CHAPTER 6

Planar Symmetries

As has been mentioned before, one of the most serious deficiencies in Euclid's axiomatic development of geometry was his failure to provide an explicit discussion of rigid motions, despite the fact that they play an important role in several of his proofs, beginning with that of Proposition 4 of Book I. These transformations are not mentioned in Hilbert's axiomatization either, where they are replaced by several congruence axioms. Other axiom systems, notably that of Mario Pieri (1860 - 1925), do refer to such motions explicitly. The 19th century also witnessed the creation of many alternative geometries, each with its own collection, or *group*, of rigid motions. This proliferation of geometries called for their classification and in 1872 Felix Klein (1849 - 1925) promulgated his *Erlanger Program* in which he suggested that they be classified by their groups of rigid motions.

This chapter is devoted primarily to the classification of the rigid motions of the Euclidean plane and the allied topic of planar symmetry. Some information is also obtained about the rigid motions of the hyperbolic plane.

1. Translations, Rotations, and Fixed points

Informally speaking, a rigid motion of the plane is a transformation that does not alter the distances between the points. More formally, a *rigid motion* is a function f of the plane into itself such that for any two points P and Q

$$PQ = P'Q',$$

where $P' = f(P)$ and $Q' = f(Q)$. The prototypical rigid motion is the *translation* that "slides" the plane on itself so that all straight lines remain parallel to their original positions. More precisely, given any two points A and B, the translation that carries A onto B is denoted by τ_{AB} and if P is any point then

$$\tau_{AB}(P) = Q$$

where Q is the unique point such that $AB = PQ$, $AB \parallel PQ$, and the segments AB and PQ are similarly directed. If P does not lie on AB then this, by virtue of Proposition 3.1.7, is tantamount to saying that the quadrilateral $ABQP$ is a parallelogram. In Figure 6.1 $\tau_{AB}(P_i) = Q_i$ for all i = 1, 2, 3, 4. Note that in this figure P_iP_j is both parallel and equal to Q_iQ_j whenever $i \neq j$ and hence τ_{AB} is indeed a rigid motion.

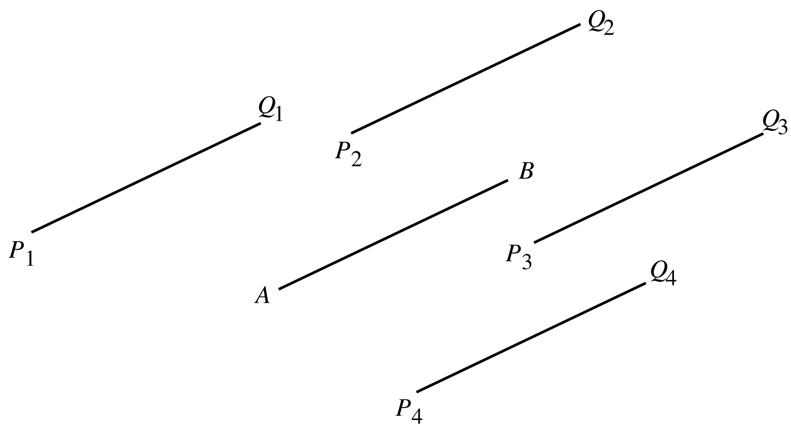

Figure 6.1 A translation.

The same translation can be represented in many different ways. Thus, the translation τ_{AB} of Figure 6.1 can also be denoted by $\tau_{P_1Q_1}$, $\tau_{P_2Q_2}$, and so on. Two rigid motions f and g are said to be *equal* provided that

$$f(P) = g(P) \quad \text{for all points } P \text{ in the plane.}$$

In other words, if the rigid motion is visualized as a physical movement of the plane, then the intermediary stages of the motion are immaterial: all that matters are the final positions of the points.

This chapter's goal is the classification of all the rigid motions of the plane and the most important tool in this text's approach is the composition of rigid motions. The reader is reminded that if f and g are functions of any set into itself, then their *composition* $g \circ f$ is a function of the same set into itself such that

$$g \circ f(P) = g(f(P)) .$$

The *identity* transformation Id is defined by the equation

$$Id(P) = P \qquad \text{for every point } P$$

and has the property that for any rigid motion f,

$$f \circ Id = Id \circ f = f.$$

The operation of composition is associative in the sense that for any three such functions $f, g,$ and $h,$

$$(f \circ g) \circ h = f \circ (g \circ h) .$$

We begin with the composition of translations.

PROPOSITION 6.1.1. If A, B, C are any points of the plane, then

$$\tau_{BC} \circ \tau_{AB} = \tau_{AC} .$$

PROOF: Let P be any point of the plane and set (see Fig. 6.2)

$$P' = \tau_{AB}(P) \qquad\qquad P'' = \tau_{BC}(P') = \tau_{BC} \circ \tau_{AB}(P) .$$

It is necessary to show that $P'' = \tau_{AC}(P)$. However, as was noted above, $ABP'P$ and

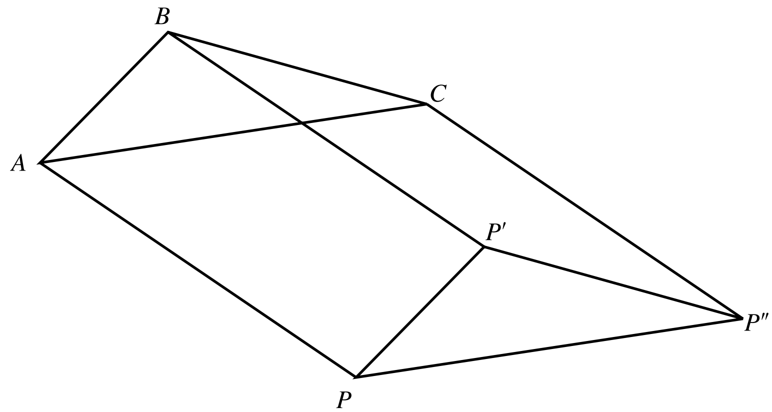

Figure 6.2 The composition of translations.

$BCP''P'$ are both parallelograms. It follows from Proposition 3.1.8 that AP and BP' are equal to and parallel to BP' and CP'' respectively. Hence, by Proposition 3.1.7 $ACP''P$ is a parallelogram and so $P'' = \tau_{AC}(P)$.

Q.E.D.

It follows from this proposition that the composition of any two translations is itself a translation. For if f and g are any translations and P is any point, then we could set $P' = f(P)$, $P'' = g(P')$ and conclude that

$$g \circ f = \tau_{P'P''} \circ \tau_{PP'} = \tau_{PP''}.$$

The *inverse* f^{-1} of the rigid motion f is a rigid motion such that

$$f \circ f^{-1} = f^{-1} \circ f = Id.$$

It is clear that for any two points A and B, $\tau_{AB}^{-1} = \tau_{BA}$.

Another type of rigid motion is the *rotation*. If C is any point of the plane and α is some directed angle, then the rotation $R_{C,\alpha}$ is the rotation that moves the general point P to the point $P' = R_{C,\alpha}(P)$ where $CP' = CP$ and $\angle P'CP = \alpha$ (Fig. 6.3). Exercise

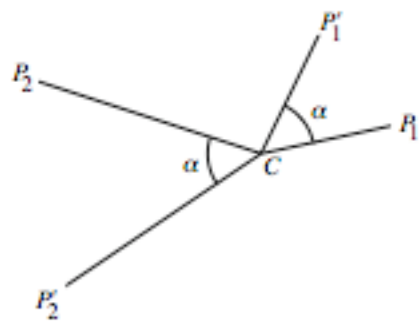

Figure 6.3 The rotation $R_{C,\alpha}$

1 calls for the formal proof of the rigidity of rotations. The point C is the *pivot point* of the rotation $R_{C,\alpha}$. The angle α of the rotation is understood to be *oriented*, in the sense that it can be either positive or negative, and the rotation accordingly proceeds either counterclockwise or clockwise. Moreover, if n is any integer and $\beta = \alpha + n{\cdot}360^{\circ}$, then $R_{c,\beta} = R_{c,\alpha}$. Consequently, in describing any rotation $R_{c,\alpha}$ the angle will generally be chosen so that $0 \le \alpha < 360^{\circ}$. Note that $R^{-1}{}_{C,\alpha} = R_{C,-\alpha}$.

