Axioms, definitions and theorems for plane geometry

Math 409, Spring 2009

March 9, 2009

1 Definitions

Definition 1. A collection of three or more points is collinear if there is some line containing all those points.

Definition 2. Two lines are parallel if they never meet.

Definition 3. When two lines meet in such a way that the adjacent angles are equal, the equal angles are called right angles, and the lines are called perpendicular to each other.

Definition 4. A circle is the set of all points equally distant from a given point. That point is called the center of the circle.

Definition 5. Given three distinct collinear points A, B, C, we say that B is between A and C if $AC > AB$ and $AC > BC$.

Definition 6. The line segment $\overline{AB}$ between two points A and B consists of A and B themselves, together with the set of all points between them.

Definition 7. A midpoint of a line segment $\overline{AB}$ is a point C on $\overline{AB}$ such that $AC = BC$ and $2 \cdot AC = AB$.

Definition 8. A bisector of an angle $\angle BAC$ is a line $\overleftrightarrow{AD}$ such that D is between B and C and $m\angle BAD = m\angle DAC = \frac{1}{2} m\angle BAC$.

Definition 9. Two things are congruent iff one of them can be moved rigidly so that it coincides with the other. In particular, if one of them consists of line segments then so does the other, and corresponding sides have the same measure. We write $\mathcal{F} \cong \mathcal{G}$ to mean that $\mathcal{F}$ and $\mathcal{G}$ are congruent.

Definition 10. Two things are similar iff one of them is proportional to the other. In particular, if one of them consists of line segments then so does the other, and corresponding sides have proportional measures. We write $\mathcal{F} \sim \mathcal{G}$ to mean that $\mathcal{F}$ and $\mathcal{G}$ are similar.

Definition 11. A quadrilateral $Q = ABCD$ consists of four points A, B, C, D and the line segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$. The diagonals of Q are the line segments $\overline{AC}$ and $\overline{BD}$. The quadrilateral is called convex if the diagonals cross each other, but $\overline{AB}$ does not meet $\overline{CD}$ and $\overline{BC}$ does not meet $\overline{DA}$. All quadrilaterals we'll consider will be convex.

Definition 12. A quadrilateral $Q = ABCD$ is a parallelogram if $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ and $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{AD}$. It is a rectangle if $\angle ABC, \angle BCD, \angle CDA, \angle DAB$ are all right angles. It is a rhombus if $AB = BC = CD = DA$. It is a square if it is both a rectangle and a rhombus.

2 Axioms

Axiom 1. *If A, B are distinct points, then there is exactly one line containing both A and B, which we denote $\overleftrightarrow{AB}$ (or $\overleftrightarrow{BA}$).*

Axiom 2. *If A and B are any two points, then $AB = BA$.*

Axiom 3. $AB = 0$ *iff* $A = B$.

Axiom 4. *If point C is between points A and B, then $AC + BC = AB$.*

Axiom 5. *(Triangle inequality) If C is not between A and B, then $AC + BC > AB$.*

Axiom 6. *(a.) $m(\angle BAC) = 0°$ iff B, A, C are collinear and A is not between B and C.*

(b.) $m(\angle BAC) = 180°$ iff B, A, C are collinear and A is between B and C.

Axiom 7. *Whenever two lines meet to make four angles, the measures of those four angles add up to $360°$.*

Axiom 8. *Suppose that A, B, C are collinear points, with B between A and C, and that X is not collinear with A, B and C. Then $m(\angle AXB) + m(\angle BXC) = m(\angle AXC)$. Moreover, $m(\angle ABX) + m(\angle XBC) = m(\angle ABC)$. (We know that $\angle ABC = 180°$ by Axiom 6.)*

Axiom 9. *Equals can be substituted for equals.*

Axiom 10. *Given a point P and a line ℓ, there is exactly one line through P parallel to ℓ.*

Axiom 11. *If ℓ and ℓ' are parallel lines and m is a line that meets them both, then alternate interior angles are equal.*

Axiom 12. *For any positive whole number n, and distinct points A, B, there is some C between A, B such that $n \cdot AC = AB$.*

Axiom 13. *For any positive whole number n and angle $\angle ABC$, there is a point D between A and C such that $n \cdot m(\angle ABD) = m(\angle ABC)$.*

Axiom 14. *(SSS) Two triangles are congruent iff their corresponding sides are equal. That is, if ΔABC and $\Delta A'B'C'$ are two triangles such that $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$, then $\Delta ABC \cong \Delta A'B'C'$.*

