CHAPTER 8

Symmetry in Space

The focus now shifts to a discussion of some symmetries in three (and more) dimensions. Attention is restricted mostly to the symmetries of the five regular solids. The chapter concludes with a discussion of some recent discoveries in group theory.

1. Regular and Semiregular Polyhedra

Proposition 1 of Book I of *The Elements* states that it is possible to construct equilateral triangles. The thirteenth and last of these books is concerned exclusively with the constructibility of the three dimensional analogs of the regular polygons. Euclid's decision to both begin and end his text with a discussion of highly symmetrical figures was in all likelihood conscious and testifies to a concern with esthetic issues that goes back to the roots of geometry in Pythagorean mysticism.

A *polyhedron* is a solid body of finite extent whose surface consists of several polygons, called *faces*. The sides and vertices of these polygonal faces are respectively the *edges* and *vertices* of the polyhedron. The vertices, edges, and faces of a polyhedron are collectively referred to as its *cells*.

A *regular polyhedron* is a polyhedron whose cells satisfy the following constraints:

1. *All the faces are the same regular polygon*;
2. *All the vertices are equivalent in the sense that for any two vertices* u *and* v *there is a rotation of the polyhedron that replaces the vertex* u *with the vertex* v *and also replaces all the edges emanating from* u *with the edges emanating from* v.

As proved by Euclid, there are five regular polyhedra. The easiest regular polyhedron to visualize is of course the *cube* (Fig. 8.1) whose faces consist of 6 congruent squares. It has 12 edges and 8 vertices. Almost as immediate as the cube is the *tetrahedron*, a triangle-based pyramid, whose faces consist of 4 equilateral triangles. It has 6 edges and 4 vertices. The *octahedron* a double square-based pyramid, has 8 equilateral triangles as its faces. It has 12 edges and 6 vertices. The *dodecahedron* has 12 regular pentagons as its faces, 30 edges and 20 vertices. The *icosahedron* has 20 equilateral triangles as its faces, 30 edges and 12 vertices. These counts are tabulated in Table 8.1.

	v = vertices	e = edges	f = faces	$v - e + f$
Cube	8	12	6	$8 - 12 + 6 = 2$
Tetrahedron	4	6	4	$4 - 6 + 4 = 2$
Octahedron	6	12	8	$6 - 12 + 8 = 2$
Dodecahedron	20	30	12	$20 - 30 + 12 = 2$
Icosahedron	12	30	20	$12 - 30 + 12 = 2$

Table 8.1

It is commonly accepted that the Pythagorean were aware of all five regular polyhedra. Theaetetus (415? - 369? BC) is credited with being the first mathematician to formally prove their existence. While the existence of the cube, tetrahedron, and octahedron hardly requires justification, the existence of the dodecahedron and icosahedron is much less obvious. One way of demonstrating the existence of all these polyhedra is by means of coordinates. Assume that space has been endowed with a Cartesian coordinate system so that each point is described by a triple in

$$\mathrm{R}^3 = \{(x, y, z) \text{ such that } x, y, \text{ and } z \text{ are real numbers}\} .$$

Then the following coordinates describe vertices of regular polyhedra.

Cube:	$(\pm 1, \pm 1, \pm 1)$
Tetrahedron:	$(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$
Octahedron:	$(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$
Dodecahedron:	$(0, \pm\tau, \pm 1/\tau)$, $(\pm 1/\tau, 0, \pm\tau)$, $(\pm\tau, \pm 1/\tau, 0)$, $(\pm 1, \pm 1, \pm 1)$
Icosahedron:	$(\pm\tau, 0, \pm 1)$, $(0, \pm 1, \pm\tau)$, $(\pm 1, \pm\tau, 0)$

where τ is the golden ratio $(\sqrt{5} - 1)/2$ of Proposition 3.4.1.

