CHAPTER 7

Inversions

Transformations that are not rigid can be interesting too, even though they are not as natural as the rigid motions of the previous chapter. The inversions of this chapter are particularly appealing because they play important roles in both Euclidean and non-Euclidean geometry.

1. Inversions as Transformations

Given a point C and a positive real number k, the *inversion* $I_{C,k}$ is a transformation of the plane that maps any point $P \neq C$ of the plane into the point $P' = I_{C,k}(P)$ such that

a) C, P' are collinear with C outside the segment PP',

and

b) $CP \cdot CP' = k^2$.

Figure 7.1 illustrates the action of a typical inversion.

It is clear that in general $I_{C,k}(P) = P'$ if and only if $I_{C,k}(P') = P$ and hence $I_{C,k}^2 = Id$. Moreover, $I_{C,k}(P) = P$ if and only if P is on the circle $(C; k)$. Otherwise the point

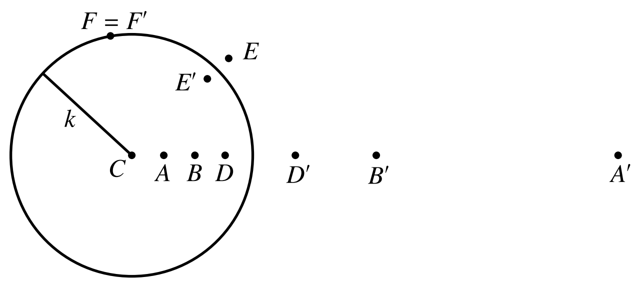

Figure 7.1 The inversion $I_{C,k}$

P is inside the circle $(C; k)$ if and only P' is outside it. If $g = (C; k)$ the inversion $I_{C,k}$ will also be denoted by I_g. Note that $I_{C,k}$ is undefined for C and only for C. The point C is called the *center* of the inversion $I_{C,k}$. Figure 7.2 displays the relation between P and $P' = I_{C,k}(P)$ geometrically. The circle of this figure has radius k and the lines SP' and TP' are tangent to it (see Exercise 8).

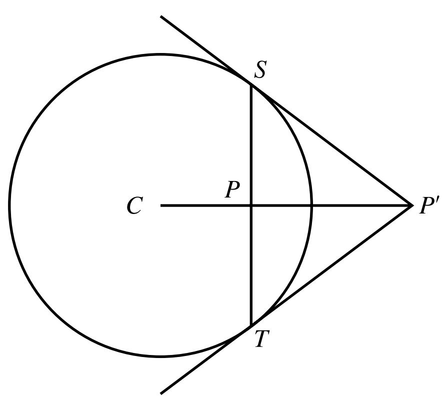

Figure 7.2

Any two points P, Q inside the circle $(C; k)$ are transformed by the inversion $I_{C,k}$ into two points P', Q' such that

$$P'Q' > PQ.$$

In fact, the closer P and Q are to the center C, the greater the discrepancy between PQ and $P'Q'$. It follows that $I_{C,k}$ is not a rigid motion. Nevertheless, it does share some of the properties of rigid motions. Rigid motions transform straight lines into straight line (Proposition 6.1.2) and circles into circles (Exercise 6.1.2). Inversions, too, transform straight lines and circles into straight lines and circles, although not necessarily respectively: some straight lines are bent into circles and some circles are straightened out. The next theorem describes these phenomena in detail.

THEOREM 7.1.1. *The inversion $I_{C,k}$ maps*

a) straight lines through C onto themselves,

b) straight lines not through C onto circles through C,

c) circles through C onto straight lines not through C,

d) circles not through C onto circles not through C.

PROOF:

a) This follows directly from the definition of inversions.

b) Let m be a straight line not through C, and let M be that point of m such

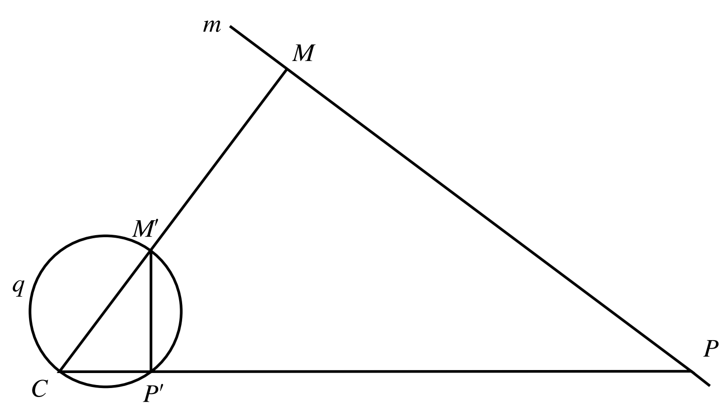

Figure 7.3

that $CM \perp m$ (Figure 7.3) whereas P is an arbitrary point of m. Set

$$M' = I_{C,k}(M) \text{ and } P' = I_{C,k}(P) .$$

Since $CM \cdot CM' = k^2 = CP \cdot CP'$ it follows that

$$\frac{CM'}{CP} = \frac{CP'}{CM} .$$

Moreover, since $\angle PCM$ is common to both ΔCPM and $\Delta CM'P'$, it follows from Proposition 3.5.9 that these triangles are similar and consequently

$$\angle M'P'C = \angle PMC = 90^{\circ}.$$

Since the points C and M' are fixed (whereas P is arbitrary on m) it follows from Exercise 4.2B.5 that P' falls on the fixed circle with diameter CM'.

