. But since $BC = BD$ (by equation (2)), this means that $CD = 0$.

In either case, Axiom 3 tells us that $C = D$. □

Notice that the two cases were essentially the same, In a written proof, you might the author say something like, "Without loss of generality, we'll just consider the first case"—that's what this means.

Definition 8. A bisector of an angle $\angle BAC$ is a line $\overleftrightarrow{AD}$ such that D is between B and C and $m\angle BAD = m\angle DAC = \frac{1}{2} m\angle BAC$.

Theorem 3. *Every angle $\angle BAC$ has exactly one bisector.*

Proof. By Axiom 13, $\angle BAC$ has at least one bisector. We have to show that it has only one.

Suppose that $\overleftrightarrow{AD}$ and $\overleftrightarrow{AE}$ are both bisectors. Then D and E are points on $\overline{BC}$. So D is either between B and E, or between E and C. Without loss of generality, we'll consider the first case. By Axiom 8,

$$m\angle BAD + m\angle DAE = m\angle BAE.$$

On the other hand, since $\overleftrightarrow{AD}$ and $\overleftrightarrow{AE}$ are both bisectors, we know from Definition 8 that

$$m\angle BAD = m\angle BAE = \frac{1}{2} m\angle BAC.$$

Substituting this into the previous equation gives $m\angle BAD + m\angle DAE = m\angle BAD$, which implies that $m\angle DAE = 0$. By Axiom 6, the points A, D, E are collinear — but that means that $\overleftrightarrow{AD} = \overleftrightarrow{AE}$, so both bisectors are the same. □

As an immediate corollary, we get the following uniqueness result:

Theorem 4. *If C is between A and B, then there is exactly one line ℓ passing through C that is perpendicular to $\overline{AB}$.*

Proof. Suppose that ℓ is such a line. Then ℓ is a bisector of $\angle ACB$. By Theorem 3, there is exactly one such line. ☐

Before we start looking at congruence and similarity, we need to establish a few more theorems.

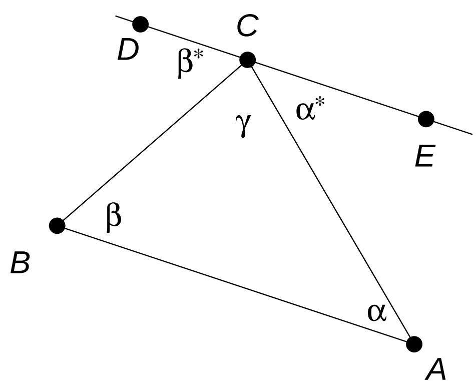
Theorem 5. *Any two distinct lines intersect in at most one point.*

Proof. Let m and n be lines that have two different points P, Q in common. By Axiom 1, there is exactly one line containing P and Q. Both m and n must be that line. Therefore, $m = n$. ☐

Just like Axiom 1, this is a statement that seems utterly obvious, but fails in spherical geometry, where every pair of distinct lines meets in two (polar opposite) points.

Theorem 6. *The sum of the interior angles of any triangle is* $180°$. *That is, if* ΔABC *is any triangle, then* $m\angle ABC + m\angle BAC + m\angle ACB = 180°$.

Proof. Draw a line ℓ through C parallel to $\overleftrightarrow{AB}$. (By Axiom 10, there is exactly one such line.) Put points D and E on ℓ so that C is between D and E.