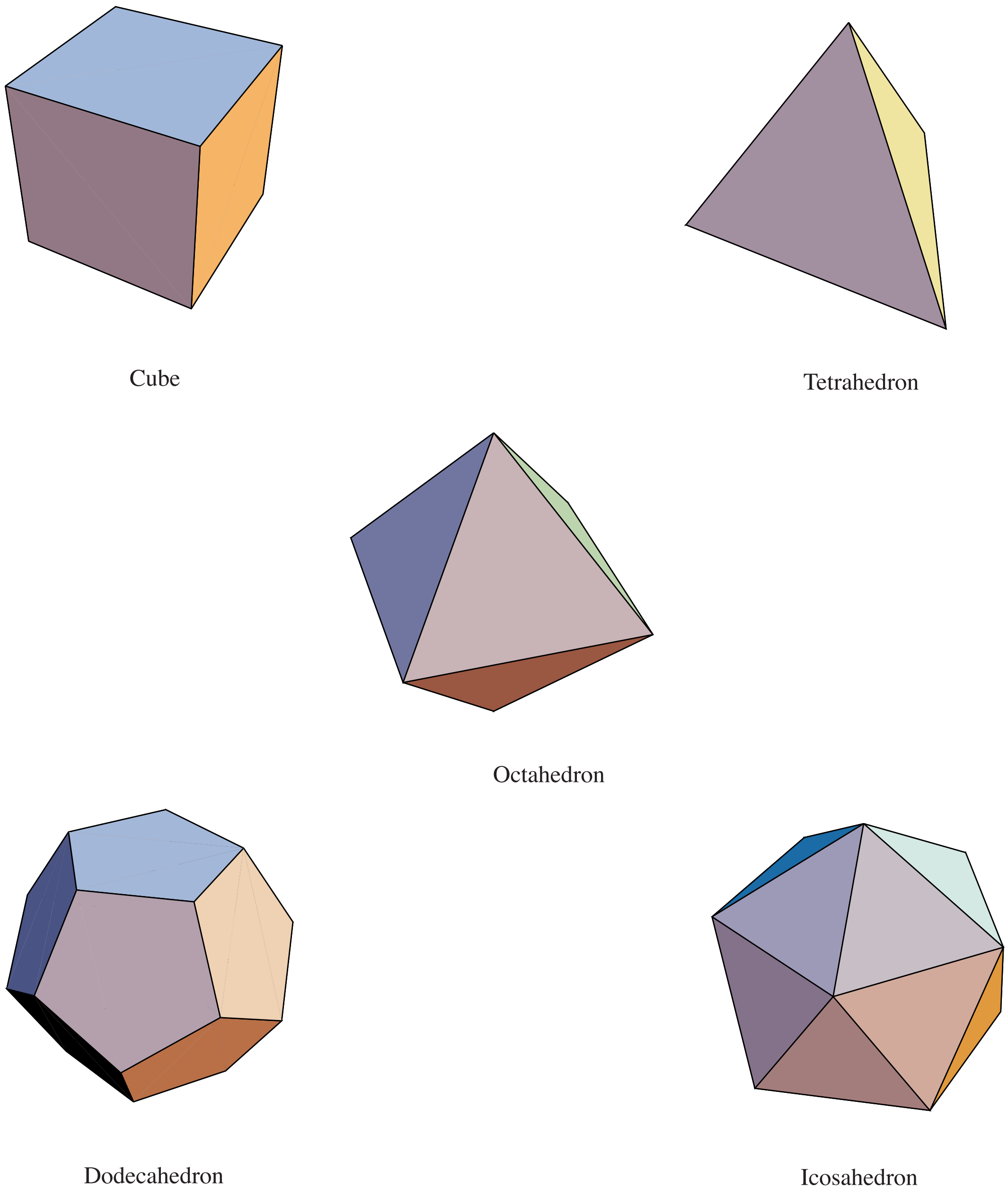


Figure 8.1 The Platonic or regular polyhedra

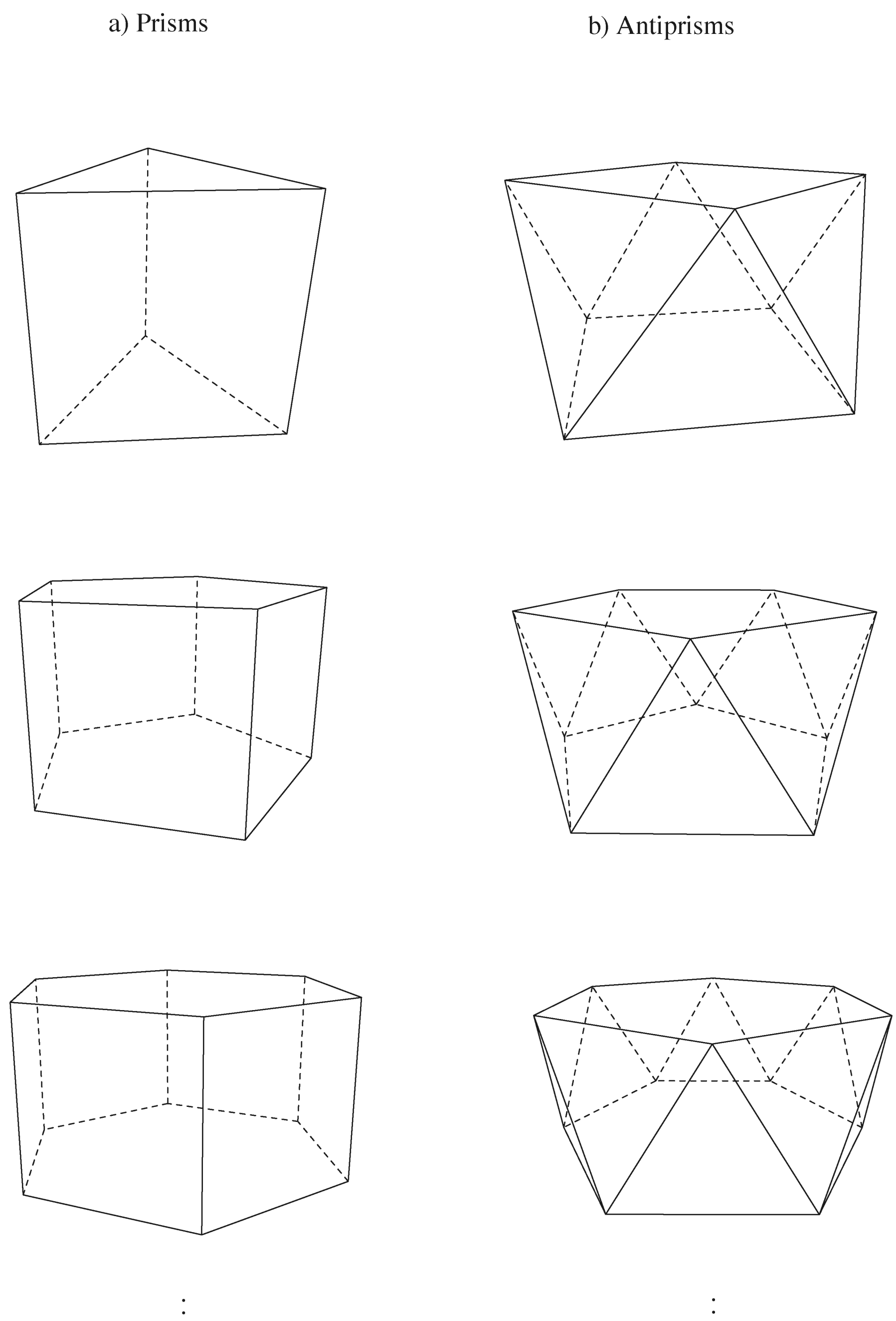


Figure 8.2 Some prisms and antiprisms

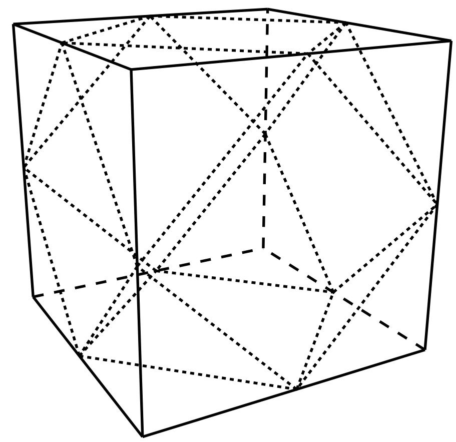
Figure 8.3 See Wikipedia.

A polyhedron is said to be *semiregular* if

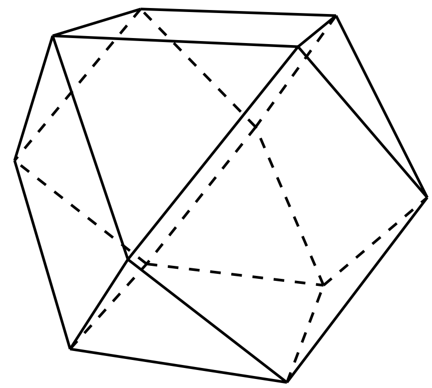
1. *All the faces are regular polygons*;
2. *All the vertices are equivalent in the sense that for any two vertices u and v there is a rotation of the polyhedron that replaces the vertex u with the vertex v and also replaces all the edges emanating from u with the edges emanating from v*.

The semiregular polyhedra differ from the regular ones only in that the faces need not all be the <u>same</u> regular polygon. While one of the semiregular polyhedra was mentioned by Plato (427-347 BC), their first serious study is commonly attributed to Archimedes. They consist of the *prisms* and *antiprisms* (Fig 8.2) as well as the *Archimedian polyhedra*. His work on this topic was lost and it was Johannes Kepler (1571-1630) who once again constructed all the semiregular polyhedra and discussed their relation to the regular polyhedra. Some of these semiregular polyhedra can be derived by truncating the corners of the regular polyhedra. This process is demonstrated here for the cube. In this description the cells of the original cube are referred to as the *old* cells and those of the derived solid as the *new* ones.