c) Let q be a circle through C, let CM' be a diameter of q, set $M = I_{C,k}(M')$ and let m be the line through M perpendicular to CM' (Fig. 7.3). It follows from part b above that $I_{C,k}(m) = q$ and hence

$$I_{C,k}(q) = I^2_{C,k}(m) = m .$$

d) Let p be a circle not through C with P an arbitrary point on p. Let DE be a diameter of p whose extension contains C, and suppose the given inversion $I_{C,k}$ maps the points D, E, P onto the points D', E', P' (Figure 7.4). Since

$$CP \cdot CP' = CD \cdot CD' = CE \cdot CE' = k^2$$

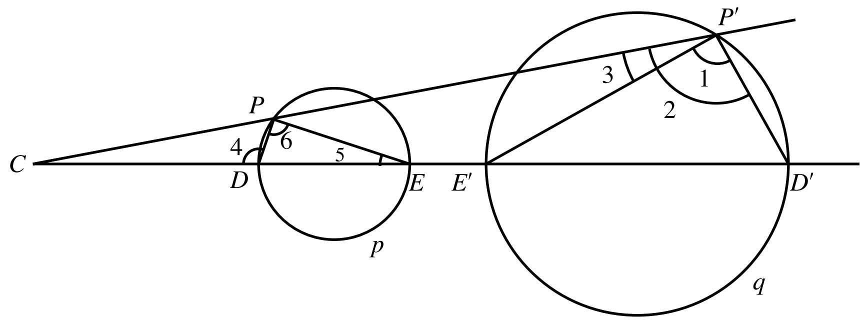

Figure 7.4

it follows that

$$\frac{CD}{CP'} = \frac{CP}{CD'} \quad \text{and} \quad \frac{CE}{CP'} = \frac{CP}{CE'} .$$

Since $\angle DCP$ is common to ΔDCP, $\Delta P'CD'$, ΔECP, and $\Delta P'CE'$ it follows that the first two are similar to each other, as are the last two. Consequently,

$$\angle 4 = \angle 2 \quad \text{and} \quad \angle 5 = \angle 3$$

and

$$\angle 1 = \angle 2 - \angle 3 = \angle 4 - \angle 5 = \angle 6 = 90^{\circ} .$$

Since D' and E' are fixed points it follows that P' lies on the circle that has $D'E'$ as its diameter (see Exercise 4.2B.5).

Q.E.D.

The following observations are implicit in the proof of the above theorem:

When the inversion $I_{C,k}$ *transforms a circle into a circle, their centers are collinear with* C. *When the inversion transforms a straight line into a circle, or vice versa, the line through* C *perpendicular to the straight line contains the center of the circle.*

EXAMPLE 7.1.2. Let $I = I_{O,6}$, where O denotes the origin and let m denote the line $y = -6$ (Fig. 7.5). By Theorem 7.1.1b, $I(m)$ is a circle q that contains O and whose center lies on the y-axis. Since I fixes the point (-6, 0), this point must lie on q. It follows that q is the circle ((-3, 0); 3).

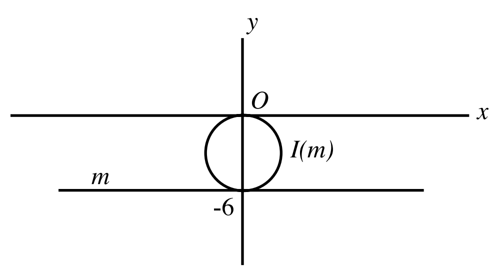

Figure 7.5

EXAMPLE 7.1.3. Let $I = I_{O,6}$ and let q = ((4,0); 1) (Fig. 7.6). It follows from Theorem 7.1.1d that $I(q)$ is also a circle, with center on the x-axis, which of necessity contains the points

$$I((5,0)) = ((\frac{36}{5}),0) = (7.2,0) \quad \text{and} \quad I((3,0) = (\frac{36}{3},0) = (12,0) .$$

Consequently $I(q)$ = ((9.6, 0); 2.4).

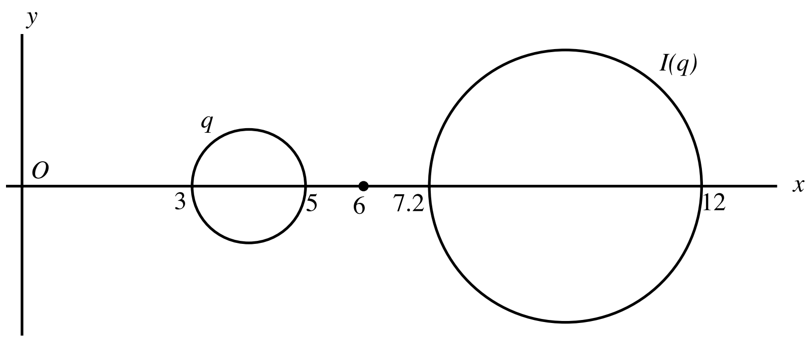

Figure 7.6

EXAMPLE 7.1.4. *Identify the inversion* $I_{C,k}$ *that maps the circle* $(O;\, 2)$ *onto the straight line* $x = 6$.

By Theorem 7.1.1c, C must be either $(2, 0)$ or $(-2, 0)$ (Fig. 7.7). Since C

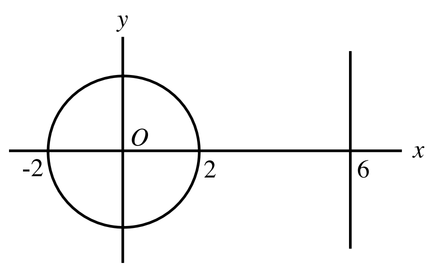

Figure 7.7

cannot lie between a point and its image, it follows that $C = (-2, 0)$. Finally, since $I_{C,k}$ maps $(2, 0)$ onto $(6, 0)$ it follows that

$$k = \sqrt{[2-(-2)]\cdot[6-(-2)]} = 4\sqrt{2}.$$

Another way in which inversions resemble rigid motions is that they too preserve the measures of angles (see Exercise 6.1.8). Of course, since inversions bend straight lines it is necessary to allow for non-rectilineal angles. As is customary in calculus, the measure of the angle determined by two intersecting curves is defined to be the measure of the angle between their respective tangents at that intersection.

PROPOSITION 7.1.5. *Inversions preserve angles (but reverse their senses).*

PROOF: Figure 7.8 describes how an inversion centered at C transforms the angle α formed by the curves h and j into the angle α' formed by the image curves $h' = I_{C,k}(h)$ and $j' = I_{C,k}(j)$. It is clear that the sense of the angle is reversed by the inversion.