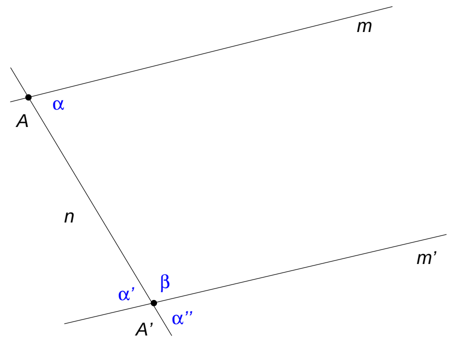
By Axiom 8 and Axiom 6,

$$\alpha^* + \gamma + \beta^* \;=\; (\alpha^* + \gamma) + \beta^* \;=\; m\angle ECB + m\angle BCD \;=\; m\angle ECD = 180°.$$

By Axiom 11, $\alpha = \alpha^*$ and $\beta = \beta^*$. Substituting into the last equation gives $\alpha + \gamma + \beta = 180°$. ☐

Theorem 7. *Suppose that two distinct lines m, m' both intersect a third line n. If alternate interior angles are equal, or if corresponding angles are equal then m and m' are parallel.*

Proof. Here's the picture:



Let's prove the "alternate interior angles" case. We are given that $\alpha = \alpha'$, and we want to prove that m and m' are parallel.

Suppose that m and m' meet at a point Z. Then we have a triangle $\Delta AZZ'$. By Theorem 6,

$$m\angle AZZ' + m\angle AZ'Z + m\angle ZAZ' = \alpha + \beta + m\angle ZAZ' = 180°.$$

On the other hand, $\alpha + \beta = 180°$ by Axiom 8. Therefore, $m\angle ZAZ' = 0$, which says that Z, A, Z' are collinear—but they're not. This is a contradiction, and we conclude that m and m' do not meet.

As for the "corresponding angles" case, alternate interior angles (such as α and α') are equal if and only if corresponding angles (such as α and α'') are equal. This is because of the Vertical Angle Theorem (which says that $\alpha' = \alpha''$). □

5 Congruence and similarity

Definition 9. Two things are <u>congruent</u> iff one of them can be moved rigidly so that it coincides with the other. In particular, if one of them consists of line segments then so does the other, and corresponding sides have the same measure. We write $\mathcal{F} \cong \mathcal{G}$ to mean that $\mathcal{F}$ and $\mathcal{G}$ are congruent.

Definition 10. Two things are <u>similar</u> iff one of them is proportional to the other. In particular, if one of them consists of line segments then so does the other, and corresponding sides have proportional measures. We write $\mathcal{F} \sim \mathcal{G}$ to mean that $\mathcal{F}$ and $\mathcal{G}$ are similar.

Notice that "congruent" is a stronger relationship than "similar". If two things are congruent, then they are necessarily similar, but two similar things don't have to be congruent.

Axiom 14. *(SSS) Two triangles are congruent iff their corresponding sides are equal. That is, if ΔABC and $\Delta A'B'C'$ are two triangles such that $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$, then $\Delta ABC \cong \Delta A'B'C'$.*

We already know from Definition 9 that <u>if</u> the triangles are congruent, <u>then</u> corresponding sides are equal. What is new in Axiom 14 is the reverse implication: <u>if</u> corresponding sides are equal, <u>then</u> the triangles are congruent.

Axiom 15. *(AAA) Two triangles are similar iff their corresponding angles are equal. That is, if $m\angle BAC = m\angle B'A'C'$, $m\angle ABC = m\angle A'B'C'$, and $m\angle BCA = m\angle B'C'A'$, then $\Delta ABC \sim \Delta A'B'C'$.*

The abbreviations SSS and AAA are short for "Side-Side-Side" and "Angle-Angle-Angle". It is natural to ask about other criteria for congruence of triangles.