<u>Truncated cube I</u>: All the corners of the cube are cut off in such a manner that the cutting planes meet at the midpoints of the old edges (see Fig. 8.4). There are 12 new vertices, one for each of the old edges. Each of the 8 old vertices contributes 3 new edges, for a total of 24. Each of the 6 old square faces has been trimmed down to a smaller new square face and each of the 8 truncated corners has left a new triangular face, for a total of 14 new faces. The new polyhedron will be recognized as the cuboctahedron of Figure 8.3.



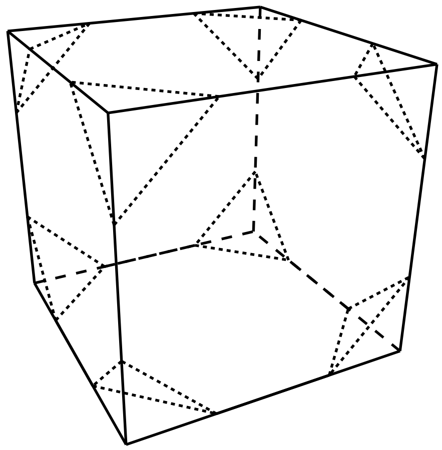
Cutting off corners



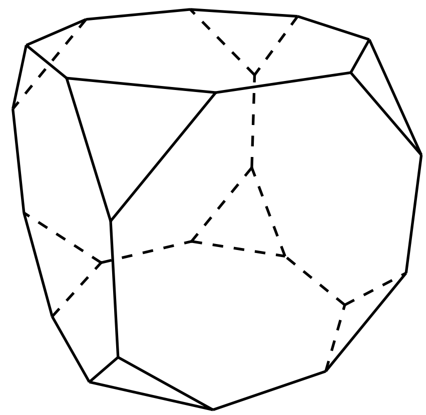
The truncated cube I

Figure 8.4

<u>Truncated cube II</u>: Again all the corners of the cube are cut off but this time the cutting planes do not meet at the midpoints of the edges. Instead, a central portion of the old edge is left whose length equals that of the edge of the new triangular face created by the truncation process (see Figure 8.5). There are 24 new vertices, two for each of the old edges. Each of the 8 old vertices contributes 3 new edges, and there are also the 12 remnants of the old edges. These add up to a total of 36 new edges. Each of the 6 old square faces has been trimmed down to an octagon and each of the 8 truncated corners has left a triangular face. Hence this polyhedron has 14 faces and is the one titled *truncated cube* in Figure 8.3.



Cutting off corners



The truncated cube II

Figure 8.5

The rightmost column of the tally of the cells of the regular polyhedra (Table 8.1) indicates that these counts are subject to a very simple and surprising relationship. This relationship actually holds for all polyhedra provided that their definition excludes the possibility of such troublesome features as the hole in the torus and the point juncture of two cubes (Fig. 8.6). For such trouble-free polyhedra, whose precise definition falls outside the bounds of this text, the following proposition holds.

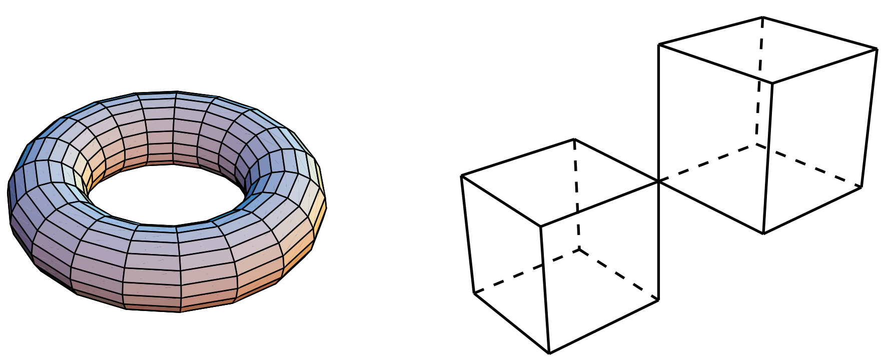


Figure 8.6 Counterexamples to Euler's equation.

PROPOSITION 8.1.1 (Euler's equation, 1758). *For any polyhedron,* $v - e + f = 2$.

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This equation is named after its discoverer, Euler, who has already been mentioned several times in this text. The discovery of this equation initiated a flourishing branch of mathematics now known as topology. Note that the cells of the above truncated cubes also satisfy Euler's equation. The pyramid and the prism that are based on n-sided polygons are offered as further examples of non-regular polyhedra for which Euler's equation holds . The pyramid has $n+1$ vertices, $2n$ edges, and $n+1$ faces so that

$$(n+1) - 2n + (n+1) = 2$$

and the prism has $2n$ vertices, $3n$ edges, and $n+2$ faces so that again

$$2n - 3n + (n+2) = 2.$$

EXERCISES 8.1

1. Answer the following questions for each of the two polyhedra obtained from the octahedron by the two truncation methods described in Figures 8.4-5 (parts a, b, c are to be answered without reference to Euler's equation),
   a) How many vertices does it have?
   b) How many edges does it have?
   c) What regular polygons appear as its faces and how many times?
   d) Identify the truncated polyhedron in Figure 8.1 or in Figure 8.3.
   e) Verify that the cells of this polyhedron satisfy Euler's equation.
2. Repeat Exercise 1 for the tetrahedron.
3. Repeat Exercise 1 for the dodecahedron.
4. Repeat Exercise 1 for the icosahedron.
5. The truncation procedure that produced the truncated cube I can be applied to arbitrary polyhedra so as to obtain new polyhedra. Without using Euler's equation find the number of vertices, edges,

and faces of the polyhedra obtained by applying this procedure to each of the polyhedra below. Also verify Euler's equation for each derived polyhedron.

a) the two truncated cubes;

b) the two truncated tetrahedra;

c) the two truncated octahedra;

d) the two truncated dodecahedra;

e) the two truncated icosahedra;

f) a polyhedron with v vertices, e edges, and f faces, in which each vertex is incident to 3 edges.

6. The truncation procedure that produced the truncated cube II can be applied to arbitrary polyhedra so as to obtain new polyhedra. Without using Euler's equation find the number of vertices, edges, and faces of the polyhedra obtained by applying this procedure to each of the polyhedra below. Also verify Euler's equation for each derived polyhedron.

a) the two truncated cubes;

b) the two truncated tetrahedra;

c) the two truncated octahedra;

d) the two truncated dodecahedra;

e) the two truncated icosahedra;

f) a polyhedron with v vertices, e edges, and f faces, in which each vertex is incident to d edges.

7. Use your favorite mathematical computer application to draw the following regular polyhedron from the coordinates given in this section:

a) tetrahedron b) octahedron c) cube

d) dodecahedron e) icosahedron.