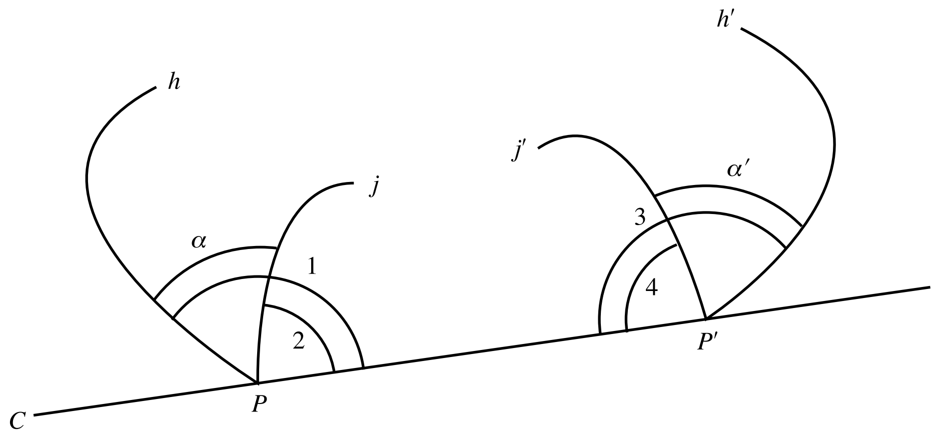

Figure 7.8

Since

$$\alpha = \angle 1 - \angle 2 \quad \text{and} \quad \alpha' = \angle 3 - \angle 4\,,$$

it will suffice to prove the special case that $\angle 1 = \angle 3$, or, that $\alpha = \alpha'$ in Figure 7.9. However, the equation

$$CP \cdot CP' = k^2 = CQ \cdot CQ'$$

implies that

$$\Delta\, CPQ \sim \Delta\, CQ'P'$$

and hence

$$\angle\ CQ'P' = \angle\ CPQ\ .$$

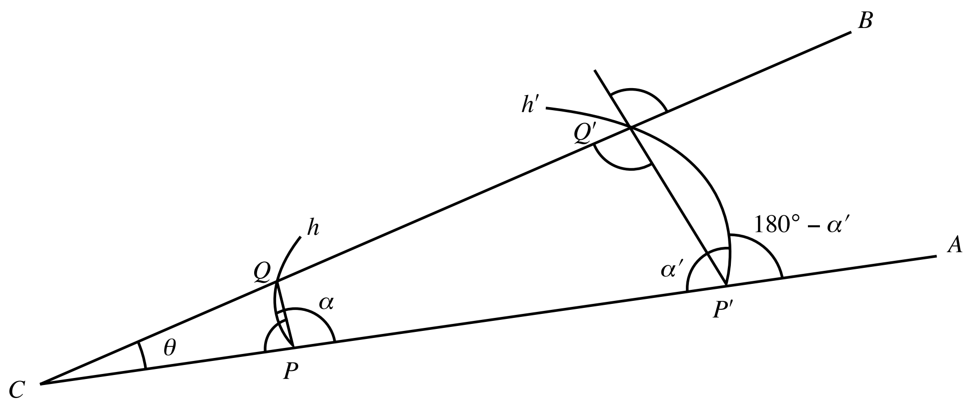

Figure 7.9

Since the limiting values, as $\theta \to 0$, of $\angle\ CQ'P'$ and $\angle\ CPQ$ are $180^{\circ} - \alpha'$ and $180^{\circ} - \alpha$ respectively, it follows that $\alpha = \alpha'$.

Q.E.D.

Two intersecting circles are said to be *orthogonal* if their respective tangents at the point of intersection are perpendicular to each other. However, by Proposition 4.1.4, the tangent line is perpendicular to the radius through the point of contact and hence it follows that two intersecting circles are orthogonal if and only if their tangents at the point of intersection pass through each other's centers (see Fig. 7.10). In fact, orthogonality is guaranteed by one of the tangents passing through the center of the other. Exercise 3 relates orthogonality to inversions.

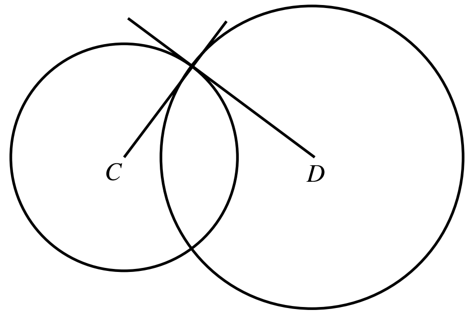

Figure 7.10 Two orthogonal circles with respective centers C *and* D.

A straight line is said to be *orthogonal* to a circle if it is perpendicular to the tangents at the intersection points. This is equivalent to saying that the straight line contains a diameter of the circle.

EXERCISES 7.1

1. If O denotes the origin, to what point or curve does the inversion $I_{O,4}$ transform the sets below?

a) The point $(3, 0)$ b) The point $(0, -2)$
c) The point $(2, 2)$ d) The point $(-1, 1)$
e) The line $y = -2x$ f) The line $x + y = 4$
g) The line $x = 4$ h) The line $y = -4$
i) The line $x = 2$ j) The line $y = -8$
k) The line $y = x + 8$ l) The line $y = -x + 4$
m) The line $y = x - 2$ n) The line $y = -x - 8$
o) The circle $(O; 3)$ p) The circle $(O; 8)$
q) The circle $((3, 0); 1)$ r) The circle $((3, 0); 6)$
s) The circle $((0, 8); 2)$ t) The circle $((0, 8); 4)$
u) The circle $((0, 8); 6)$ v) The circle $((0, 8); 8)$
w) The circle $((0, 8); 10)$ x) The circle $((4, 4); 4)$
y) The circle $((5, 5); 5)$ z) The circle $((5, 5); \sqrt{26})$

2. For each of the following pairs of curves, decide whether there exists an inversion that transforms one onto the other. Identify the inversion if it exists.

a) The x axis and the line $y = 2$

b) The circle $(O; 5)$ and the line $x = 3$

c) The circle $(O; 5)$ and the line $x = 5$

d) The circle $(O; 5)$ and the line $x = 10$

e) The circle $(O; 5)$ and the circle $(O; 10)$

f) The circle $(O; 5)$ and the circle $((5, 0); 5)$

g) The circle $(O; 5)$ and the circle $((35, 0); 30))$

3. Let p be a circle C a point and k a positive real number. Prove that $I_{C,k}(p) = p$ if and only if the circles p and $(C; k)$ are orthogonal.