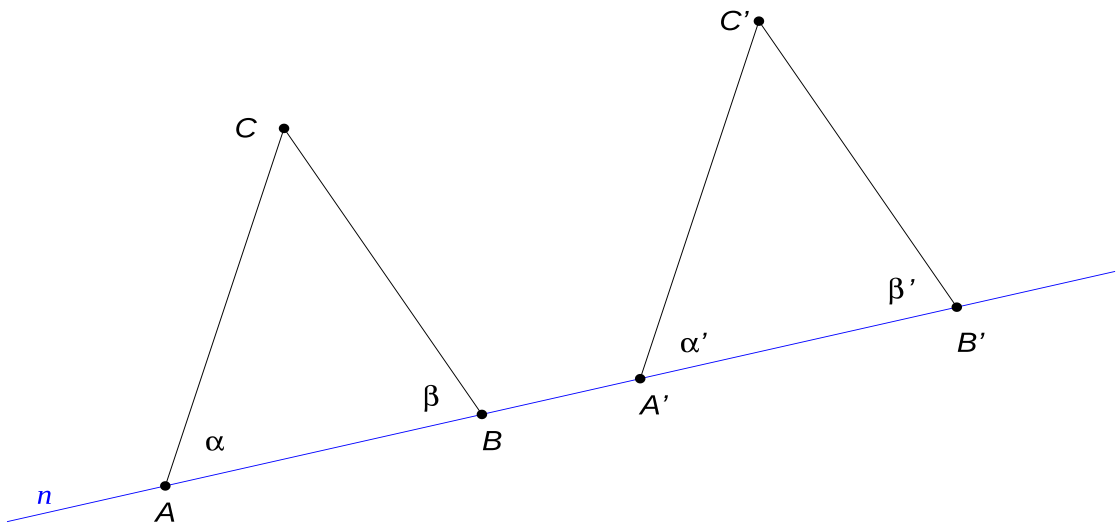
Theorem 8. *(ASA) Two triangles are congruent iff two pairs of corresponding angles, and the sides between them, are equal. That is,*

$$\Delta ABC \cong \Delta A'B'C' \quad \textit{iff} \quad m\angle BAC = m\angle B'A'C',\ m\angle ABC = m\angle A'B'C',\ \textit{and } AB = A'B'.$$

Proof. To prove a theorem with an "iff" in its statement, we need to prove that both implications hold. In this case, it is easy to prove that if the triangles are congruent, then the three equalities actually hold — this is an immediate consequence of Definition 9.

So suppose we have two triangles that satisfy the three equalities. Draw then so that the points A, B, A', B' lie along the same line n, in that order. Also, let

$$\alpha = m\angle BAC, \qquad \beta = m\angle ABC$$
$$\alpha' = m\angle B'A'C', \qquad \beta' = m\angle A'B'C'.$$



By Theorem 7, we know that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel (because they both intersect n, and the corresponding angles α, α' are equal). By similar logic, we know that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel.

Now, slide $\Delta A'B'C'$ along n so that the segments $\overline{A'B'}$ and $\overline{AB}$ coincide with each other, i.e., $A = A'$ and $B = B'$. (We know we can do this because $AB = A'B'$ by hypothesis.) Theorem 7 still applies, but here it says that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are actually the same line (they can't be parallel because they both contain the point $A = A'$, so according to theorem the only other possibility is that they weren't distinct lines to begin with). Similarly, $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ coincide.

Let $\ell = \overleftrightarrow{AC} = \overleftrightarrow{A'C'}$ and $m = \overleftrightarrow{BC} = \overleftrightarrow{B'C'}$. The lines ℓ and m intersect in a unique point, by Theorem 5. But that unique point must be both C and C' — so we conclude that $C = C'$, and the proof is done. □

An equivalent way of stating the ASA theorem is as follows:

A triangle is determined, up to congruence, by one side and the two angles adjacent to it.

In other words, if you know the length of $\overline{AB}$ and the angles α and β in the figure above, then there's only one possible point where C can be, hence only one possible triangle ΔABC.

Here's another congruence theorem.

Theorem 9. *(SAS) Two triangles are congruent iff two pairs of corresponding sides, and the angles between those sides, are equal. That is,*

$$\Delta ABC \cong \Delta A'B'C' \quad \textit{iff} \quad AB = A'B',\ AC = A'C',\ \textit{and}\ m\angle BAC = m\angle B'A'C'.$$

Proof. Again, if the triangles are congruent, then the three equalities do indeed hold by Definition 9.