The composition of the rotations $R_{C,\alpha}$ and $R_{C,\beta}$ is clearly $R_{C,\alpha+\beta}$, but what about the composition of $R_{C,\alpha}$ with $R_{D,\beta}$ where C and D are distinct points? In order to answer this natural question, it is first necessary to deal with the issue of identifying rigid motions in general. The following sequence of propositions aims to answer the question of

> *How much information is it necessary to have about a rigid motion before we can say that it is known?*

It will soon be seen that surprisingly little is needed.

PROPOSITION 6.1.2. *Every rigid motion transforms straight lines into straight lines.*

PROOF: Let f be a rigid motion, let m be a straight line with two distinct points A and B on it, and set $A' = f(A)$ and $B' = f(B)$ (Fig 6.4). If P is any point of m between A

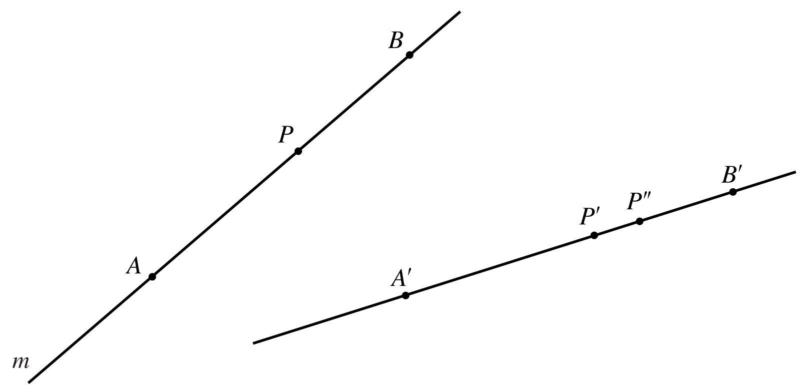

Figure 6.4

$$A'P + P'B' = AP + PB = AB = A'B'.$$

and B, and $P' = f(P)$, then it follows from Proposition 2.3.25 that P' is on the line segment $A'B'$. A similar argument (Exercise 5) demonstrates that as long as P is on m then P' is on the line $\overleftrightarrow{A'B'}$ even when P is not between A and B.

Conversely, let P' be any point of $\overleftrightarrow{A'B'}$ that lies on the line segment $A'B'$. Since $A'B' = AB$ there is a unique point P of m such that

$$AP = A'P' \quad and \quad BP = B'P'.$$

If $P'' = f(P)$, then

$$A'P'' = AP = A'P' \quad and \quad B'P'' = BP = B'P'$$

so that P' and $P'' = f(P)$ must be identical. The same holds even when P' is on $\overleftrightarrow{A'B'}$ but not between A' and B'. This means that every point of $\overleftrightarrow{A'B'}$ is covered by some point of m. In other words, $f(m) = \overleftrightarrow{A'B'}$.

Q.E.D.

PROPOSITION 6.1.3. *If two rigid motions agree on two distinct points then they agree at every point of the straight line joining them.*

PROOF: Let f and g be two rigid motions and A and B two distinct points such that

$$f(A) = g(A) = A' \quad \text{and} \quad f(B) = g(B) = B'.$$

If P is any point of AB, then, by Proposition 6.1.2, $f(P)$ and $g(P)$ are both points of $A'B'$ whose distances from A' and B' are respectively equal. It follows that $f(P) = g(P)$.

Q.E.D.

THEOREM 6.1.4. *If two rigid motions agree at three noncollinear points then they agree everywhere.*

PROOF: Let f and g be two rigid motions that agree at the three noncollinear points A, B, C. By Proposition 6.1.3, f and g agree at every point on the straight lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$. If P is any point of the plane then there clearly exists a straight line through P that intersects the union of these three straight lines in some two distinct points X and Y. Since f and g agree at X and Y, it follows from Proposition 6.1.3 that they must also agree at P.

Q.E.D.

Thus, in order to pin down a rigid transformation it suffices to know how it affects some triple of noncollinear points.

A *fixed point* of the transformation f is a point P such that

$$f(P) = P .$$

It is clear that the point C is a fixed point of the rotation $R_{C,\alpha}$ and is in fact the only fixed point of that rotation. It is equally clear that, with the exception of the identity, translations have no fixed points whatsoever. On the other hand, every point is a fixed point of the identity. The following corollary is an immediate consequence of Theorem 6.1.4.

COROLLARY 6.1.5. *If a rigid motion fixes three noncollinear points, then it must be the identity.*

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EXERCISES 6.1

1. Prove that every rotation is a rigid motion.
2. Prove that every rigid motion transforms circles into circles.
3. Prove that if A, B, C are any three points, then $\tau_{CA} \circ \tau_{BC} \circ \tau_{AB} = Id$.
4. Let ΔABC be a clockwise triangle with oriented interior angles α, β, γ at A, B, C respectively. Prove that $R_{C,2\gamma} \circ R_{B,2\beta} \circ R_{A,2\alpha} = Id$.
5. Complete the proof of Proposition 6.1.2 by providing the details for the case where P is on the infinite line AB but outside the segment AB.
6. Let $A(a_1, a_2)$ and $B(b_1, b_2)$ be two points. Explain why the transformation $f(P) = Q$ that takes the point $P(x, y)$ to the point $Q(x', y')$ where

$$x' = x + b_1 - a_1$$
$$y' = y + b_2 - a_2$$

is in fact the translation τ_{AB}.

7. Let α be an angle. Explain why the transformation $f(P) = Q$ that maps the point $P(x, y)$ to the point $Q(x', y')$ where

$$\begin{aligned} x' &= x \cos \alpha - y \sin \alpha \\ y' &= x \sin \alpha + y \cos \alpha \end{aligned}$$

is in fact the rotation $R_{O,\alpha}$, where O is the origin.

8. Prove that rigid motions preserve angles. In other words show that if f is a rigid motion and m and n are straight lines that form an angle of measure α, then $f(m)$ and $f(n)$ are also straight lines that form an angle of measure α.

2. Reflections

Given a straight line m, the *reflection* ρ_m is the transformation that fixes every point of m and associates to each point P not on m the unique point P' such that m is the perpendicular bisector of PP' (see Fig. 6.5 and Exercise 28). It follows directly from the definition that $\rho_m \circ \rho_m = Id$ and hence $\rho_m^{-1} = \rho_m$. This text's classification of the

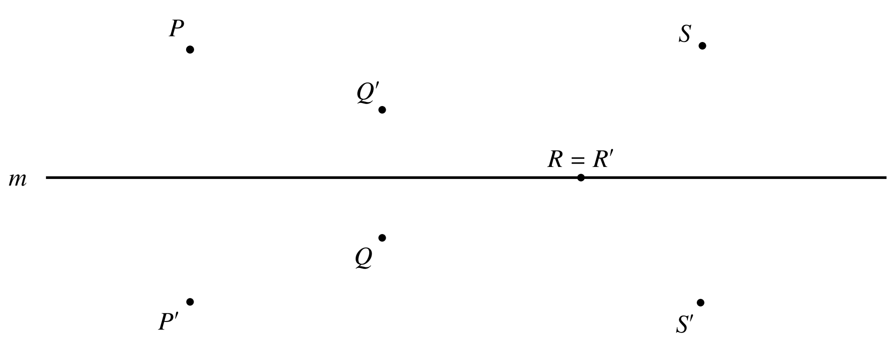


Figure 6.5

rigid motions is based on the fact that these reflections are the building blocks of all the rigid motions in the sense that every rigid motion can be expressed as the composition of some reflections. The next two propositions show that such is indeed the case for translations and rotations.