4. Let I be an inversion and let p be a circle such that $I(p)$ is also a circle. When do p and $I(p)$ have different radii?

5. Let p and q be two circles with different radii. Show that there is an inversion I such that $I(p) = q$.

6. Let m be a straight line. Characterize all the circles p such that there exists an inversion I for which $I(m) = p$.

7. Let p be a circle. Characterize all the straight lines m such that there exists an inversion I for which $I(p) = m$.

8. Prove that if the radius of the circle of Figure 7.2 is k, then $I_{C,k}(P) = P'$.

9(C). Write a script that will take a circle $(C; k)$ and a point P as input and yield $I_{C,k}(P)$ as output.

10(C). Use a computer application to verify the following parts of Theorem 7.1.1: a) a b) b c) c d) d.

7.2 Inversions to the Rescue

Inversions can be very useful in transforming problems about circles into simpler problems about straight lines. Two examples of this procedure are offered.

EXAMPLE 7.2.1. Let two circles p and q intersect in A and B, and let the extensions of the diameters of p and q through B intersect q and p in the points C and D respectively. Show that the line $\overleftrightarrow{AB}$ contains a diameter of the circle that circumscribes ΔBCD.

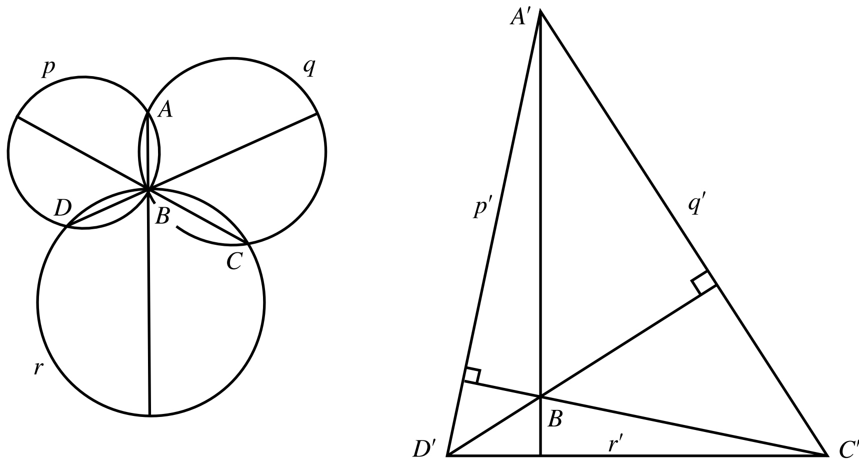

Figure 7.11

Let k be any real number and suppose the inversion $I = I_{B,k}$ is applied to the given configuration of circles so that $A' = I(A)$, $C' = I(C)$, $D' = I(D)$, $p' = I(p)$, $q' = I(q)$, $r' = I(r)$ (Fig.7.11). Since BC and BD are orthogonal to p and q respectively, it follows from Proposition 7.1.5 that $BC' \perp p'$ and $BD' \perp q'$. The concurrence of the three altitudes of the triangle (Exercise 4.2B.11) now implies that $BA' \perp r'$ and hence BA contains a diameter of r.

Proposition 7.2.3 below was first proved by Ptolemy . It was an important tool in his construction of the table of chords which appears in his definitive book on Greek astronomy, the *Almagest*. The proof of the required lemma is relegated to Exercise 1.

LEMMA 7.2.2. *Suppose* $P' = I_{C,k}(P)$ *and* $Q' = I_{C,k}(Q)$. *Then*

$$P'Q' = \frac{k^2 PQ}{CP \cdot CQ}$$

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PROPOSITION 7.2.3. *In a cyclic quadrilateral the product of the diagonals equals the sum of the products of the two pairs of opposite sides.*

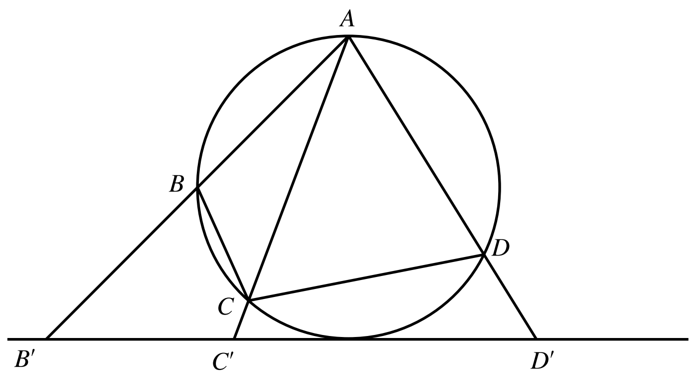

Figure 7.12

PROOF: Let $ABCD$ be a cyclic quadrilateral inscribed in a circle of diameter k (Fig. 7.12). It follows from Theorem 7.1.1b that $I_{A,k}$ inverts this circle into a tangent line that contains the points $B' = I_{A,k}(B)$, $C' = I_{A,k}(C)$, and $D' = I_{A,k}(D)$. Since $B'C' + C'D' = B'D'$ it follows from the lemma that

$$\frac{k^2BC}{AB\cdot AC}+\frac{k^2CD}{AC\cdot AD}=\frac{k^2BD}{AB\cdot AD}$$

or, upon multiplying this equation by $(AB\cdot AC\cdot AD)/k^2$,

$$BC\cdot AD + CD\cdot AB = BD\cdot AC\,.$$

Q.E.D.