As in the proof of Theorem 8 before, we draw the triangles so that A, B, A', B' are collinear, and we can conclude from Theorem 7 that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel (but not $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$, since we don't know whether or not β equals β'). Once again, slide $\Delta A'B'C'$ along n so that $A = A'$ and $B = B'$, so that now $\overleftrightarrow{AC} = \overleftrightarrow{A'C'}$.

Now, either C' is between A and C or C is between A and C'. Without loss of generality, suppose the first case. Then Axiom 4 says that $AC' + C'C = AC$. Also, A and A' are the same point, so $AC' = A'C'$, and substituting into the previous equation, we get $A'C' + C'C = AC$. But $A'C' = AC$ by hypothesis, so $C'C = 0$. Therefore $C = C'$ by Axiom 3. □

An equivalent way of stating the SAS theorem is as follows:

A triangle is determined, up to congruence, by two sides and the angle between them.

It is important that the angle has to be between the sides (that's why we call it SAS and not SSA). Specifying two sides and an angle opposite one of the sides does **not** determine the triangle up to congruence — see problem SA 20.

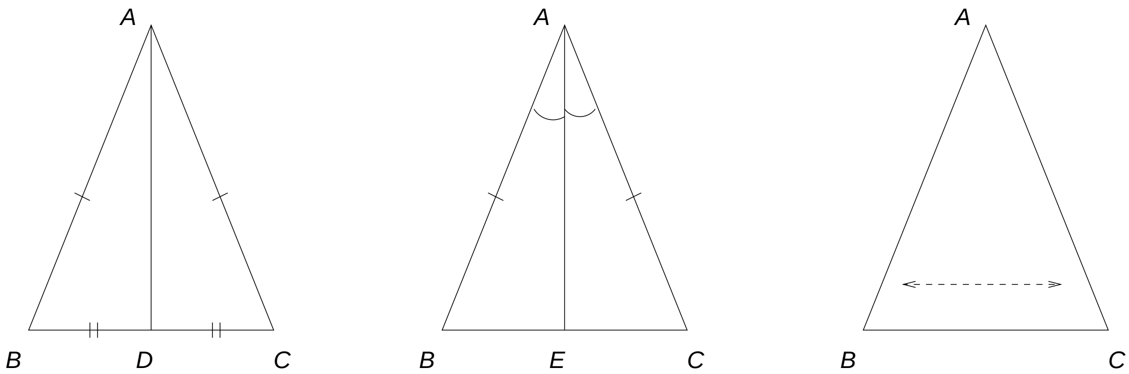
Here are some important consequence of the angle congruence theorems.

Theorem 10. *The base angles of an iosceles triangle are equal. That is, if $AB = AC$ then $\angle ABC \cong \angle ACB$.*

First proof. Let D be the midpoint of $\overline{BC}$. Then $AD = AD$; $AB = AC$ (given); and $BD = CD$ (by definition of midpoint). So $\Delta ABD \cong \Delta ACD$ by SSS. By definition of congruence, $\angle ABC \cong \angle ACB$. □

Second proof. Let ℓ be the bisector of angle $\angle BAC$, and let E be the point where ℓ meets $\overline{BC}$. Then $AE = AE$; $AB = AC$ (given); and $\angle BAE \cong \angle CAE$ (by definition of angle bisector So $\Delta ABD \cong \Delta ACD$ by SAS. Again, by definition of congruence, $\angle ABC \cong \angle ACB$. □

Third (and slickest) proof. Observe that $AB = AC$, $AC = AB$, and $BC = CB$. Therefore $\Delta ABC \cong \Delta BAC$ by SSS. In particular, $\angle ABC \cong \angle ACB$. (The big idea here is that the triangle is congruent to a reflected copy of itself.) □



Proof 1 Proof 2 Proof 3

The points D and E are actually the same point, but we can't assume that from the start — so it is a logical mistake to say something like, "Let ℓ be the bisector of $\angle BAC$, and let ℓ meet $\overline{BC}$ at its midpoint". To put it another way, you can't assume that you can construct ℓ in a way that meets both those specifications. On the other hand, we can now prove that D and E coincide.