PROPOSITION 6.2.1. *Let m and n be two parallel straight lines. Let AB be a directed line segment that first intersects m and then n and whose length is twice the distance between A and B. Then*

a) $\rho_n \circ \rho_m = \tau_{AB}$,

b) $\rho_n \circ \tau_{AB} = \rho_m$,

c) $\tau_{AB} \circ \rho_m = \rho_n$.

PROOF: Let P be any point outside the infinite strip bounded by m and n such that the

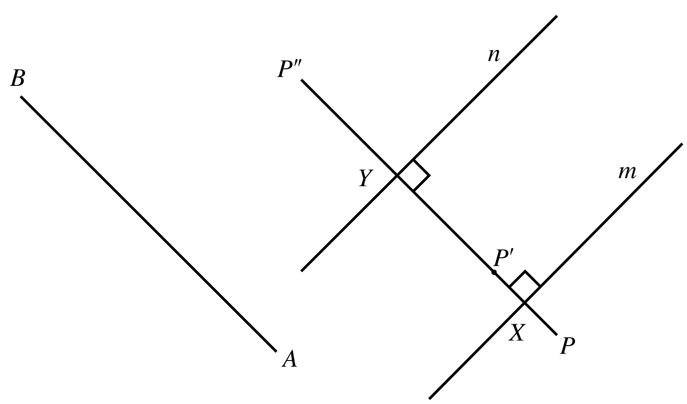


Figure 6.6

distance from P to m is less than the distance between m and n (Fig. 6.6). Set

$$P' = \rho_m(P) \quad \text{and} \quad P'' = \rho_n(P').$$

It is clear that P, P', and P'' are collinear and that

$$PP'' = PP' + P'P'' = 2XP' + 2P'Y = 2XY$$

$$= \text{twice the distance between } m \text{ and } n.$$

Hence

$$\rho_n + \rho_m(P) = \rho_n(P') = P'' = \tau_{PP''}(P) \qquad (1).$$

Since it is easy to find three noncollinear positions of P that satisfy the constraints specified in the beginning of this proof, it follows that Equation (1) holds for three noncollinear points and hence, by Theorem 6.1.4, $\rho_m + \rho_n = \tau_{PP''}$. This completes the proof of part a. Parts b and c follow immediately, since

$$\rho_n \circ \tau_{AB} = \rho_n \circ (\rho_n \circ \rho_m) = (\rho_n \circ \rho_n) \circ \rho_m = Id \circ \rho_m = \rho_m ,$$

$$\tau_{AB} \circ \rho_m = (\rho_n \circ \rho_m) \circ \rho_m = \rho_n \circ (\rho_m \circ \rho_m) = \rho_n \circ Id = \rho_n .$$

Q.E.D.

Conversely, given any translation τ_{AB}, there clearly exist two parallel straight lines that are perpendicular to $\overleftrightarrow{AB}$ and whose distance from each other equals half of AB. By the above proposition, either $\tau_{AB} = \rho_m \circ \rho_n$ or $\tau_{AB} = \rho_n \circ \rho_m$ and in either case the arbitrary translation τ_{AB} has been expressed as the composition of two reflections. This expression is, of course, not unique, since m can be any line that is perpendicular to AB.

PROPOSITION 6.2.2. *Let m and n be two straight lines that intersect at a point A, and let α be the counterclockwise angle from m to n at A. Then*

$$\rho_n \circ \rho_m = R_{A,2\alpha}.$$

PROOF: Let P be a point outside $\angle BAC = \alpha$ (Fig. 6.7) but close enough to m so that $P' = \rho_m(P)$ is inside the angle. Set $P'' = \rho_n(P') = \rho_n \circ \rho_m(P)$. Then m bisects $\angle PAP'$ and n bisects $\angle P'AP''$. Consequently,

$$\angle PAP'' = 2\angle BAP' + 2\angle P'AC = 2\alpha.$$

Hence

$$R_{A,2\alpha}(P) = P'' = \rho_n \circ \rho_m(P) \qquad (2).$$

Since it is easy to find three noncollinear positions of P that satisfy the constraints specified in the beginning of this proof, it follows that Equation (2) holds for three noncollinear points and hence, by Theorem 6.1.4, $\rho_n \circ \rho_m = R_{A,2\alpha}$.

Q.E.D.

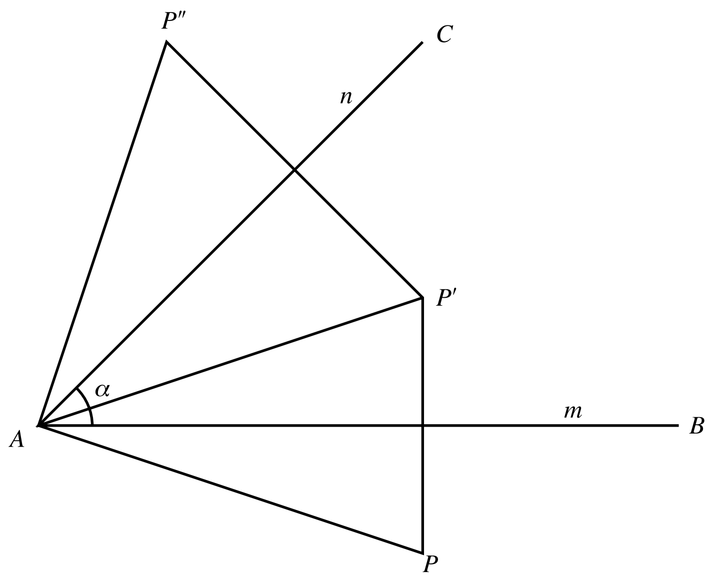


Figure 6.7

It was noted above that the composition of rotations that share their pivot points is a rotation about the same point but that the nature of the composition of rotations with distinct pivot points was unclear. We are now ready to dispose of this and other similar issues.

PROPOSITION 6.2.3. *Let A and B be two points and let α and β be two oriented angles. Then the composition $R_{B,\beta} \circ R_{A,\alpha}$ is*

a) *a translation if $\alpha + \beta$ is a multiple of 360^o,*

b) *a rotation $R_{\cdot,\alpha+\beta}$ if $\alpha + \beta$ is not a multiple of 360^o.*

PROOF: This is obvious if A and B are identical points as well as when either α or β is zero. It is therefore assumed that A and B are distinct, and neither α nor β is zero. Let $m = \overleftrightarrow{AB}$, let k be the line through A such that the oriented angle from k to m is $\alpha/2$, let n be the line through B such that the oriented angle from m to n is $\beta/2$ (Fig. 6.8). Then, by the above proposition,

$$R_{B,\beta} \circ R_{A,\alpha} = (\rho_n \circ \rho_m) \circ (\rho_m \circ \rho_k) = \rho_n \circ (\rho_m \circ \rho_m) \circ \rho_k = \rho_n \circ \rho_k \ .$$

which is either a translation or a rotation, depending on whether the lines k and n are parallel or not. However, these lines are parallel if and only if $\alpha/2 + \beta/2$ is a multiple of 180° which is of course equivalent to $\alpha + \beta$ being a multiple of 360°. Hence, by Proposition 6.2.1, the composition is a translation if $\alpha + \beta$ is a multiple of 360°. When $\alpha + \beta$ is not such a multiple, then, by Proposition 6.2.2, the composition is the rotation $R_{X,\alpha+\beta}$.