In order to illustrate one more application of inversions we return to the issue of constructibility. In elementary geometry classes it is customary to construct figures using rulers and compasses. These two tools, however, are qualitatively different. The compass generates a circle because of its mechanical properties whereas the ruler simply allows us to copy a given straight line onto the paper. The circular analog of a ruler would be a coin or the lid of a jar. What then is the linear analog of the compass? In other words, what mechanical device, consisting of linked rods, would constrain a pencil to move so as to draw a straight line? Such devices are called *linkages*, the simplest one being a single rod AB with fixed point A. It is clear that a pencil attached at B would be constrained to draw a circle.

The utility of linkages in drawing curves has been studied for hundreds of years, but the first one capable of drawing a straight line was invented in 1864 by A. Peaucellier (1832 - 1913), who was an engineer in the French army. In his honor this device, which contains 7 rods, is called *Peaucellier's cell.* In 1874 Harry Hart (1848 - 1920) invented a 5 rod linkage for drawing straight lines and it is unknown whether there are any such linkages with fewer than 5 rods.

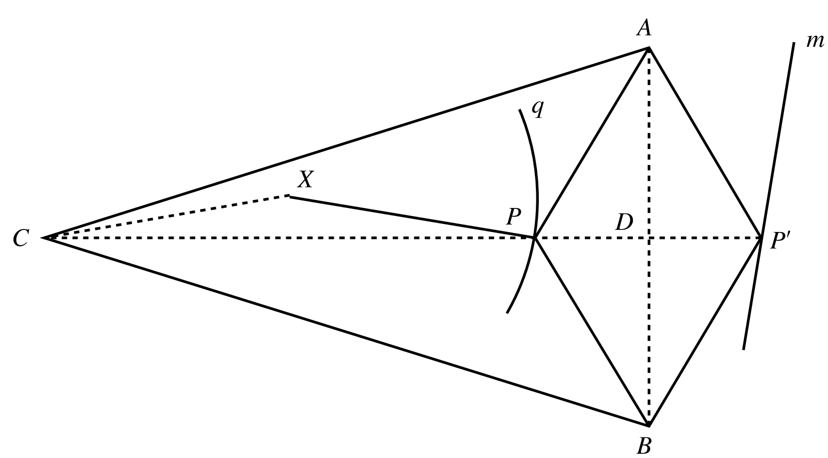

Figure 7.13 Peaucellier's cell.

Peaucellier's cell is depicted in Figure 7.13 where the points C and X are fixed, $XP = XC$, and the solid lines $XP, AP = PB = BP' = P'A$ and $CA = CB$ denote rods that are loosely linked at their endpoints. The dashed lines denote auxiliary lines that serve only for the purpose of demonstrating the properties of the linkage. Since $APBP'$ is a rhombus we have

$$CP \cdot CP' = (CD - DP)(CD + DP) = CD^2 - DP^2$$
$$= (CD^2 + DA^2) - (DP^2 + DA^2) = CA^2 + AP^2.$$

If $k = \sqrt{CA^2 + AP^2}$ then k is constant and we have

$$P' = I_{C,k}(P) .$$

Since the rod XP constrains P to move in a circle q that contains C, it follows from Proposition 7.1.1c that P' traces out a straight line m.

EXERCISES 7.2

1. Prove Lemma 7.2.2.

2. Two circles p and q have a common tangent at a point T, and a variable circle through T intersects p and q orthogonally in points P and Q. Prove that $\overset{\times}{PQ}$ passes through a fixed point.
3. Suppose $ABCD$ is a cyclic quadrilateral. If T is the point of contact of a circle containing A and B with another circle containing C and D, show that the locus of T is a circle.
4. Prove that if $ABCD$ is a convex quadrilateral, then $BC \cdot AD + CD \cdot AB \geq BD \cdot AC$. Show that equality holds if and only if $ABCD$ is cyclic.
5. Let p be a fixed circle and let P be a fixed point not on p. Prove that there exists a point P' distinct from P such that every circle through P that is orthogonal to p also passes through P'.
6. Let p be a fixed circle and P a fixed point. Show that the locus of the centers of all the circles that pass through P and are orthogonal to p is a straight line.

7(C). Use a computer application to verify Proposition 7.2.3.

7.3 Inversions as Hyperbolic Rigid Motions

In addition to the role they play in Euclidean geometry, inversions also provide a powerful tool for describing the rigid motions of non-Euclidean geometry in the context of the upper half-plane model of Section 1.2. As was the case in that section, the exposition here is informal and no proofs are given. Instead, several examples are offered that are easily implemented on computers and substantiate the discussion. Exercises 13-17 provide an opportunity for an exploration of the properties of the hyperbolic rigid motions.

It was proven in Section 6.4 that every hyperbolic rigid motion can be expressed as the composition of hyperbolic reflections. These hyperbolic reflections are now described in the context of the upper half-plane geometry.

PROPOSITION 7.3.1. *There are two kinds of reflections of the upper half-plane:*

a) Euclidean reflections whose axes are vertical;

b) Inversions whose centers are on the x-axis.

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It stands to reason that Euclidean reflections with vertical axes should also double as hyperbolic reflections. After all, the distortion of lengths that was used to create this geometry depends only on the distances from the *x*-axis, and since these particular reflections do not change these distances, it is not surprising that they constitute hyperbolic

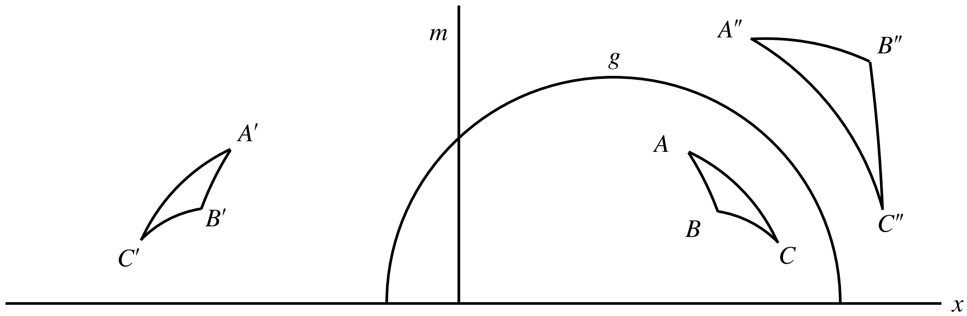

Figure 7.14 Hyperbolic reflections.

rigid motions. Figure 7.14 illustrates the effect of the reflections of both types on a triangle. Hyperbolic $\Delta\, A'B'C'$ is both the Euclidean and the hyperbolic reflection of $\Delta\, ABC$ in the vertical geodesic m, and $\Delta\, A''B''C'' = I_g(\Delta\, ABC)$ is the hyperbolic reflection of $\Delta\, ABC$ in the bowed geodesic g.