Finally, two more very useful theorems about triangles inscribed in a semicircle.

Theorem 11. *Suppose that $\overline{AB}$ is a diamater of a circle centered at O, and that C is a point on the circle. Then*

$$m\angle ACB = 90^\circ \tag{3a}$$

and

$$m\angle BOC = 2m\angle BAC. \tag{3b}$$

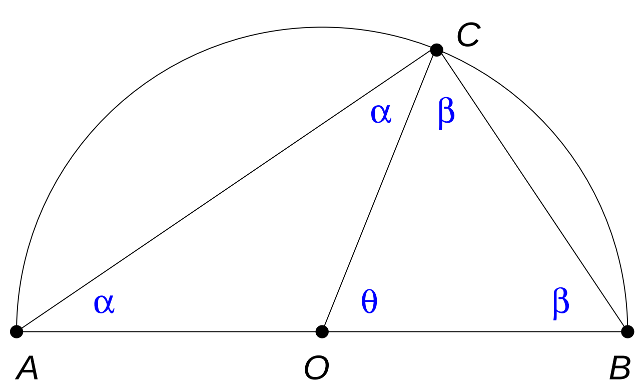
Proof. The segments $\overline{OA}, \overline{OB}, \overline{OC}$ are all radii of the circle, so the triangles ΔOAC and ΔOBC are isosceles. Therefore, by Theorem 10,

$$m\angle OAC = m\angle OCA \quad \text{and} \quad m\angle OCB = m\angle OBC.$$

Let $\alpha = m\angle OAC = m\angle OCA$ and $\beta = m\angle OCB = m\angle OBC$. By Axiom 8 and Theorem 6,

$$2\alpha + 2\beta \;=\; m\angle OAC + m\angle OCA + m\angle OCB + m\angle OBC \;=\; m\angle BAC + m\angle ACB + m\angle ABC \;=\; 180^\circ$$

from which it follows that $\alpha + \beta = 90^\circ$, proving (3a).



Now, let $\theta = m\angle BOC$. Then

$$\theta + 2\beta = 180°$$

and we already know that

$$2\alpha + 2\beta = 180°$$

and combining these two equations yields

$$\theta = 2\alpha$$

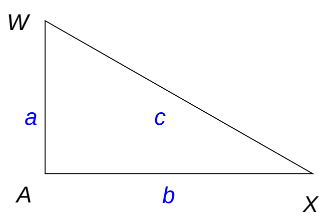
which proves (3b). ☐

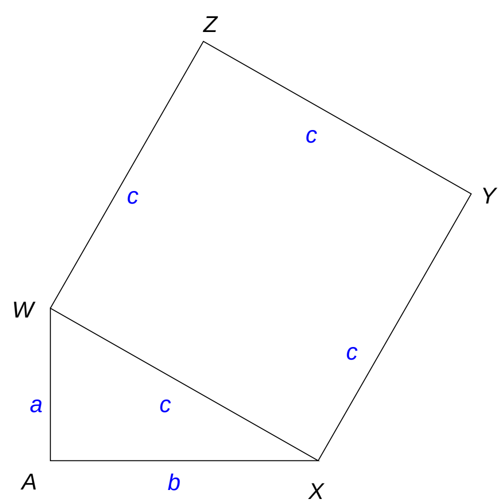
6 The Pythagorean Theorem

Theorem 12. *In a right triangle with legs of lengths a and b and hypotenuse of length c, we have*