Q.E.D.

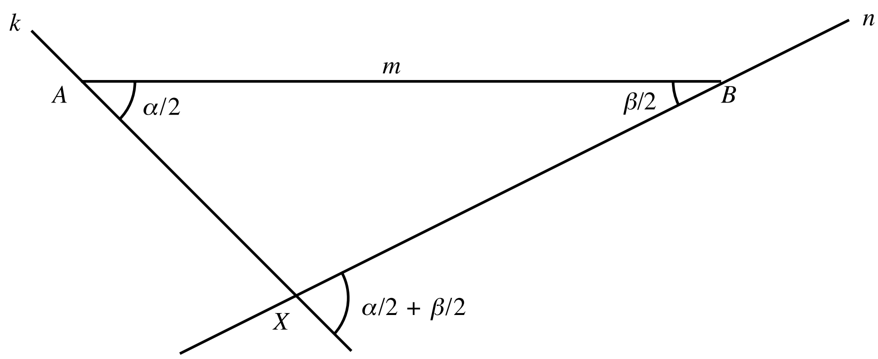


Figure 6.8

EXAMPLE 6.2.4. *Given any two points A and B, identify* $R_{B,60^o} \circ R_{A,60^o}$.

It follows from Proposition 6.2.3 that this composition is a rotation $R_{C,120^o}$. The pivot point C is located as follows: Set (see Fig. 6.9)

$$A' = R_{C,120^o}(A) = R_{B,60^o} \circ R_{A,60^o}(A) = R_{B,60^o}(A) .$$

Then C is that unique point such that $\Delta\, ACA'$ is isosceles with vertex angle $\angle\, ACA' = 120^{\circ}$. In other words, C is the center of the equilateral $\Delta\, AA'B$.

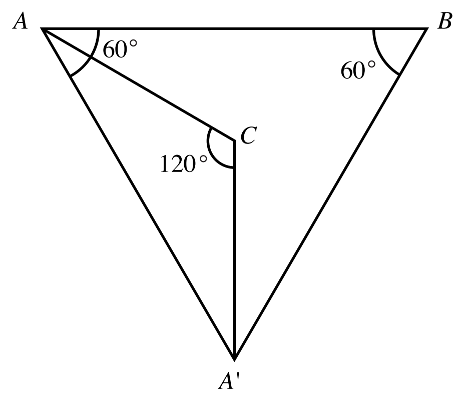


Figure 6.9

PROPOSITION 6.2.5. *Let R be a rotation which is not the identity and let τ be a translation. Then both $R + \tau$ and $\tau + R$ are rotations with the same angle as R.*

PROOF: Suppose $R = R_{A,\alpha}$, $A' = \tau(A)$ and let B be the midpoint of the segment AA' (Figure 6.10). Let k and m be the lines through B and A respectively, that are

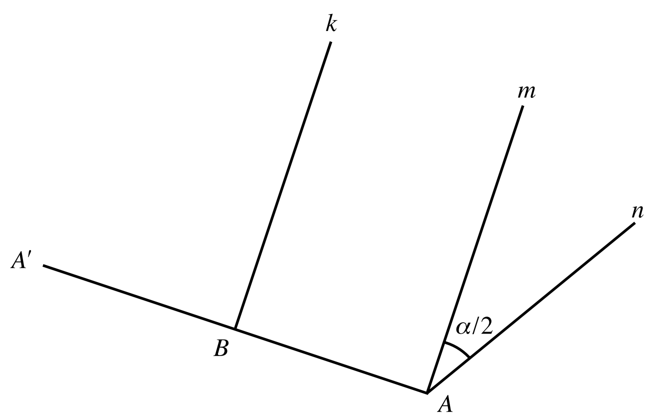


Figure 6.10

perpendicular to AA', and let n be the line through A such that the oriented angle from n to m is equal to $\alpha/2$. Then

$$\tau \circ R = (\rho_k \circ \rho_m) \circ (\rho_m \circ \rho_n) = \rho_k \circ (\rho_m \circ \rho_m) \circ \rho_n = \rho_k \circ \rho_n ,$$

which is a rotation by angle α because k and n, when extended, intersect in an angle of $\alpha/2$.

The proof that $R + \tau$ is also a rotation is relegated to Exercise 25.

Q.E.D.

EXAMPLE 6.2.6. *For the two given points A and B of Figure 6.11 identify both $\tau_{AB} \circ R_{A,90^o}$ and $R_{A,90^o} \circ \tau_{AB}$.*

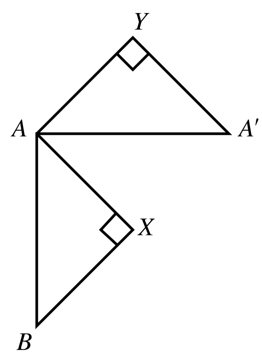


Figure 6.11

By Proposition 6.2.5 $\tau_{AB} \circ R_{A,90^o}$ is a 90^o rotation such that

$$\tau_{AB} \circ R_{A,90^o}(A) = \tau_{AB}(A) = B .$$

It follows that the pivot point of $\tau_{AB} \circ R_{A,90^o}$ is that point X such that ΔABX is an isosceles right triangle. In other words, $\tau_{AB} \circ R_{A,90^o} = R_{X,90^o}$. Similarly, $R_{A,90^o} \circ \tau_{AB}$ is a 90^o rotation such that

$$R_{A,90^o} \circ \tau_{AB}(A) = R_{A,90^o}(B) = A' .$$

It follows that the pivot point of $R_{A,90^o} \circ \tau_{AB}$ is the point Y where $\Delta AA'Y$ is an isosceles right triangle. In other words, $R_{A,90^o} \circ \tau_{AB} = R_{Y,90^o}$. Note that the two compositions $\tau_{AB} \circ R_{A,90^\circ}$ and $R_{A,90^\circ} \circ \tau_{AB}$ are not equal. In general, rigid motions do not commute.

EXERCISES 6.2

Identify the compositions of Exercises 1-18, where ABCD is the square of Figure 6.12.

1.	$R_{A,90^\circ} \circ R_{B,90^\circ}$	2.	$R_{B,90^\circ} \circ R_{A,90^\circ}$	3.	$R_{C,180^\circ} \circ R_{A,90^\circ}$
4.	$R_{A,90^\circ} \circ \tau_{BC}$	5.	$R_{A,90^\circ} \circ \tau_{CA}$	6.	$\tau_{CA} \circ R_{A,90^\circ}$
7.	$\tau_{BC} \circ \tau_{BA}$	8.	$\tau_{BC} \circ \tau_{AD}$	9.	$\tau_{DA} \circ \tau_{BC}$
10	$R_{A,270^\circ} \circ R_{C,90^\circ}$	11.	$R_{A,180^\circ} \circ R_{D,180^\circ}$	12.	$R_{A,45^\circ} \circ R_{C,135^\circ}$
13.	$R_{A,45^\circ} \circ R_{B,45^\circ}$	14.	$R_{A,60^\circ} \circ R_{B,120^\circ}$	15.	$R_{D,120^\circ} \circ R_{C,120^\circ}$
16.	$R_{A,90^\circ} \circ R_{B,30^\circ}$	17.	$\tau_{AB} \circ R_{A,60^\circ}$	18.	$R_{B,60^\circ} \circ \tau_{AB}$

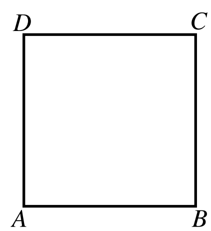


Figure 6.12

19. Let ΔABC be a clockwise triangle with oriented interior angles α, β, γ *at* A, B, C respectively. Use Proposition 6.2.2 to prove that $R_{C,2\gamma} \circ R_{B,2\beta} \circ R_{A,2\alpha} = Id$.