EXAMPLE 7.3.2. *Find a hyperbolic reflection that transforms the point P(1, 1) to the point Q(3, 5) (Fig. 7.15).*

The line $\overleftrightarrow{PQ}$ has equation $y - 1 = \frac{1}{2}(x - 1)$ and intersects the x-axis at the

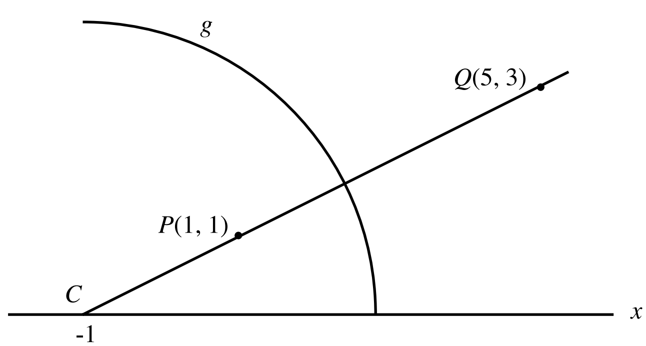

Figure 7.15

point $C(-1, 0)$. Since

$$CP \cdot CQ = \sqrt{(1 + 1)^2 + (1 - 0)^2} \cdot \sqrt{(5 + 1)^2 + (3 - 0)^2}$$

$$= \sqrt{5} \cdot \sqrt{45} = 15$$

it follows that the inversion $I_{C,\sqrt{15}}$ is the required hyperbolic reflection.

EXAMPLE 7.3.3. *Find a hyperbolic reflection that transforms the geodesic consisting of the upper half of the circle ((4, 0); 2) onto the upper half of the straight line x = -3 (Figure 7.16).*

Any inversion centered at (6, 0) will transform the given semicircle into a vertical ray. Since $(6 - 2)(6 - (-3)) = 36$, it follows that the requisite hyperbolic reflection is the inversion $I_{(6, 0),6}$ with fixed circle *g*.

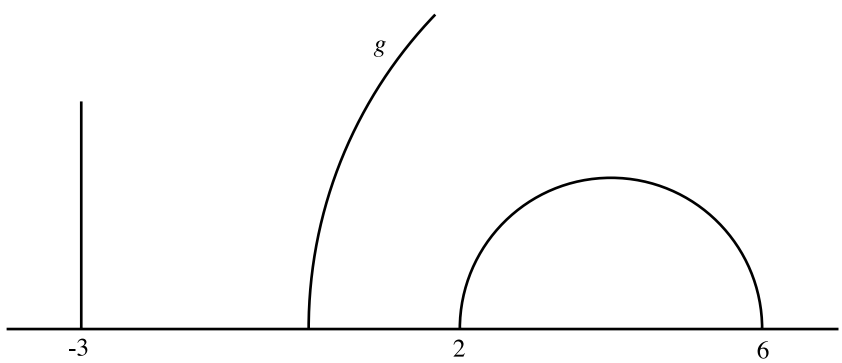

Figure 7.16 A hyperbolic reflection.

Inasmuch as the definition of a Euclidean rotation makes no reference to parallelism or any of its consequences, it can also serve as the definition of a *hyperbolic rotation*. Unfortunately, because of the distortion of distances in the upper half-plane this definition is not very helpful in trying to visualize this non-Euclidean transformation. However, Proposition 6.2.2, which states that the composition of two reflections with intersecting axes is a rotation about the point of intersection, is neutral, and so it applies here as well. Hence, if I_g denotes the inversion whose fixed circle is g (Fig. 7.17) and

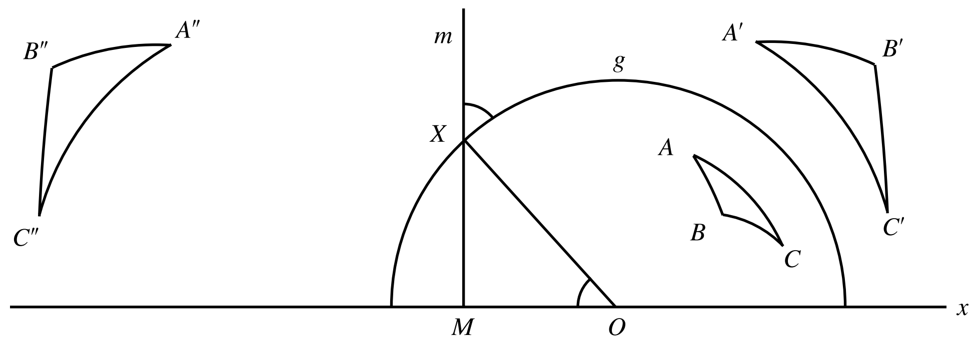

Figure 7.17 Hyperbolic reflections.

we recall that ρ_m is also a hyperbolic reflection, then $\rho_m + I_g$ is a hyperbolic rotation R about the point X. Note that

$$R(\Delta ABC) = \rho_m(I_g(\Delta ABC)) = \rho_m(\Delta A'B'C') = \Delta A''B''C'' .$$

If $g = (O; 3)$ and m is the line $x = -2$, it follows from Exercise12 that

$$\angle(g, m) = \angle XOM = \cos^{-1}(\frac{2}{3}) = 48.2^{\circ}.$$

Consequently, $R = R_{X,96.4^{\circ}}$. Figure 7.18 illustrates a hyperbolic rotation $R = R_{Y,60^{\circ}}$.