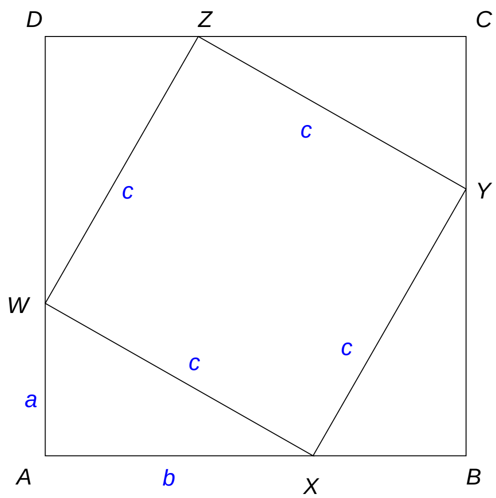
$$a^2 + b^2 = c^2.$$

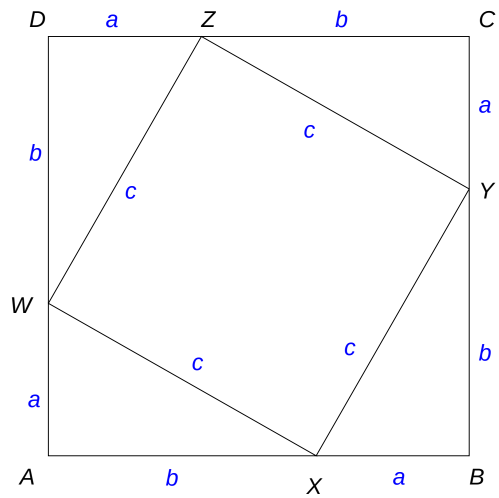
Proof. Suppose we are given such a right triangle ΔAWX, where $\overline{WX}$ is the hypotenuse (*see figure, left, below*). Construct a square $WXYZ$ using the hypotenuse as one side (*see figure, right, below*).





Extend $\overline{AW}$ and $\overline{AX}$ to segments $\overline{AD}$ and $\overline{AB}$ respectively so that $\angle ABY$ and $\angle ADZ$ are right angles, and then extend $\overline{BY}$ and $\overline{DZ}$ until they meet at C. (*See figure, left, below.*)





By Theorem 6,

$$m\angle AWX + m\angle AXW + m\angle WAX = 180^\circ, \tag{4a}$$

and $\angle WAX$ is a right angle, so

$$m\angle AWX + m\angle AXW = 90^\circ. \tag{4b}$$

On the other hand, by Axiom 8

$$m\angle AXW + m\angle WXY + m\angle BXY = 180^\circ \tag{4c}$$

and $\angle WXY$ is a right angle, so

$$m\angle AXW + m\angle BXY = 90^\circ. \tag{4d}$$

Comparing (4b) and (4d) tells us that $m\angle AWX = m\angle BXY$. Repeating these arguments, we find that

$$\angle AWX \cong \angle BXY \cong \angle CYZ \cong \angle DZW \quad \text{and}$$
$$\angle AXW \cong \angle BYX \cong \angle CZY \cong \angle DWZ.$$

Also, $WX = XY = YZ = ZW$ by construction, so by ASA,

$$\Delta AWX \cong \Delta BXY \cong \Delta CYZ \cong \Delta DZW$$

and so

$$AW = BX = CY = DZ = a, \qquad AX = BY = CZ = DW = b.$$

(*See figure, right, above.*)

Now, we calculate the area of the square $WXYZ$ two ways. On the one hand,

$$\text{area}(WXYZ) = c^2. \tag{5a}$$

On the other hand,

$$\begin{aligned} \text{area}(WXYZ) &= \text{area}(ABCD) - \text{area}(\Delta AWX) - \text{area}(\Delta BXY) - \text{area}(\Delta CYZ) - \text{area}(\Delta DZW) \\ &= (a+b)^2 - 4(ab/2) \\ &= (a^2 + 2ab + b^2) - 2ab \;=\; a^2 + b^2. \end{aligned} \tag{5b}$$