8. Show that there are infinitely many polyhedra all of whose faces are congruent squares.

9. Show that the cell counts of the polyhedron in Figure 8.7 do not satisfy Euler's equation. Explain why this is not a counterexample to this equation.

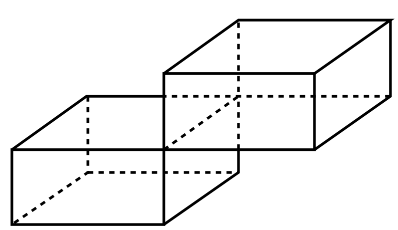

Figure 8.7

10. Show that the cell counts of the polyhedron in Figure 8.8 do not satisfy Euler's equation. Explain why this is not a counterexample to this equation.

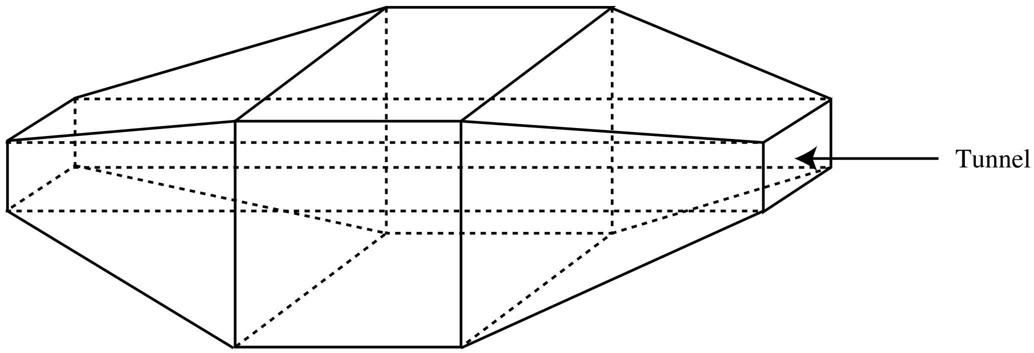

Figure 8.8

11. Show that the cell counts of the polyhedron in Figure 8.9 do satisfy Euler's equation.

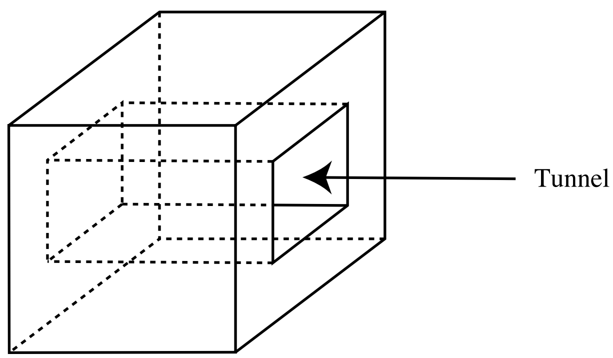

Figure 8.9

12. Show that the cell counts of the polyhedron in Figure 8.10 do not satisfy Euler's equation. Explain why this is not a counterexample to this equation.

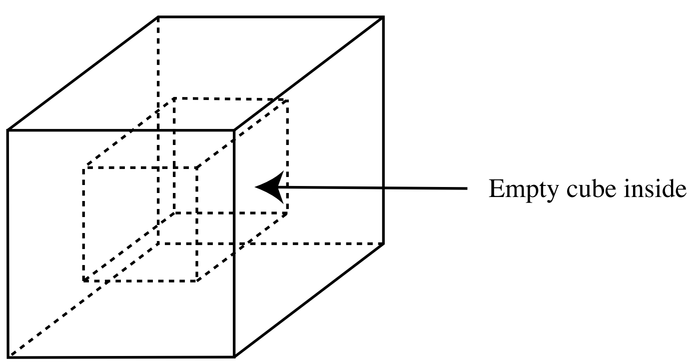

Figure 8.10

13. Each of the corners of the pyramid based on an *n*-sided polygon is truncated (in the usual two ways). Derive the number of vertices, edges, and faces of the resultant polyhedra without using Euler's equation and verify that this equation does indeed hold.
14. Each of the corners of the prism based on an *n*-sided polygon is truncated (in the usual two ways). Derive the number of vertices, edges, and faces of the resultant polyhedra without using Euler's equation and verify that this equation does indeed hold.
15. A diagonal of a polyhedron is a line segment joining two of its vertices. A polyhedron is said to be convex if it contains all of its diagonals either in its interior or on its faces. Show that there is only one convex polyhedron all of whose faces are congruent squares.
16. Construct a cube using the medium of your choice.
17. Construct a tetrahedron using the medium of your choice.
18. Construct an octahedron using the medium of your choice.
19. Construct a dodecahedron using the medium of your choice.
20. Construct an icosahedron using the medium of your choice.
21. A paper model of the dodecahedron can be constructed from thirty square sheets of paper (8.5" 3 8.5" is easy to work with). Each piece should be folded in half and then each half is to be folded in half again, accordion fashion. Next, fold each piece along the dashed lines indicated in Figure 8.11, where the two corners are <u>isosceles</u> right triangles. These last three folds should all bend towards you. The pieces are to be tucked into each other as indicated in Figure 8.11.

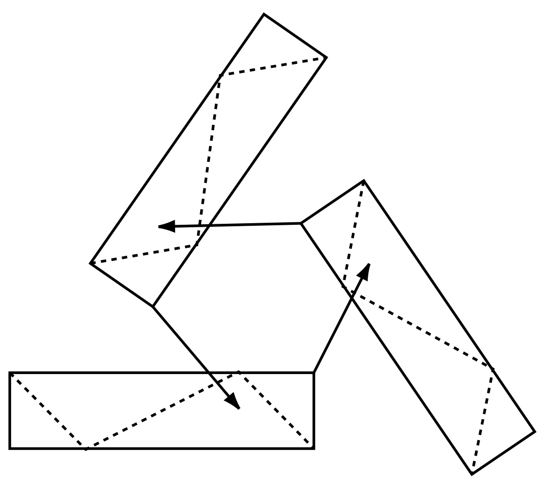

Figure 8.11

22. Construct all thirteen semiregular polyhedra using the medium of your choice.
23. Note that the torus of Figure 8.6 has 19 “meridians”. Assuming that it has n “equators”, derive the number of vertices, edges, and faces (in terms of n) and show that Euler’s equation is not satisfied.

2. Rotational Symmetries of Regular Polyhedra

A *symmetry* of a polyhedron is a rigid transformation of its ambient space that leaves the polyhedron in its original position. If the transformation is a rotation then the symmetry is said to be a *rotational symmetry*. During the rotation the polyhedron may very well pass through nearby parts of space that it did not occupy initially, but when the rotation is accomplished the solid's position must coincide exactly with its initial position. Spatial rotations are denoted by the symbol $R_{A,\alpha}$ where A indicates the axis and α the magnitude and sense of the rotation.