20. Let $A_1, A_2, \dots, A_n$ be the clockwise successive vertices of a polygon with n sides. If the interior angle at A_i is α_i, Identify the composition $R_{A_n,2\alpha_n} \circ \dots \circ R_{A_2,2\alpha_2} \circ R_{A_1,2\alpha_1}$.

21. Let $A_1, A_2, \dots, A_n$ be the midpoints of the successive sides of a polygon with n sides. Identify the composition $R_{A_n,\pi} \circ \dots \circ R_{A_2,\pi} \circ R_{A_1,\pi}$ if

    a) $n = 3$ b) $n = 4$ c) n is an arbitrary positive integer.

22. Let n be an even integer and let $A_1, A_2, \dots, A_n$ be the successive vertices of a regular n-sided polygon, and let m_i be the bisector of the interior angle at A_i. Identify the composition $\rho_{m_n} \circ \dots \circ \rho_{m_2} \circ \rho_{m_1}$.

23. Let P be any point on the straight line m and let θ be any angle. Prove that both $R_{P,\theta} \circ \rho_m$ and $\rho_m \circ R_{P,\theta}$ are reflections. What are their axes?

24. Prove that if P is a point on the straight line m then $\rho_m \circ R_{P,\theta} \circ \rho_m = R_{P,-\theta}$.

25. Complete the proof of Proposition 6.2.5 by showing that $R + \tau$ is also a rotation with the same angle as α.

26. Let $ABCD$ be a cyclic quadrilateral. Identify the composition $\rho_{DA} \circ \rho_{CD} \circ \rho_{BC} \circ \rho_{AB}$.

27. Let α be an angle and let m be the straight line through the origin with inclination α to the positive x axis. Explain why the transformation $f(P) = Q$ that maps the point $P(x, y)$ to the point $Q(x', y')$, where

$$\begin{aligned} x' &= x \cos 2\alpha + y \sin 2\alpha \\ y' &= x \sin 2\alpha - y \cos 2\alpha \end{aligned}$$

is in fact the reflection ρ_m.

28. Prove that every reflection is a rigid motion.

3. Glide-reflections

So far reflections have been used merely in order to explain how translations and rotations interact under compositions. We now examine how these two types interact with reflections. A special case of this issue was resolved by parts b and c of Proposition 6.2.1 wherein it was proved that the composition of a reflection with a translation whose direction is perpendicular to the direction of the translation is another reflection with an axis parallel to that of the given reflection. The composition of a reflection with a rotation whose pivot point lies on the reflection's axis is also a reflection (Exercise 6.2.23). However, in general, the composition of either a translation or a rotation with a reflection forms a new kind of rigid motion.

Let A and B be two distinct points. The composition $\rho_{AB} \circ \tau_{AB}$ is called a *glide-reflection* and is denoted by γ_{AB}. It is easily seen that the reverse composition $\tau_{AB} \circ \rho_{AB}$ also equals γ_{AB} and that the inverse of γ_{AB} is γ_{BA} (Fig. 6.13). In order to simplify the statements of some of the subsequent propositions, reflections will be considered as special cases of glide-reflections. The line $\overleftrightarrow{AB}$ is called the *axis* of the glide-reflection γ_{AB}, and it is easily seen that for any point P not on the axis A, the line segment joining P to $\gamma_{AB}(P)$ is bisected by $\overleftrightarrow{AB}$ (see Fig. 6.13 and Exercise 25).

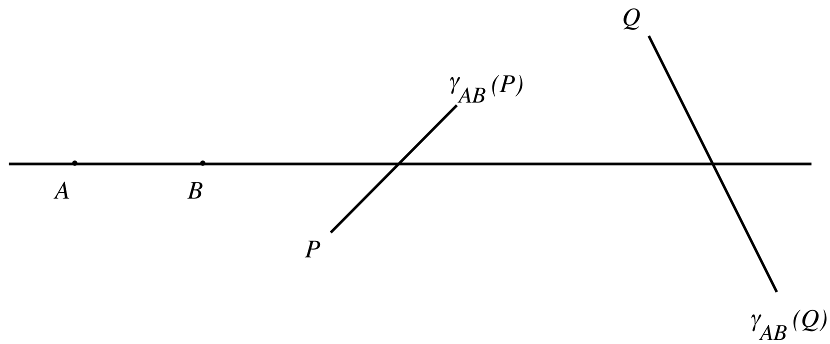


Figure 6.13

PROPOSITION 6.3.1. *Let τ be any translation and γ any glide-reflection. Then $\gamma \circ \tau$ and $\tau \circ \gamma$ are both glide-reflections.*

PROOF: Suppose $\tau = \tau_{AB}$.

If $\gamma = \rho_{AB}$, then clearly $\gamma \circ \tau = \tau \circ \gamma = \gamma_{AB}$.

If $\gamma = \rho_m$ where $m \parallel AB$ then there exist points A', B' on m such that $\tau = \tau_{AB} = \tau_{A'B'}$. Consequently, by the previous argument,

$$\gamma \circ \tau = \rho_{A'B'} \circ \tau_{A'B'} = \gamma_{A'B'} = \tau_{A'B'} \circ \rho_{A'B'} = \tau \circ \gamma.$$

If $\gamma = \rho_m$ where $m \perp AB$, then this proposition follows from Proposition 6.2.1bc.

If $\gamma = \rho_m$ where m is skew to AB, let C be a point such that $AC \parallel m$ and

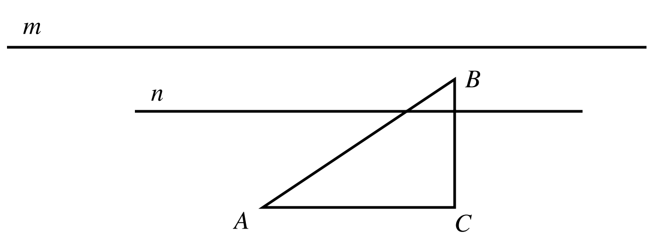


Figure 6.14

$BC \perp m$ (Fig. 6.14). By Proposition 6.1.1,

$$\gamma \circ \tau = \rho_m \circ \tau_{AB} = \rho_m \circ (\tau_{CB} \circ \tau_{AC}) = (\rho_m \circ \tau_{CB}) \circ \tau_{AC} \ .$$

By Proposition 6.2.1b, there is a line $n \perp BC$ such that $\rho_m \circ \tau_{CB} = \rho_n$, and hence

$$\gamma \circ \tau = \rho_n \circ \tau_{AC},$$

which, since $n \parallel AC$, is known to be a glide-reflection. The proof that $\tau \circ \gamma$ is also a glide reflection is relegated to Exercise 23.

Finally, let γ be an arbitrary glide-reflection. If $\gamma = \gamma_{CD} = \rho_{CD} \circ \tau_{CD}$ then, by Proposition 6.1.1,

$$\gamma \circ \tau = (\rho_{CD} \circ \tau_{CD}) \circ \tau_{AB} = \rho_{CD} \circ (\tau_{CD} \circ \tau_{AB}) = \rho_{CD} \circ \tau$$

for some translation τ. This, however, is known to be a glide-reflection. The proof that $\tau \circ \gamma$ is also a glide-reflection is relegated to Exercise 23.

Q.E.D.