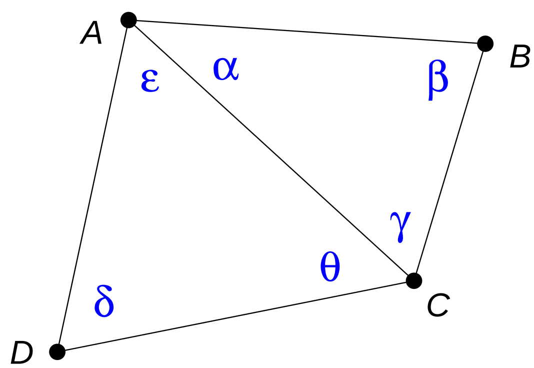
Now, equating (5a) and (5b) gives the Pythagorean Theorem. □

7 Definitions and theorems about quadrilaterals

Definition 11. A quadrilateral $Q = ABCD$ consists of four points A, B, C, D and the line segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$. The diagonals of Q are the line segments $\overline{AC}$ and $\overline{BD}$. The quadrilateral is called convex if the diagonals cross each other, but $\overline{AB}$ does not meet $\overline{CD}$ and $\overline{BC}$ does not meet $\overline{DA}$. All quadrilaterals we'll consider will be convex.

Theorem 13. *[EG 23] The angles of every quadrilateral add up to* $360°$.

Proof. Draw the quadrilateral $ABCD$ and the diagonal $\overline{AC}$. Label the angles as shown.



Then,

$$\begin{aligned} m\angle ABC + m\angle BCD + m\angle CDA + m\angle DAB &= \beta + (\gamma + \theta) + \delta + (\varepsilon + \alpha) && \text{(by Axiom 8)} \\ &= (\alpha + \beta + \gamma) + (\theta + \delta + \varepsilon) \\ &= 180° + 180° = 360° && \text{(by Theorem 6).} \end{aligned}$$ □

Definition 12. A quadrilateral $Q = ABCD$ is a parallelogram if $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ and $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{AD}$. It is a rectangle if $\angle ABC, \angle BCD, \angle CDA, \angle DAB$ are all right angles. It is a rhombus if $AB = BC = CD = DA$. It is a square if it is both a rectangle and a rhombus.

The next several theorems are about parallelograms.

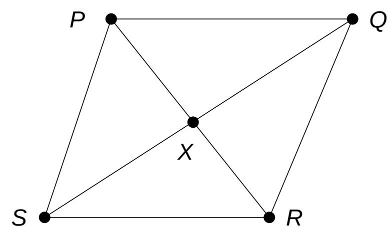
Theorem 14. *[EG 27] In a parallelogram $PQRS$, opposite sides and opposite angles are equal. That is, if $\overleftrightarrow{PQ}$ is parallel to $\overleftrightarrow{RS}$ and $\overleftrightarrow{PS}$ is parallel to $\overleftrightarrow{QR}$, then*

$$PQ = RS \quad \text{and} \quad PS = RQ \tag{6a}$$

and

$$\angle PQR \cong \angle RSP \quad \text{and} \quad \angle QRS \cong \angle SPQ. \tag{6b}$$

Proof. Draw the diagonal $\overline{PR}$. By Axiom 11, $\angle SRP \cong \angle RPQ$ and $\angle SPR \cong \angle QRP$. Also, $PR = RP$ (Axiom 2), so by ASA (Theorem 8), we have congruent triangles: $\Delta PRS \cong \Delta RPQ$. In particular, $RS = PQ$ and $PS = RQ = QR$, proving (6a). Also, $\angle RSP \cong \angle PQR$, which is one of the assertions of (6b), and we can obtain the other equality by constructing the diagonal $\overline{QS}$ and arguing similarly. □



The next theorem is a well-known fact about parallelograms, but does take a little effort to prove. Once we know it, it will be very useful in the following two results about special kinds of parallelograms (that is, rhombi and rectangles).