The set of symmetries of a polyhedron is its *symmetry group* and the set of rotational symmetries is its *rotation group*. The following theorem of Euler's implies that the rotation group of every polyhedron is closed under composition.

THEOREM 8.2.1. *If the axes of two rotations of* R^3 *intersect, then their composition is a rotation whose axis passes through the intersection point.*

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The axes of those rotations that constitute symmetries of a polyhedron are constrained by the fact that they must pass through a vertex, the midpoint of an edge, or else the center of some face. This observation can be used to devise a notation for the rotational symmetries. Accordingly, $R_{7,\cdot}$ denotes a rotation of the cube whose axis passes through the vertex at 3 (Fig. 8.12), $R_{26,\cdot}$ denotes a rotation of the cube whose axis passes through the midpoint of the edge 26 (Fig. 8.13), and $R_{1265,\cdot}$ denotes a rotation of the cube whose axis passes through the center of the square 1265 (Figs. 8.14-15). This notation is subject to some redundancy. Thus, for the cube of Figures 8.12-15, the rotation $R_{7,\cdot}$ can also be written as $R_{1,\cdot}$, the rotation $R_{26,\cdot}$ can also be written as $R_{48,\cdot}$, and the rotation $R_{1265,\cdot}$ can also be written as $R_{3487,\cdot}$.

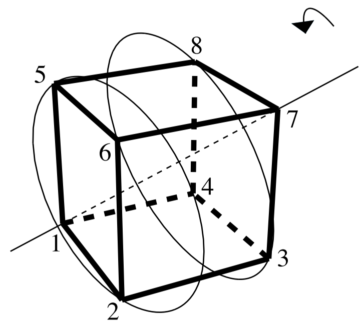

Figure 8.12 $R_{7,120^o} = R_{1,-120^o}$

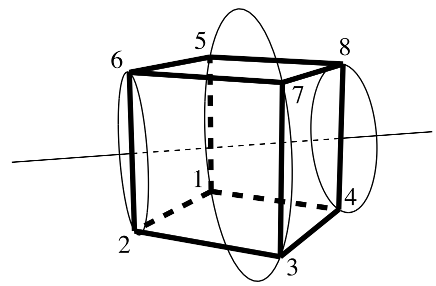

Figure 8.13 $R_{26,180^o} = R_{48,180^o}$

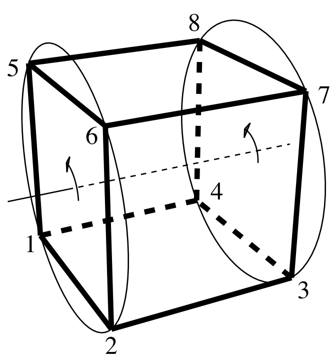

Figure 8.14 $R_{1265,90^o} = R_{3487,-90^o}$

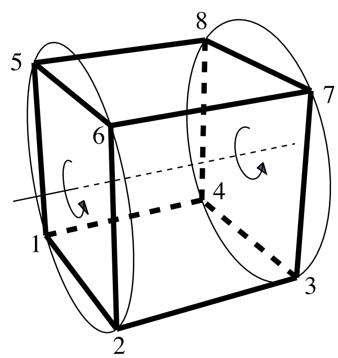

Figure 8.15 $R_{1265,270^o} = R_{3487,-270^o}$

The symbol $R_{A,\alpha}$ denotes a rotation by the oriented angle α where the orientation is understood to be determined by an observer positioned outside the polyhedron near A. Thus, $R_{1265,90^o}$ (Fig. 8.14) denotes the 90^o rotation of the cube about the axis that passes through the centers of the faces 1265 and 3487, counterclockwise from the point of view of an observer situated outside the cube near the face 1265. Note that since -90^o denotes a clockwise rotation it follows that $R_{1265,90^o} = R_{3487,-90^o}$. The circles in the illustrations are meant to help visualize the rotation; they are the "tracks" in which the vertices move to their new positions.

Just like their two dimensional counterparts, spatial symmetries also have permutation representations. Thus, since the rotation $R_{1265,90^o}$ cycles the vertices in positions 1, 2, 6, 5 and also the vertices in positions 3, 7, 8, 4, it has the permutation representation (1 2 6 5)(3 7 8 4) . Similarly, the 180^o rotation $R_{1265,180^o}$ has the permutation representations (1 6)(2 5)(7 4)(8 3).

A qualitatively different symmetry of the cube is obtained by a 180^o rotation about an axis that passes through the midpoints of two diametrically opposite edges of the cube. Such, for example, is the rotation $R_{26,180^o} = R_{48,180^o}$ (Fig. 8.13) It has the permutation representation (1 7)(2 6)(3 5)(4 8). While this permutation looks very much like that of $R_{1265,180^o}$ above, there is a significant geometrical difference between them. The permutation representation (1 7)(2 6)(3 5)(4 8) of $R_{26,180^o}$ has cycles that are in fact cells of the cube, namely (2 6) and (4 8). On the other hand, none of the cycles of the permutation representation (1 6)(2 5)(3 8)(4 7) of $R_{1265,180^o}$ are cells of the cube.

The permutation representation of the rotation $R_{7,120^o}$ is (1)(2 4 5)(3 8 6)(7).

The compositions of rotational symmetries are easily computed by means of their permutation representations. Accordingly, since $R_{1265,90^o}$ = (1 2 6 5)(3 7 8 4) and $R_{7,120^o}$ = (1)(2 4 5)(3 8 6)(7) it follows that

$$(R_{1265,90^o})\circ(R_{7,120^o}) = (1\ 2\ 6\ 5)(3\ 7\ 8\ 4)\circ(1)(2\ 4\ 5)(3\ 8\ 6)(7)$$

$$= (1\ 2\ 3\ 4)(5\ 6\ 7\ 8) = R_{1234,-90^o}.$$

On the other hand,

$$(R_{7,120^o})\circ(R_{1265,90^o}) = (1)(2\ 4\ 5)(3\ 8\ 6)(7)\circ(1\ 2\ 6\ 5)(3\ 7\ 8\ 4)$$

$$= (1\ 4\ 8\ 5)(2\ 3\ 7\ 6) = R_{1485,-90^o}.$$

Similarly, since $R_{26,180^o}$ = (1 7)(2 6)(3 5)(4 8) and $R_{34,180^o}$ = (1 7)(2 8)(3 4)(5 6) it follows that

$$(R_{26,180^o})\circ(R_{34,180^o}) = (1\ 7)(2\ 6)(3\ 5)(4\ 8)\circ(1\ 7)(2\ 8)(3\ 4)(5\ 6)$$

$$= (1)(2\ 4\ 5)(3\ 8\ 6)(7) = R_{7,120^o}\,.$$

The number of symmetries in a group is that group's *order*.