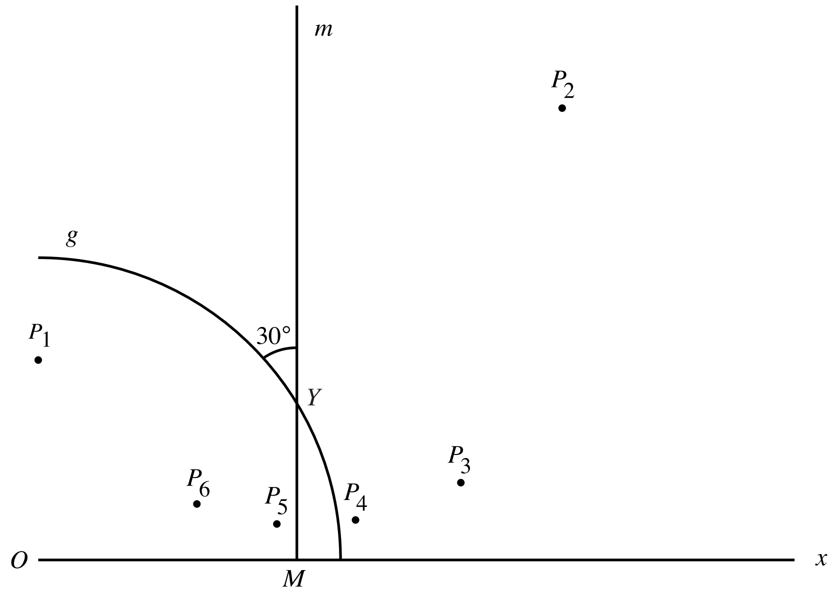

Figure 7.18 A hyperbolic rotation.

In this figure $P_{i+1} = R(P_i)$ for i = 1, 2, ..., 5 and $P_1 = R(P_6)$. Obviously, these points all lie on a hyperbolic circle centered at Y. Exercise 15(C) examines this issue further.

The definition of a Euclidean translation does involve parallelism and is therefore of no use in the hyperbolic context. Instead, motivated by Proposition 6.2.1, *a hyperbolic translation* is defined as the composition of two hyperbolic reflections whose axes do not intersect. If the axes of both the hyperbolic reflections are straight geodesics, then their composition is a horizontal Euclidean translation. This is illustrated in Figure 7.19 where $\tau = \rho_n + \rho_m$ and $\tau(P_i) = P_{i+1}$ for i = 1, 2, 3, 4 .

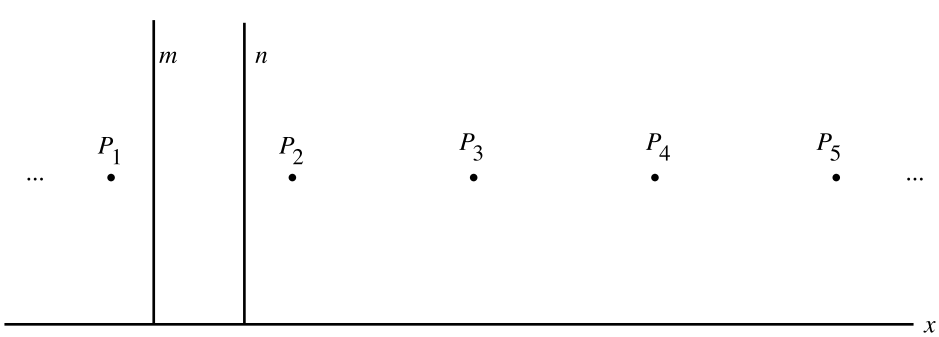

Figure 7.19 Both a Euclidean and a hyperbolic translation.

Such horizontal Euclidean translations constitute hyperbolic rigid motions for the same reason that the Euclidean reflections ρ_m and ρ_n do. They do not alter the distances of points from the x-axis.

Figure 7.20 illustrates the composition of hyperbolic reflections of mixed types with non-intersecting axes. If $\tau = I_g + \rho_m$ then $P_{i+1} = \tau(P_i)$ for $i = 1, 2, 3, 4$. Note

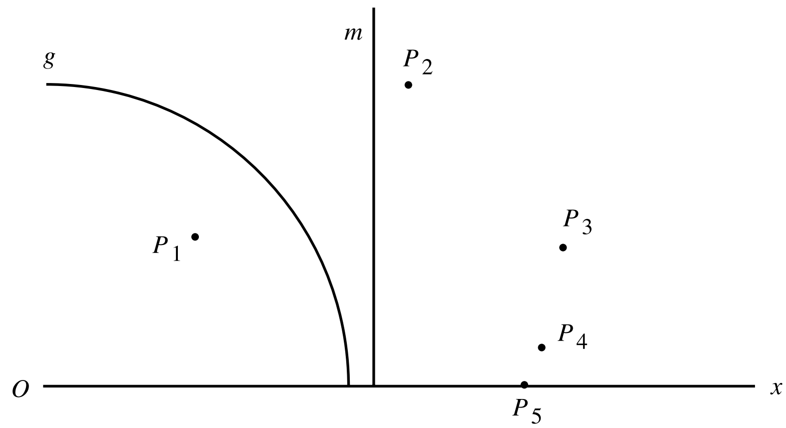

Figure 7.20 A hyperbolic translation.

that the orbit of this translation τ does go out to hyperbolic infinity just as Euclidean translations go out to Euclidean infinity. This stands in marked contrast with the orbit of the hyperbolic rotation R of Figure 7.18.

Figure 7.21 displays an orbit of the hyperbolic glide-reflection $\gamma = \tau + I_{O,5}$ where τ is the horizontal shift $(x, y) \to (x + 2, y)$.