Theorem 15. *The diagonals of every parallelogram bisect each other. That is, if $PQRS$ is any parallelogram, and $X = \overline{PR} \cap \overline{QS}$ is the point where its diagonals meet, then $PX = RX$ and $QX = SX$.*

Proof. By Axiom 11 and the fact that P, X, R are collinear, we have

$$\underline{\angle XRS} = \angle PRS \cong \angle RPQ = \underline{\angle XPQ}$$

and similarly by Axiom 11 and the fact that Q, X, S are collinear,

$$\underline{\angle XSR} = \angle QSR \cong \angle SQP = \underline{\angle XQP}.$$

Moreover, Theorem 14 tells us that $PQ = SR$. Together with the underlined angle equalities and ASA, we conclude that

$$\Delta XSR \cong \Delta XQP \tag{7}$$

from which it follows that $PX = RX$ and $QX = SX$. □

Theorem 16 (EG 28)**.** *The diagonals of parallelogram $PQRS$ meet at a right angle if and only if the parallelogram is a rhombus.*

Proof. **Part I:** Suppose that the diagonals $\overline{PR}$, $\overline{QS}$ meet at a right angle. Then

$$PX = PX, \quad QX = SX, \quad \text{and} \quad \angle PXS \cong \angle PXQ,$$

the second equality by Theorem 15 and the third by Theorem 1. So $\Delta PXS \cong \Delta PXQ$ by SAS, and in particular $PS = PQ$. By the same argument, $PQ = QR = RS = SP$. That is, the parallelogram is a rhombus.

Part II: Suppose that the parallelogram is a rhombus. Then the triangles

$$\Delta PXQ, \quad \Delta RXQ, \quad \Delta RXS, \quad \Delta PXS \tag{8}$$

are mutually congruent by SSS. In particular,

$$\angle PXQ \;\cong\; \angle RXQ \;\cong\; \angle RXS \;\cong\; \angle PXS.$$

But these four angles add up to 360° by Axiom 7, so each of the four must equal 90°. □

Theorem 17 (EG 29)**.** *The diagonals of a parallelogram are congruent to each other if and only if the parallelogram is a rectangle.*

Proof. **Part I:** Suppose that $PR = QS$. Then $PX = QX = RX = SX$ by Theorem 15. So each of the triangles

$$\Delta PXQ, \quad \Delta RXQ, \quad \Delta RXS, \quad \Delta PXS$$

is isosceles, so

$$\angle XPQ \cong \angle XQP, \quad \angle XRQ \cong \angle XQR, \quad \angle XRS \cong \angle XSR, \quad \angle XPS \cong \angle XSP.$$

The argument of Theorem 15 says that $\Delta XSR \cong \Delta XQP$ (see (7)) and likewise $\Delta XPS \cong \Delta XRQ$, so $\angle XRS \cong \angle XPQ$ and $\angle XPS \cong \angle XRQ$. Combining with the previous equalities, we know that

$$\alpha = m\angle XPQ = m\angle XQP = m\angle XRS = m\angle XSR,$$
$$\beta = m\angle XRQ = m\angle XQR = m\angle XPS = m\angle XSP.$$

On the other hand, adding up the eight angles just listed gives $360°$ by Theorem 13. Therefore $\alpha + \beta = 90°$, and each angle of the quadrilateral is $\alpha + \beta$ (for example, $m\angle PQR = m\angle PQX + m\angle XQR = \alpha + \beta$) by Theorem 8. Therefore, the parallelogram is a rectangle.

Part II: Suppose that $PQRS$ is a rectangle. We could prove that $PR = QS$ by methods similar to the previous results, but there's a much easier way: apply the Pythagorean Theorem, which says that

$$QS = \sqrt{(PQ)^2 + (PS)^2} \quad \text{and} \quad PR = \sqrt{(RS)^2 + (PS)^2}.$$

On the other hand, $PQ = RS$ by Theorem 15, so $QS = PR$. □