PROPOSITION 8.2.2. *The rotation group of the cube has order 24 and its rotations are classified as:*

Id
8 rotations of the type $R_{vertex,120^o}$
6 rotations of the type $R_{edge,180^o}$
6 rotations of the type $R_{face,90^o}$
3 rotations of the type $R_{face,180^o}$.

SKETCH OF PROOF: This follows from the fact that the cube has 4 axes that join opposite vertices, 6 axes that join opposite edges, and 3 axes that join opposite faces.

Q.E.D.

Whereas the axis of any rotational symmetry of the cube joins the midpoints of cells of the same dimension, the tetrahedron presents us with a new alternative. The axis of the symmetry $R_{3,-120^o}$ joins the vertex 3 to the center of the triangular face with vertices 1, 2, and 4. This rotation has the permutation representation (1 4 2)(3). The only qualitatively different rotational symmetry of the tetrahedron is the 180^{o} rotation about the line joining the midpoints of two opposite edges. Such, for example, is $R_{24,180^{o}} = (1\ 3)(2\ 4)$. Note that

$$(R_{3,-120^{o}})\circ(R_{24,180^{o}}) \quad = \quad (1\ 4\ 2)(3)\circ(1\ 3)(2\ 4) \quad = \quad (1\ 3\ 4)(2)$$

$$= \quad R_{2,-120^{o}}$$

The rotational symmetries of the tetrahedron are illustrated in Figures 8.16-17 and summarized in a proposition.

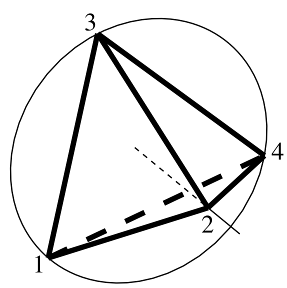

Figure 8.16 $R_{3,240^{o}}$

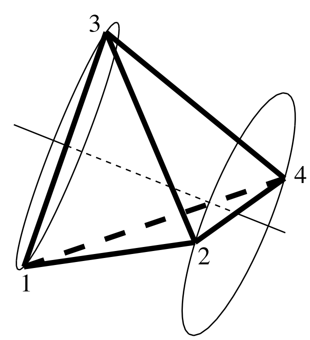

Figure 8.17 $R_{13,180^{o}}$

PROPOSITION 8.2.3. *The rotation group of the tetrahedron has order 12 and its rotations are classified as:*

Id

4 rotations of each of the types $R_{vertex,120^{o}}$ *and* $R_{vertex,240^{o}}$

3 rotations of the type $R_{edge,180^{o}}$.

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The verification of the analogous proposition regarding the octahedron, is relegated to Exercise 1.

PROPOSITION 8.2.4. *The rotation group of the octahedron has order 24 and its rotations are classified as:*

Id

6 rotations of the type $R_{vertex,90^o}$

3 rotations of the type $R_{vertex,180^o}$

6 rotations of the type $R_{edge,180^o}$

8 rotations of each of the types $R_{face,120^o}$.

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EXERCISES 8.2

1. Classify the rotational symmetries of the octahedron.
2. Classify the rotational symmetries of the dodecahedron.
3. Classify the rotational symmetries of the icosahedron.
4. Suppose $A = R_{1,120^o}$, $B = R_{26,180^o}$, $C = R_{2376,90^o}$, $D = \mathrm{R}_{1234,180^o}$ are symmetries of the cube of Figures 8.12-15.

   a) Find the permutation representations of A, B, C, D

   b) Identify the following symmetries:

i)	AoA	ii)	AoB	iii)	AoC	iv)	AoD
v)	BoA	vi)	BoB	vii)	BoC	viii)	BoD
ix)	CoA	x)	CoB	xi)	CoC	xii)	CoD
xiii)	DoA	xiv)	DoB	xv)	DoC	xvi)	DoD

5. Repeat Exercise 4 with $A = R_{2,-120^o}$, $B = R_{15,180^o}$, $C = R_{3487,270^o}$, $D = R_{5678,180^o}$.
6. Suppose $A = R_{1,120^o}$, $B = R_{24,180^o}$, $C = R_{2,240^o}$, $D = R_{14,180^o}$ are symmetries of the tetrahedron of Figures 8.16-17.

   a) Find the permutation representations of A, B, C, D

   b) Identify the following symmetries:

i)	$A \circ A$	ii)	$A \circ B$	iii)	$A \circ C$	iv)	$A \circ D$
v)	$B \circ A$	vi)	$B \circ B$	vii)	$B \circ C$	viii)	$B \circ D$
ix)	$C \circ A$	x)	$C \circ B$	xi)	$C \circ C$	xii)	$C \circ D$
xiii)	$D \circ A$	xiv)	$D \circ B$	xv)	$D \circ C$	xvi)	$D \circ D$

7. Repeat Exercise 6 with $A = R_{3,120^\circ}$, $B = R_{23,180^\circ}$, $C = R_{3,240^\circ}$, $D = \mathrm{R}_{12,180^\circ}$.

8. Suppose $A = R_{1,90^\circ}$, $B = R_{25,180^\circ}$, $C = R_{235,240^\circ}$, $D = R_{4,180^\circ}$ are symmetries of the octahedron of Figure 8.18.

   a) Find the permutation representations of A, B, C, D

   b) Identify the following symmetries:

i)	$A \circ A$	ii)	$A \circ B$	iii)	$A \circ C$	iv)	$A \circ D$
v)	$B \circ A$	vi)	$B \circ B$	vii)	$B \circ C$	viii)	$B \circ D$
ix)	$C \circ A$	x)	$C \circ B$	xi)	$C \circ C$	xii)	$C \circ D$
xiii)	$D \circ A$	xiv)	$D \circ B$	xv)	$D \circ C$	xvi)	$D \circ D$

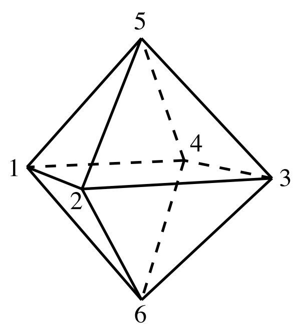

Figure 8.18 Octahedron

9. Repeat Exercise 8 with $A = R_{1,180^\circ}$, $B = R_{46,180^\circ}$, $C = R_{345,120^\circ}$, $D = R_{6,90^\circ}$.

10. Suppose $A = R_{5,120^\circ}$, $B = R_{57,180^\circ}$, $C = R_{57jbf,72^\circ}$, $D = R_{1d5f2,144^\circ}$ are symmetries of the dodecahedron of Figure 8.19.

   a) Find the permutation representations of A, B, C, D

   b) Identify the following symmetries:

i)	$A \circ A$	ii)	$A \circ B$	iii)	$A \circ C$	iv)	$A \circ D$
v)	$B \circ A$	vi)	$B \circ B$	vii)	$B \circ C$	viii)	$B \circ D$
ix)	$C \circ A$	x)	$C \circ B$	xi)	$C \circ C$	xii)	$C \circ D$
xiii)	$D \circ A$	xiv)	$D \circ B$	xv)	$D \circ C$	xvi)	$D \circ D$