EXAMPLE 6.3.2. *Identify the compositions* $\gamma_{AD} \circ \tau_{AB}$ *and* $\tau_{AB} \circ \gamma_{AD}$ *where* $ABCD$ *is the square of Figure 6.15.*

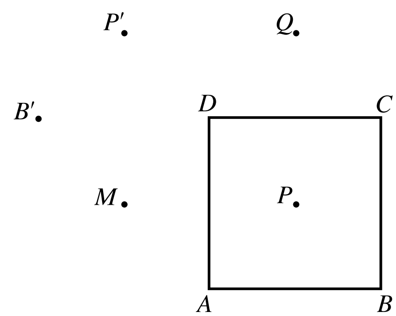


Figure 6.15

By the previous proposition, these compositions are both glide-reflections. Moreover, since

$$\gamma_{AD} \circ \tau_{AB}(A) \;=\; \gamma_{AD}(B) \;=\; B'$$

it follows that the axis of $\gamma_{AD} + \tau_{AB}$ must contain the midpoint M of the segment AB'. In addition,

$$\gamma_{AD} \circ \tau_{AB}(M) \;=\; \gamma_{AD}(P) \;=\; P'.$$

It follows that MP' is the axis of $\gamma_{AD} \circ \tau_{AB}$ and in fact $\gamma_{AD} \circ \tau_{AB} = \gamma_{MP'}$.

Again,

$$\tau_{AB} \circ \gamma_{AD}(A) \;=\; \tau_{AB}(D) \;=\; C$$

and hence the axis of $\tau_{AB} \circ \gamma_{AD}$ contains the midpoint P of AC. In addition,

$$\tau_{AB} \circ \gamma_{AD}(P) \;=\; \tau_{AB}(P') \;=\; Q$$

and hence $\tau_{AB} \circ \gamma_{AD} \;=\; \gamma_{PQ}$.

PROPOSITION 6.3.3. *Let R be any rotation and γ any glide-reflection. Then both $\gamma \circ R$ and $R \circ \gamma$ are glide-reflections.*

PROOF: Let $R = R_{A,\alpha}$ and suppose first that $\gamma = \rho_k$. Let m be the straight line through A that is parallel to k, and let n be the straight line through A such that the oriented angle from n to m is $\alpha/2$ (Figure 6.16). Then

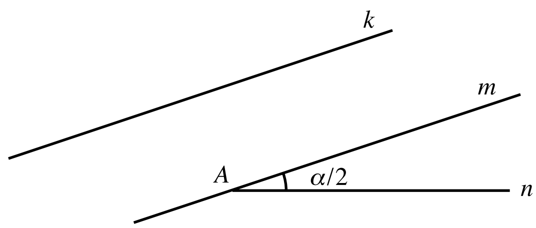


Figure 6.16

$$\gamma \circ R = \rho_k \circ R_{A,\alpha} = \rho_k \circ (\rho_m \circ \rho_n) = (\rho_k \circ \rho_m) \circ \rho_n .$$

Since $k \parallel m$ it follows from Proposition 6.2.1 that $\rho_k \circ \rho_m$ is a translation and hence, by Proposition 6.3.1, $\gamma \circ R = (\rho_k \circ \rho_m) \circ \rho_n$ is a glide-reflection.

If γ is the arbitrary glide-reflection $\tau_{CD} \circ \rho_{CD}$ then

$$\gamma \circ R = (\tau_{CD} \circ \rho_{CD}) \circ R_{A,\alpha} = \tau_{CD} \circ (\rho_{CD} \circ R_{A,\alpha}) .$$

By the first part of the proof $\rho_{CD} \circ R_{A,\alpha}$ is a glide-reflection and hence it follows from Proposition 6.3.1 that $\gamma \circ R = \tau_{CD} \circ (\rho_{CD} \circ R_{A,\alpha})$ is also a glide-reflection.

The proof that $R \circ \gamma$ is a glide reflection is relegated to Exercise 24.

Q.E.D.

EXAMPLE 6.3.4. *Identify the composition $\gamma_{AD} \circ R_{A,90^o}$ and $R_{A,90^o} \circ \gamma_{AD}$ where $ABCD$ is the square of Figure 6.17.*

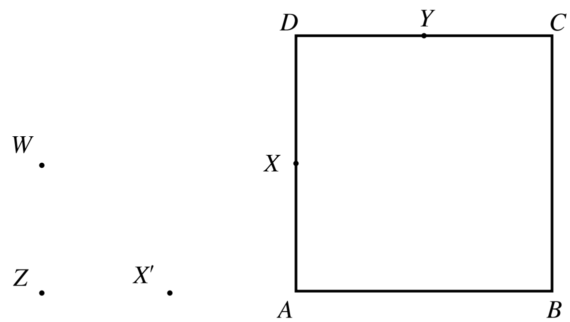


Figure 6.17

By the previous proposition, both of these compositions are glide-reflections. Moreover,

$$\gamma_{AD} \circ R_{A,90^o}(A) = \gamma_{AD}(A) = D$$

so that the axis of this composition contains the midpoint X of AD. Since

$$\gamma_{AD} \circ R_{A,90^o}(X) = \gamma_{AD}(X') = Y\,,$$

it follows that $\gamma_{AD} \circ R_{A,90^o} = \gamma_{XY}$. Similarly,

$$R_{A,90^o} \circ \gamma_{AD}\,(A) = R_{A,90^o}(D) = Z$$

so that the axis of this composition contains the midpoint X' of AZ. Since

$$R_{A,90^o} \circ \gamma_{AD}\,(X') = R_{A,90^o}(Y) = W\,,$$

it follows that $R_{A,90^o} \circ \gamma_{AD} = \gamma_{X'W}$.

PROPOSITION 6.3.5. *Let γ_{AB} and γ_{CD} be two glide-reflections. The composition $\gamma_{AB} \circ \gamma_{CD}$ is*

a) *a translation if $AB \parallel CD$*

b) *a rotation of angle 2α otherwise, where α is the oriented angle from CD to AB.*

PROOF: Note that

$$\gamma_{AB} \circ \gamma_{CD} = (\tau_{AB} \circ \rho_{AB}) \circ (\rho_{CD} \circ \tau_{CD}) = \tau_{AB} \circ (\rho_{AB} \circ \rho_{CD}) \circ \tau_{CD}$$

$$= \tau_{AB} \circ f \circ \tau_{CD}$$

where, by Propositions 6.2.1-2, f is a translation if $AB \parallel CD$ and a rotation by angle 2α otherwise. The desired results now follow from Proposition 6.1.1 in the first case and from Proposition 6.2.5 in the second case.

Q.E.D.

EXAMPLE 6.3.6. *Identify the compositions $\gamma_{AD} \circ \gamma_{AB}$ and $\gamma_{AB} \circ \gamma_{CD}$ where $ABCD$ is the square of Figure 6.18.*

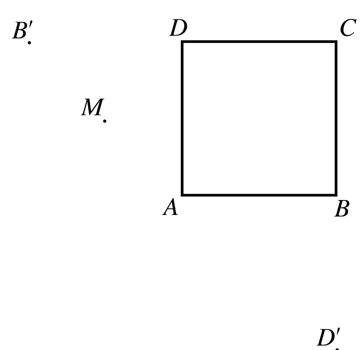


Figure 6.18

By Proposition 6.3.5b, $\gamma_{AD} \circ \gamma_{AB}$ is a 180° rotation. Since

$$\gamma_{AD} \circ \gamma_{AB}(A) = \gamma_{AD}(B) = B'$$

it follows that the pivot point of this rotation is the midpoint M of AB'. Hence $\gamma_{AD} \circ \gamma_{AB} = R_{M,180^\circ}$. By Proposition 6.3.5a, $\gamma_{AB} \circ \gamma_{CD}$ is a translation. Since

$$\gamma_{AB} \circ \gamma_{CD}(C) = \gamma_{AB}(D) = D',$$

it follows that $\gamma_{AB} \circ \gamma_{CD} = \tau_{CD'}$.

EXERCISES 6.3

Identify the compositions of Exercises 1-18, where ABCD is the square of Figure 6.12.