Axiom 15. *(AAA) Two triangles are similar iff their corresponding angles are equal. That is, if $m\angle BAC = m\angle B'A'C'$, $m\angle ABC = m\angle A'B'C'$, and $m\angle BCA = m\angle B'C'A'$, then $\Delta ABC \sim \Delta A'B'C'$.*

3 Theorems

Theorem 1. *All right angles have the same measure, namely $90°$.*

Theorem 2. *Every line segment $\overline{AB}$ has exactly one midpoint.*

Theorem 3. *Every angle $\angle BAC$ has exactly one bisector.*

Theorem 4. *If C is between A and B, then there is exactly one line ℓ passing through C that is perpendicular to $\overline{AB}$.*

Theorem 5. *Any two distinct lines intersect in at most one point.*

Theorem 6. *The sum of the interior angles of any triangle is $180°$. That is, if ΔABC is any triangle, then $m\angle ABC + m\angle BAC + m\angle ACB = 180°$.*

Theorem 7. *Suppose that two distinct lines m, m' both intersect a third line n. If alternate interior angles are equal, or if corresponding angles are equal, then m and m' are parallel.*

Theorem 8. *(ASA) Two triangles are congruent iff two pairs of corresponding angles, and the sides between them, are equal. That is,*

$$\Delta ABC \cong \Delta A'B'C' \quad \text{iff} \quad m\angle BAC = m\angle B'A'C', \ m\angle ABC = m\angle A'B'C', \ \text{and } AB = A'B'.$$

Equivalently, a triangle is determined, up to congruence, by one side and the two angles adjacent to it.

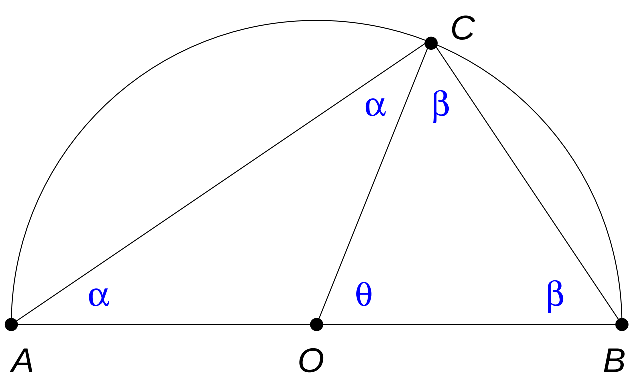
Theorem 9. *(SAS) Two triangles are congruent iff two pairs of corresponding sides, and the angles between those sides, are equal. That is,*

$$\Delta ABC \cong \Delta A'B'C' \quad \text{iff} \quad AB = A'B', \ AC = A'C', \ \text{and } m\angle BAC = m\angle B'A'C'.$$

Equivalently, A triangle is determined, up to congruence, by two sides and the angle between them.

Theorem 10. *The base angles of an isosceles triangle are equal. That is, if $AB = AC$ then $\angle ABC \cong \angle ACB$.*

Theorem 11. *Suppose that $\overline{AB}$ is a diamater of a circle centered at O, and that C is a point on the circle, as in the following figure.*



Then

$$m\angle ACB = 90° \quad \text{and} \quad m\angle BOC = 2m\angle BAC.$$

Theorem 12. *(The Pythagorean Theorem) In a right triangle with legs of lengths a and b and hypotenuse of length c,*

$$a^2 + b^2 = c^2.$$

Theorem 13. *[EG 23] The angles of every quadrilateral add up to 360°.*

Theorem 14. *[EG 27] In a parallelogram $PQRS$, opposite sides and opposite angles are equal. That is, if $\overleftrightarrow{PQ}$ is parallel to $\overleftrightarrow{RS}$ and $\overleftrightarrow{PS}$ is parallel to $\overleftrightarrow{QR}$, then*

$$PQ = RS, \quad PS = RQ, \quad \angle PQR \cong \angle RSP, \quad \text{and} \quad \angle QRS \cong \angle SPQ.$$

Theorem 15. *The diagonals of every parallelogram bisect each other. That is, if $PQRS$ is any parallelogram, and $X = \overline{PR} \cap \overline{QS}$ is the point where its diagonals meet, then $PX = RX$ and $QX = SX$.*

Theorem 16 (EG 28)**.** *The diagonals of parallelogram $PQRS$ meet at a right angle if and only if the parallelogram is a rhombus.*

Theorem 17 (EG 29)**.** *The diagonals of a parallelogram are congruent to each other if and only if the parallelogram is a rectangle.*