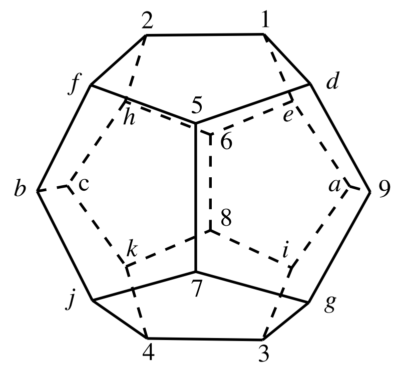

Figure 8.19 Dodecahedron

11. Repeat Exercise 10 with $A = R_{3,240^o}$, $B = R_{bf,180^o}$, $C = R_{68iae,144^o}$, $D = R_{2fbch,72^o}$.

12. Suppose $A = R_{1,72^o}$, $B = R_{4a,180^o}$, $C = R_{349,240^o}$, $D = R_{4,144^o}$ are symmetries of the icosahedron of Figure 8.20.

a) Find the permutation representations of A, B, C, D

b) Identify the following symmetries:

i)	AoA	ii)	AoB	iii)	AoC	iv)	AoD
v)	BoA	vi)	BoB	vii)	BoC	viii)	BoD
ix)	CoA	x)	CoB	xi)	CoC	xii)	CoD
xiii)	DoA	xiv)	DoB	xv)	DoC	xvi)	DoD

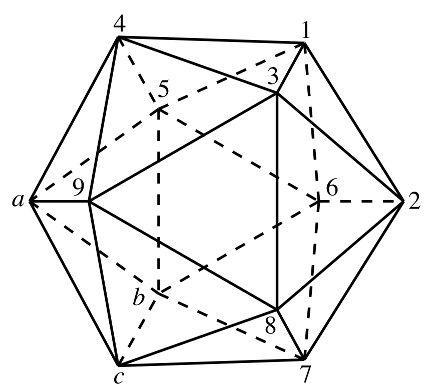

Figure 8.20 Icosahedron

13. Repeat Exercise 12 with $A = R_{6,144^o}$, $B = R_{89,180^o}$, $C = R_{126,120^o}$, $D = R_{c,144^o}$.

14. How many rotational symmetries does a rectangular box have if all of its dimensions are different?

15. How many rotational symmetries does a rectangular box have if exactly two of its dimensions are the same?

16. A triangular prism has an equilateral base. How many rotational symmetries does it have?

17. A triangular prism has a base with sides 6, 6, 4. How many rotational symmetries does it have?

18. A prism has a base that is a regular n-gon. How many rotational symmetries does it have?

3. Monstrous Moonshine

The previous section described the rotational symmetries of the cube and the octahedron separately. Since the cube's symmetries are permutations of 8 vertices whereas those of the octahedron are permutations of 6 vertices, these polyhedra's rotation groups look quite different. Nevertheless, there is a natural sense in which these two groups are identical. Observe that in Figures 8.21-23 a cube has been placed inside an octahedron so that each of the vertices of the first is the center of a triangular face of the latter. The

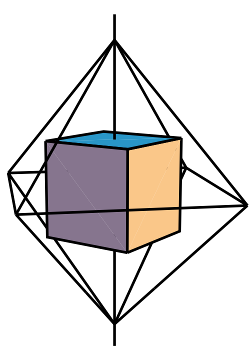

Figure 8.21

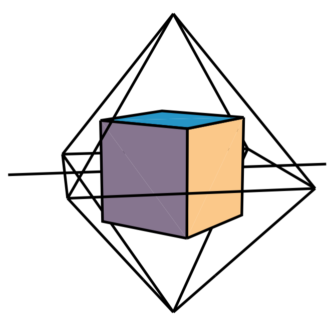

Figure 8.22

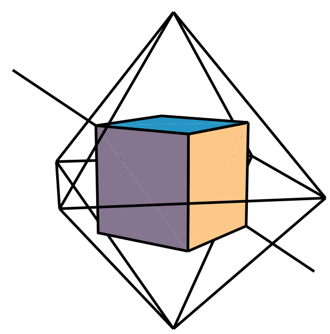

Figure 8.23

feasibility of this placement implies that every R_{face}-symmetry of the cube is also an R_{vertex}-symmetry of the octahedron (Fig. 8.21). Similarly, every R_{edge}-symmetry of the cube is also an R_{edge}-symmetry of the octahedron (Fig. 8.22) and every R_{vertex}-symmetry of the cube is an R_{face}-symmetry of the octahedron (Fig. 8.23). Thus, the rotation groups of the cube and the octahedron are now revealed as being identical. Technically, they are said to be *isomorphic*. It is clear that isomorphic groups must have the same order and so the rotation groups of the cube and the tetrahedron are not isomorphic. Groups of the same order need not be isomorphic either. This is demonstrated by the symmetry group of the regular 12-gon. This polygon has 24 symmetries and the cube has 24 rotational symmetries. However, the 30° rotation of this polygon has order 12 and no rotational symmetry of the cube has such an order. Hence the symmetry group of the regular 12-gon and the rotation group of the cube are not isomorphic, even though they have the same orders (see Exercise 6).

Group theory, the mathematical theory of symmetry, has its origins in the work of Joseph Louis Lagrange (1736 - 1813) on the theory of equations. It was later used by Niels Henrik Abel (1802 - 1829) and Évariste Galois (1811 - 1832) to settle the question of which equations could be solved by explicit algebraic formulas and which could only be solved by means of successive approximations. The subsequent investigations of Felix Klein (1849 - 1925) and Henri Poincaré (1854 - 1912) pointed out the central role that symmetry also plays in geometry.

One of the main goals of group theory is the classification of all groups up to isomorphism. While there is no expectation that this goal will be achieved in the foreseeable future, a significant milestone was passed less than twenty years ago when the *finite simple groups* were completely classified. There is nothing simple about the simple groups, nor is it possible to characterize them in this text. The symmetries of the icosahedron (and the dodecahedron) constitute a simple group whereas those of the cube and tetrahedron are not simple, but this difference does not have a geometrical interpretation. Algebraically, though, the difference is extremely important. The simplicity of the dodecahedral group turns out to be responsible for the non-existence of a formulaic solution for the general fifth degree equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0.$$

Conversely, the non-simplicity of the symmetry group of the tetrahedron is ultimately tantamount to the existence of such a formula for the fourth degree equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0.$$

The classification of the finite simple groups constitutes the most monumental task ever accomplished by mathematicians. Its proof is spread over 500 articles comprising more than 14,000 journal pages written by hundreds of researchers. This classification asserts that the finite simple groups fall into two categories: several infinite families of graphs that possess clear patterns and 26 exceptional groups, known as the *sporadic groups*, for which no general pattern has been found.