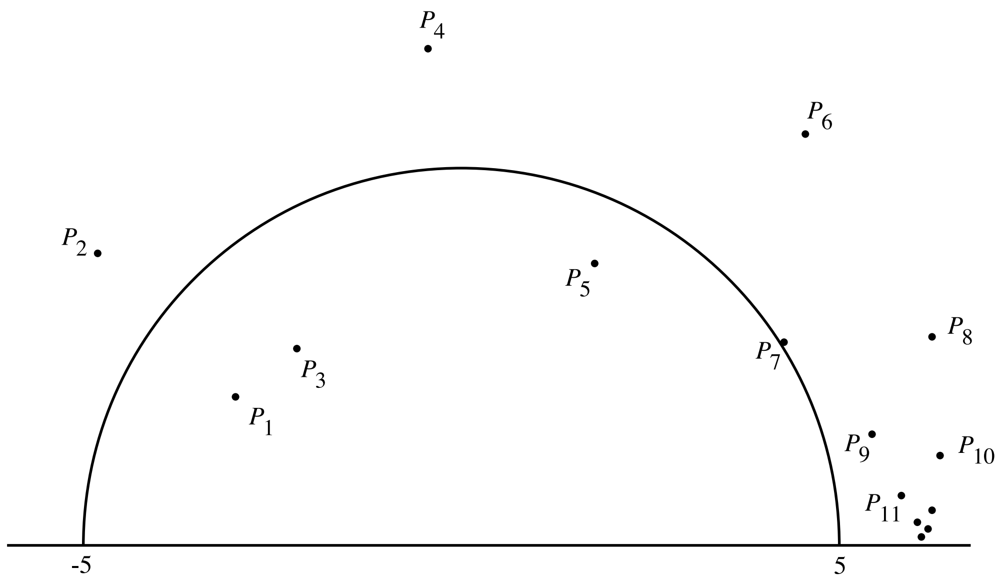

Figure 7.21 A hyperbolic glide-reflection.

Since the proof of Theorem 6.4.2 is neutral, it follows that every hyperbolic rigid motion is the composition of no more than three hyperbolic reflections. The group of all the rigid motions of the upper half-plane is one of the best studies structures of advanced mathematics. It has also proved to be an indispensable tool in such diverse areas as geometry (of course), number theory, and analysis. An elementary discussion of this topic and its surprising connection with complex numbers can be found in the author's *The Poincaré Half-Plane: A Gateway to Modern Geometry*.

Given a point P and a transformation f, the *orbit* of P generated by f is the sequence of points $P, f(P), f^2(P), f^3(P), \ldots$. Thus, the sequences $P_1, P_2, P_3, \ldots$ of Figures 7.18-21 are all orbits. These orbits help visualize the action of the transformation that generated them. It is clear that orbits of Euclidean translations are contained in straight lines whereas orbits of Euclidean rotations are circular in nature. Exercise 15 offers the reader the opportunity to explore the orbits generated by hyperbolic transformations.

EXERCISES 7.3

1. Find a hyperbolic reflection that transforms the point (1, 1) to the point (-3, 5).
2. Find a hyperbolic reflection that transforms the point (1, 1) to the point (-3, 1).
3. Find a hyperbolic reflection that transforms the upper half of $x = 4$ to the upper half of the circle $(O; 4)$.
4. Find a hyperbolic reflection that transforms the upper half of $x = 4$ to the upper half of the circle $(O; 2)$.
5. Find a hyperbolic reflection that transforms the upper half of $x = 4$ to the upper half of the circle $(O; 5)$.
6. Find a hyperbolic reflection that transforms the upper half of $x = 4$ to the upper half of $x = 17$.
7. Find a hyperbolic reflection that transforms the upper half of $(O; 5)$ to the upper half of $(O; 3)$.
8. Find a hyperbolic reflection that transforms the upper half of $(O; 5)$ to the upper half of $((9, 0); 2)$.
9. Prove that given any two points of the upper half plane, there is a hyperbolic reflection that transforms one onto the other.
10. Prove that given any two intersecting geodesics of the upper half plane there is a hyperbolic reflection that transforms one onto the other.
11. Prove that given any two non intersecting geodesics of the upper half plane there is a hyperbolic reflection that transforms one onto the other.
12. Prove that in Figure 7.17 $\angle(g, m) = \angle XOM$.

13(C). Write a script that will reflect any two points of the upper half-plane in any bowed geodesic. Use the script of Exercise 1.2.17 to substantiate the claim that this transformation is indeed a hyperbolic rigid motion.

14(C). Write a script that will reflect any triangle of the upper half-plane in any bowed geodesic.

15(C). Let P an arbitrary point of the upper half-plane, m an arbitrary vertical straight line and g an arbitrary circle centered on the x-axis. Write a script that takes P, m, and g as its input and yields several iterations of the action of $\rho_m \circ I_g$ on P. Use this script to explore the following questions:

a) When m and g intersect, what is the geometrical nature of the orbits of $\rho_m \circ I_g$?

b) What does a hyperbolic circle look like to a Euclidean observer?

c) When m and g do not intersect, what is the geometrical nature of the orbits of $\rho_m \circ I_g$? To be specific, what is the nature of each orbit of $\rho_m \circ I_g$ and how do these orbits relate to each other?

16(C). Use a computer application to model non-Euclidean glide-reflections.

17*(C). Use a computer application to explore the notion of a hyperbolic inversion.

CHAPTER REVIEW EXERCISES

1. Prove that if f is any Euclidean rigid motion and I is any inversion, then $f \circ I \circ f^{-1}$ is also an inversion.
2. Suppose F is any Euclidean rigid motion and I is any inversion. Is $I \circ f \circ I^{-1}$ necessarily a Euclidean rigid motion?
3. Is the composition of two inversions ever an inversion?
4. When is the composition of two inversions a Euclidean rigid motion?
5. Are the following statements true or false? Justify your answers.
   a) Every inversion is a rigid motion of Euclidean geometry.
   b) Some inversions are rigid motions of Euclidean geometry.
   c) Every inversion is a rigid motion of hyperbolic geometry.
   d) Some inversions are rigid motions of hyperbolic geometry.
   e) Every rigid motion of hyperbolic geometry is an inversion.
   f) Some rigid motions of hyperbolic geometry are inversions.
   g) Inversions transform circles into circles.
   h) Given a straight line, there exists no inversion that will transform it into another, distinct, straight line.
   i) Peaucellier's cell was a notorious torture chamber in the Bastille.
   j) Given any two circles there exists either a Euclidean rigid motion or an inversion that transforms one into the other.