1.	$R_{D,90^\circ} \circ \gamma_{DC}$	2.	$\gamma_{DC} \circ R_{D,90^\circ}$	3.	$R_{P,180^\circ} \circ \gamma_{AB}$
4.	$\gamma_{AB} \circ R_{P,180^\circ}$	5.	$\tau_{AB} \circ \gamma_{DC}$	6.	$\gamma_{CD} \circ \tau_{AB}$
7.	$\tau_{AB} \circ \gamma_{BC}$	8.	$\gamma_{BC} \circ \tau_{BA}$	9.	$\gamma_{AD} \circ \gamma_{BC}$
10.	$\gamma_{AD} \circ \gamma_{CB}$	11.	$\gamma_{BA} \circ \gamma_{BC}$	12.	$\gamma_{CB} \circ \gamma_{BA}$
13.	$\gamma_{AC} \circ \gamma_{BD}$	14.	$\rho_{AD} \circ \tau_{AB}$	15.	$\rho_{AD} \circ R_{C,90^\circ}$
16.	$\gamma_{CD} \circ$ s $\gamma_{BC} \circ \gamma_{AB}$	17.	$\gamma_{CD} \circ \tau_{BC} \circ \gamma_{AB}$	18.	$\tau_{CD} \circ \gamma_{BC} \circ \tau_{AB}$

19. If k, m, n are the perpendicular bisectors of the sides AB, BC, CA of ΔABC respectively, show that $\rho_k \circ \rho_m \circ \rho_n$ is a reflection. What is the axis of this reflection?

20. Let A and B be any two distinct points. Prove that the composition $R_{B,180^\circ} \circ \rho_{AB} \circ R_{A,180^\circ}$ is a glide-reflection and find its axis.

21. Show that the composition of the reflections in the three angle bisectors of a triangle is a reflection whose axis is perpendicular to one of the triangle's sides.

22. Let n be an odd integer and let $A_1, A_2, ..., A_n$ be the successive vertices of a regular n-sided polygon, and let m_i be the bisector of the interior angle at A_i. Identify the composition $\rho_{m_n} \circ ... \circ \rho_{m_2} \circ \rho_{m_1}$.

23. Complete the proof of Proposition 6.3.1.

24. Complete the proof of Proposition 6.3.3.

25. Prove that if γ_{AB} is a glide-reflection and $\gamma_{AB}(P) = P'$, then the axis AB contains the midpoint of PP'.

26. Show that $\rho_k \circ \rho_m \circ \rho_n = \rho_n \circ \rho_m \circ \rho_k$ whenever the lines k, m, n are either concurrent or parallel.
27. Show that the composition of an even number of glide-reflections is either a rotation or a translation.
28. Show that the composition of an odd number of glide-reflections is a glide reflection.

4. The Main Theorems

Enough tools are now available to demonstrate that there are no Euclidean rigid motions above and beyond those described above.

PROPOSITION 6.4.1. *Suppose $\Delta ABC \cong \Delta DEF$. Then there exists a sequence of no more than three reflections such that the composition of these reflections maps the points A, B, C onto the points D, E, F respectively.*

PROOF: Suppose the two given triangles are identical, then the composition of two identical reflections will clearly accomplish the required task.

If the two triangles share exactly two vertices, then it may be assumed that their relative position is described by Figure 6.19. In that case ρ_{AB} itself constitutes the

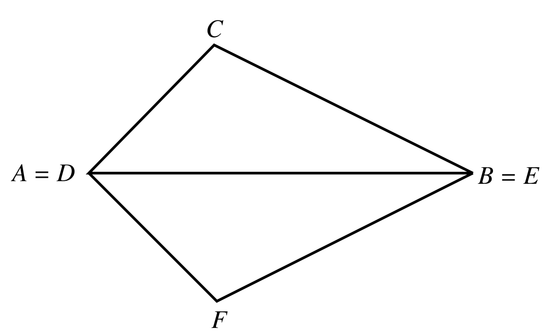


Figure 6.19

required sequence of reflections.

If the two triangles share exactly one vertex, say $A = D$, let M be the midpoint of the segment BE. Then the reflection ρ_{AM} transforms $\Delta\, ABC$ into $\Delta\, DEF'$ that shares at least two vertices with $\Delta\, DEF$. It follows from the previous argument that at most one more reflection will be required to transform $\Delta\, DEF'$ into $\Delta\, DEF$.

Finally, if the two triangles share no vertices, let M be the midpoint of the segment AD. The reflection ρ_m then transforms $\Delta\, ABC$ into $\Delta\, DE'F'$ that shares at least one vertex with $\Delta\, DEF$. By the above argument at most two more reflections will transform $\Delta\, DE'F'$ into $\Delta\, DEF$. It follows that at most three reflections are required to transform $\Delta\, ABC$ into $\Delta\, DEF$.

Q.E.D.

The following is this chapter's main theorem.

THEOREM 6.4.2. *Every rigid motion is the composition of at most three reflections.*

PROOF: Let f be a rigid motion, let A, B, C be three noncollinear points, and set $A' = f(A), B' = f(B), C' = f(C)$. Since $\Delta\, ABC \cong \Delta\, A'B'C'$, it follows from Proposition 6.4.1 that there exist at most three reflections whose composition, say g, also transforms A, B, C onto A', B', C', respectively. It follows from Theorem 6.1.4 that $f = g$.

Q.E.D.

The following classification theorem is a consequence of the above.

THEOREM 6.4.3. *Every rigid motion is either a translation, a rotation, or a glide-reflection.*

PROOF: The composition of no reflections is the identity which can be viewed is either a rotation $R_{A,0^\circ}$ or a translation τ_{AA}. The composition of one reflection is a glide-reflection. The composition of two reflections is, by Propositions 6.2.1-2, either a translation or a rotation. It follows that the composition of three reflections is also the composition of a reflection with either a translation or a rotation, which, by Propositions 6.3.1 and 6.3.3, is a glide-reflection.

Q.E.D.

EXAMPLE 6.4.4 *Let f and g be two rigid motions. Prove that g is a reflection if and only if $f \circ g \circ f^{-1}$ is a reflection.*

Suppose first that g is a reflection. By Theorem 6.4.3 f is either a translation, a rotation, or a glide-reflection. In the first two cases it follows from Proposition 6.3.1 and 6.3.3 that $f \circ g \circ f^{-1}$ is also a glide-reflection. The same conclusion can be drawn in the third case, if f is a glide-reflection, but this time Proposition 6.3.5 is also needed. In order to show that $f \circ g \circ f^{-1}$ is a reflection it suffices to show that it has a fixed point. Let P be any fixed point of the reflection g and set $P' = f(P)$. Then

$$f \circ g \circ f^{-1}(P') = f \circ g \circ f^{-1} \circ (f(P)) = f \circ g(P) = f(P) = P',$$

so that P' is the requisite fixed point of $f \circ g \circ f^{-1}$.

Conversely, suppose $f \circ g \circ f^{-1}$ is a reflection. It then follows from the above that g is also a reflection because

$$f^{-1} \circ (f \circ g \circ f^{-1}) \circ (f^{-1})^{-1} = f^{-1} \circ f \circ g \circ f^{-1} \circ f = g .$$

In conclusion we point out that the definitions of rotations and reflections as well as the proofs of Propositions 6.1.4, 6.4.1-2 are all neutral and hence they also hold for the hyperbolic plane. In particular, *Every rigid motion of the hyperbolic plane is the composition of at most three hyperbolic reflections*. These hyperbolic reflections will be described in the next chapter.

EXERCISES 6.4

In the exercises below, f and g denote two rigid motions.