The first of the sporadic groups was discovered in 1861 and the last two almost simultaneously in 1980. The largest of these was nicknamed MONSTER because of its order which is

$$808{,}017{,}424{,}794{,}512{,}875{,}886{,}459{,}904{,}961{,}710{,}757{,}005{,}754{,}368{,}000{,}000{,}000$$

MONSTER, discovered by Bernd Fischer and Robert L. Griess, is the group of symmetries of a (non-regular) polyhedron in 196,883 dimensions. When word of this discovery reached John McKay he pointed out the remarkable coincidence that 196,884 is one of the coefficients in the series

$$\begin{aligned} j \quad = \quad q^{-1} + 744 + 196{,}884q + 21{,}493{,}760q^2 + 864{,}299{,}970q^3 \\ + 20{,}245{,}856{,}256q^4 + 333{,}202{,}640{,}600q^5 + 4{,}252{,}023{,}300{,}096q^6 \\ + 44{,}656{,}994{,}071{,}935q^7 + 401{,}490{,}886{,}656{,}000q^8 + \ldots \end{aligned}$$

which arises in the seemingly unrelated context of doubly periodic functions on the hyperbolic plane. For the purposes of this exposition, these functions can be thought of as patterns that underlie non-Euclidean wallpaper designs. As mathematicians were at a loss to explain this conjunction they dubbed the following equation as *McKay's formula*:

$$196{,}884 \quad = \quad 1 \; + \; 196{,}883.$$

Shortly thereafter John Thompson noted that actually much more was true. The number 196,883 is the second one of an important sequence of 194 integers, the *degrees of the characters* of MONSTER, of which the first twelve are

1
196,883
21,296,876
842,609,326
18,538,750,076
19,360,062,527
293,553,734,298
3,879,214,937,598
36,173,193,327,999
125,510,727,015,275
190,292,345,709,543
222,879,856,734,249

and the last one is

258,823,477,531,055,064,045,234,375

The significance of these integers is that, except for 1, each denotes the number of dimensions required by a new polyhedron whose symmetry group is isomorphic to

MONSTER. What Thompson discovered was that, with the exception of 744, all the other early coefficients in the *j*-series also have simple expressions in terms of MONSTER'S degrees. For example,

$$21{,}493{,}760 \quad = \quad 1 + 196{,}883 + 21{,}296{,}876$$
$$864{,}299{,}970 \quad = \quad 2\cdot 1 + 2\cdot 196{,}883 + 21{,}296{,}876 + 842{,}609{,}326$$

John H. Conway assigned the name *Moonshine* to these and other related unexplained phenomena in 1979 "*... intending the word to convey our feelings that they are seen in a dim light, and that the whole subject is rather vaguely illicit*." It should be remembered that at that time the existence of MONSTER had only been conjectured so that even the aforementioned list of degrees was questionable, not to mention their purported relation with the coefficients of the *j*-series.

As was noted above, the existence of MONSTER was conclusively demonstrated in 1980 by R. Griess who tried, unsuccessfully, to have its named changed to *The Friendly Giant*. Monstrous Moonshine mathematics was finally explained by Richard E. Borcherds who found the connection in the theory of Vertex Algebras, a discipline developed recently for the purpose of providing a mathematical foundation to the new Superstring Theory of physics. For this work Borcherds received the 1998 Fields Medal, the most prestigious award bestowed by the mathematical community.

EXERCISES 8.3

In the exercises below $f_1 = 1, f_2 = 196{,}883, f_3 = 21{,}296{,}876, \ldots$ *denote the degrees of the characters of MONSTER and* a_k *denotes the coefficient of* q^k *in the j-series, so that* $a_{-1} = 1, a_0 = 744, a_1 = 196{,}884, a_2 = 21{,}493{,}760, \ldots$.

1. Find integers $x_1, x_2, \ldots, x_6$ such that $a_4 = x_1f_1 + x_2f_2 + \ldots + x_6f_6$.
2. Find integers $x_1, x_2, \ldots, x_7$ such that $a_5 = x_1f_1 + x_2f_2 + \ldots + x_7f_7$.

3. Find integers $x_1, x_2, x_3, \ldots$ such that $a_6 = x_1 f_1 + x_2 f_2 + x_3 f_3 + \ldots$.
4. Find integers $x_1, x_2, x_3, \ldots$ such that $a_7 = x_1 f_1 + x_2 f_2 + x_3 f_3 + \ldots$.
5. Find integers $x_1, x_2, x_3, \ldots$ such that $a_8 = x_1 f_1 + x_2 f_2 + x_3 f_3 + \ldots$.
6. Show that the rotation groups of the cube and the prism whose base is a regular 12-gon are not isomorphic even though they have the same orders.

CHAPTER REVIEW EXERCISES

In the exercises below, for each integer $n \geq 3$, P_n denotes the pyramid obtained by joining the vertices of a regular n-gon to a point outside the plane of the n-gon and lying directly above its center. The polyhedron PP_n is the double pyramid obtained by fitting together two copies of the pyramid P_n at their base.

1. Verify that for each $n \geq 3$, Euler's equation holds for the pyramid P_n and for both of the polyhedra obtained from it by truncation.
2. Verify that for each $n \geq 3$, Euler's equation holds for the double pyramid PP_n and for both of the polyhedra obtained from it by a type II truncation.
3. Describe the rotational symmetries of the pyramid P_3. Note that there are two cases to be considered.
4. Describe the rotational symmetries of the pyramid P_n, $n \geq 4$.
5. Describe the rotational symmetries of the double pyramid PP_4. Note that there are two cases to be considered.
6. Describe the rotational symmetries of the double pyramid PP_n, $n \neq 4$.
7. Explain why the regular n-gon and the double pyramid PP_n have isomorphic rotation groups for $n \geq 5$ (when considered in 3-space).
8. Prove that the icosahedron and dodecahedron have isomorphic rotation groups.
9. Are the following statements true or false? Justify your answers.
   a) Every polyhedron has at least one symmetry.
   b) Every regular polyhedron has at least two symmetries.
   c) Every regular polyhedron has at least as many symmetries as vertices.
   d) Every regular polyhedron has at least as many symmetries as edges.
   e) The composition of every two rotations of R^3 is also a rotation.
   f) If two rotations of R^3 have intersecting axes, then their composition is also a rotation.
   g) All the semiregular polyhedra can be obtained from the regular polyhedra by truncation.
   h) Euler discovered the regular polyhedra.
   i) 196,883 is an interesting integer.

j) MONSTER is a symmetry group.

k) MONSTER was discovered by John H. Conway.

l) Every two symmetry groups are isomorphic.

m) The rotation groups of the cube and the octahedron are isomorphic.

n) The rotation groups of the cube and the tetrahedron are isomorphic.

o) There are at least 193 distinct polyhedra whose symmetry groups are isomorphic to MONSTER.