APPENDIX F

Permutations

If S is a finite set, then a *permutation* of S is a function $f{:}S \to S$ that has the following two properties:

1. if a and b are distinct elements of S then $f(a)$ and $f(b)$ are also distinct elements of S;

2. for every element y of S there is an element x of S such that $y = f(x)$.

It is customary to display permutations as a collection of cycles. *A cycle* of a permutation f is a cyclic sequence

$$(a_1 \ a_2 \ \dots \ a_k)$$

where

$$a_{i+1} = f(a_i) \qquad \text{for } i = 1, 2, \dots, k\text{-}1$$

and

$$a_1 = f(a_k).$$

EXAMPLE F.1 If $S = \{1, 2, 3, 4, 5, 6, 7\}$ and $f(1) = 6, f(2) = 5, f(3) = 7, f(4) = 4,$ $f(5) = 3, f(6) = 1, f(7) = 2,$ then $(1\ 6)$, $(5\ 3\ 7\ 2)$, and (4) are cycles of f, as are $(6\ 1)$ and $(3\ 7\ 2\ 5)$. However, since cycles are, by their definition, cyclically ordered it follows that

$$(1\ 6) = (6\ 1) \text{ and } (5\ 3\ 7\ 2) = (\ 3\ 7\ 2\ 5) = (7\ 2\ 5\ 3) = (\ 2\ 5\ 3\ 7)\ .$$

Hence $(1\ 6)$, $(5\ 3\ 7\ 2)$, and (4) are the complete set of cycles of f and we write

$$f = (1\ 6)(5\ 3\ 7\ 2)(4).$$

The order of the cycles is immaterial. Thus,

$$f = (1\ 6)(5\ 3\ 7\ 2)(4) = (5\ 3\ 7\ 2)(4)(1\ 6) = (4)(1\ 6)(5\ 3\ 7\ 2) = (3\ 7\ 2\ 5)(4)(6\ 1) = \dots\ .$$

EXAMPLE F.2 If $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c\}$ and $f(1) = a, f(2) = 1, f(3) =$ $c, f(4) = 8, f(5) = 9, f(6) = 7, f(7) = 3, f(8) = 4, f(9) = 6, f(a) = 2, f(b) = b, f(c) = 5,$ then

$$f = (2\ 1\ a)(7\ 3\ c\ 5\ 9\ 6)(4\ 8)(b)\ .$$

If f and g are permutations of the same set S, then the *composition* $f{+}g$ is also a permutation of S such that

$$(f{+}g)(x) = f(g(x)) \qquad \text{for all } x \text{ in } S.$$

EXAMPLE F.3 If $f = (1\ 6)(5\ 3\ 7\ 2)(4)$ and $g = (1\ 7\ 2\ 6\ 3\ 5\ 4)$ then

$$(f{+}g)(1) = f(g(1)) = f(7) = 2$$

$$(f{+}g)(2) = f(g(2)) = f(6) = 1$$

$$(f{+}g)(3) = f(g(3)) = f(5) = 3$$

$$(f{+}g)(4) = f(g(4)) = f(1) = 6$$

$$(f{+}g)(5) = f(g(5)) = f(4) = 4$$

$$(f{+}g)(6) = f(g(6)) = f(3) = 7$$

$$(f{+}g)(7) = f(g(7)) = f(2) = 5\ .$$

Consequently, $f{+}g = (1\ 2)(3)(4\ 6\ 7\ 5)$. Similarly,

$$(g{+}f)(1) = g(f(1) = g(6) = 3$$

$$(g{+}f)(2) = g(f(2) = g(5) = 4$$

$$(g{+}f)(3) = g(f(3) = g(7) = 2$$

$$(g{+}f)(4) = g(f(4) = g(4) = 1$$

$$(g{+}f)(5) = g(f(5) = g(3) = 5$$

$$(g{+}f)(6) = g(f(6) = g(1) = 7$$

$$(g{+}f)(7) = g(f(7) = g(2) = 6\ .$$

Consequently, $g{+}f = (1\ 3\ 2\ 4)(5)(6\ 7)$.

EXERCISES F

Rewrite the functions of exercises 1-5 in terms of their cycles.

1. $f(1) = 6, f(2) = 5, f(3) = 7, f(4) = 2, f(5) = 3, f(6) = 1, f(7) = 4.$
2. $f(1) = 6, f(2) = 5, f(3) = 7, f(4) = 8, f(5) = 3, f(6) = 1, f(7) = 2,\ f(8) = 4.$
3. $f(1) = 9, f(2) = 5, f(3) = 7, f(4) = 8, f(5) = 3, f(6) = 1, f(7) = 2,\ f(8) = 4, f(9) = 6.$
4. $f(1) = 9, f(2) = 5, f(3) = 7, f(4) = 8, f(5) = 3, f(6) = 1, f(7) = a,\ f(8) = 4, f(9) = 6,$ $f(a) = 2.$
5. $f(1) = 9, f(2) = 5, f(3) = b, f(4) = 8, f(5) = 3, f(6) = 1, f(7) = a,\ f(8) = 4, f(9) = 6,$ $f(a) = 2, f(b) = 7.$
6. Suppose $f = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$, $g = (4\ 3\ 2\ 1)(5)(9\ 8\ 7)(6)$, $h = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9)$. Display the following compositions in terms of their cycles.

   a) $f+g$ b) $g+f$ c) $f+h$ d) $h+f$

   e) $g+h$ f) $h+g$ g) $f+f$ h) $g+g$

   i) $h+h$