1. Prove that g is a glide-reflection if and only if $f \circ g \circ f^{-1}$ is a glide-reflection.
2. Prove that g is a rotation if and only if $f \circ g \circ f^{-1}$ is a rotation.
3. Prove that g is a translation if and only if $f \circ g \circ f^{-1}$ is a translation.
4. Is it true that g is a reflection if and only if $f \circ g \circ f^{-1}$ is a reflection? Justify your answer.
5. Prove that $f \circ g$ is a glide-reflection if and only if $g \circ f$ is a glide-reflection.
6. Prove that $f \circ g$ is a rotation if and only if $g \circ f$ is a rotation.
7. Prove that $f \circ g$ is a translation if and only if $g \circ f$ is a translation.
8. Is it true that $f \circ g$ is a reflection if and only if $g \circ f$ is a reflection? Justify your answer?
9. Is it true that $f \circ g$ is a translation if and only if both f and g are translations? Justify your answer.
10. Is it true that $f \circ g$ is a rotation if and only if both f and g are rotations? Justify your answer.
11. Is it true that $f \circ g$ is a reflection if and only if both f and g are reflections? Justify your answer.

5. Symmetries of Polygons

A *(mathematical) symmetry* of a figure Φ is a rigid motion f such that

$$f(\Phi) = \Phi .$$

Thus, the square of Figure 6.20 possesses the symmetries $\rho_d, \rho_e, \rho_m,$ and ρ_n.

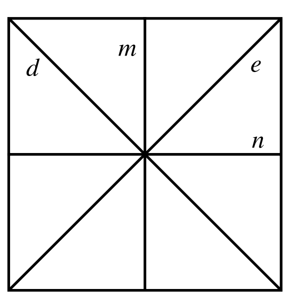


Figure 6.20 Some symmetries of the square.

This is the mathematical formalization of the more intuitive observation that the square is symmetrical about its diagonals and about the lines joining the midpoints of its opposite sides. However, the mathematical definition of symmetry is broader than the common usage of the term. If C denotes the geometrical center of the square then the rotations $R_{C,90^o}$, $R_{C,180^o}$, $R_{C,270^o}$ (see Fig. 6.21) all rotate the square back onto itself and so they too constitute mathematical symmetries, even though they wouldn't be recognized as symmetries by the proverbial person in the street. The identity rigid motion Id is another

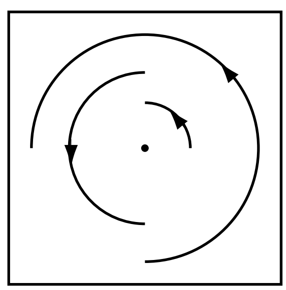


Figure 6.21 More symmetries of the square.

such symmetry. The set of all the symmetries of a figure is called its *symmetry group* or just *group*. Thus, the symmetry group of the square is

$$\{Id, \rho_d, \rho_e, \rho_m, \rho_n, R_{C,90^o}, R_{C,180^o}, R_{C,270^o}\}$$

By definition, every plane figure Φ has a symmetry group that contains at least the identity motion Id. The isosceles triangle of Figure 6.22 has $\{Id, \rho_v\}$ as its symmetry group whereas that of the equilateral triangle of Figure 6.23 is $\{Id, \rho_d, \rho_e, \rho_f, R_{C,120^o}, R_{C,240^o}\}$.

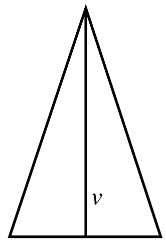


Figure 6.22 The symmetries of an isosceles triangle.

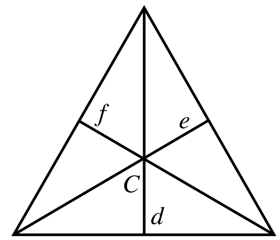


Figure 6.23 The reflections of an equilateral triangle.

While figures of finite extent cannot have either translations or glide-reflections as their symmetries, infinitely extended figures do admit such symmetries and a variety of interesting examples will be discussed in the next two sections. This section, however, is concerned with finite figures only and for the symmetries of such polygons there is a useful algebraic description that is obtained by restricting attention to the action of the symmetry on the polygon's vertices. This action is described by means of the positions occupied by the vertices. Thus, if the positions occupied by the four vertices of the square are labeled 1, 2, 3, 4 respectively (Fig. 6.24), then any symmetry f of the square can be thought of as a

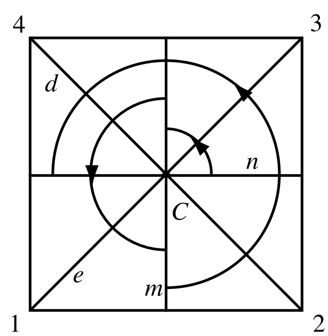


Figure 6.24 The symmetries of the square.

function

$$f: \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$$

where, for each $i = 1, 2, 3, 4$, $f(i)$ denotes the new position of the vertex that was, prior to the execution of f, in position i. Accordingly (see Appendix F), the eight symmetries of the square have the following permutation representations:

Id	=	(1)(2)(3)(4)	ρ_d	=	(1 3)(2)(4)
ρ_e	=	(1)(2 4)(3)	ρ_m	=	(1 2)(3 4)
ρ_n	=	(1 4)(2 3)	$R_{C,90^\circ}$	=	(1 2 3 4)
$R_{C,180^\circ}$	=	(1 3)(2 4)	$R_{C,270^\circ}$	=	(1 4 3 2)

Note that this involves some abuse of notation as the same symbol f is being used to denote both the symmetry as it acts on the whole plane and its restriction to the vertices alone. This will lead to no difficulties and obviates the need for a new notation.

Mathematicians and physical scientists have a great interest in groups of symmetries of solids in spaces of an arbitrary number of dimensions and their classifications. The composition operation plays an important role in the classification of both the rigid motions of the plane and the symmetry groups. The advantage of the permutation representations is that they allow for an algebraic representation of composition. Thus, since $\rho_e = (1)(2\ 4)(3)$ and $\rho_m = (1\ 3)(2\ 4)$ it follows that

$$\rho_e \circ \rho_m \quad = \quad (1)(2\ 4)(3) \circ (1\ 3)(2\ 4) \quad = \quad (1\ 3)(2)(4) \quad = \quad \rho_d$$

and since $R_{C,90^\circ} = (1\ 2\ 3\ 4)$ it follows that

$$\rho_m \circ R_{C,90^\circ} \quad = \quad (1)(2\ 4)(3) \circ (1\ 2\ 3\ 4) \quad = \quad (1\ 4)(2\ 3) \quad = \quad \rho_n.$$

Similarly, the composition of the reflection $\rho_{14} = (1)(2\ 6)(3\ 5)(4)$ and the rotation $R_{C,-60^\circ} = (1\ 6\ 5\ 4\ 3\ 2)$, both symmetries of the regular hexagon of Figure 6.25, is

$$\rho_{14} \circ R_{C,-60^\circ} \quad = \quad (1)(2\ 6)(3\ 5)(4) \circ (1\ 6\ 5\ 4\ 3\ 2) \quad = \quad (1\ 2)(3\ 6)(4\ 5) \quad = \quad \rho_m.$$

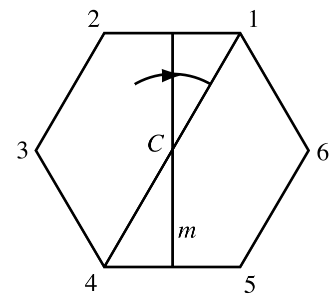


Figure 6.25 Symmetries of the regular hexagon

EXERCISES 6.5

1. Write down the symmetry groups of the following figures:

   a) the rectangle with unequal sides; b) the regular pentagon;

   c) the regular hexagon; d) the regular heptagon;

   e) the regular octagon.

2. Identify the following compositions of the symmetries of the square of Figure 6.23. Describe them both geometrically and with permutation representations.

   a) $\rho_m \circ \rho_e$ b) $(R_{C,90^o}) \circ \rho_m$ c) $\rho_n \circ \rho_e$

   d) $\rho_e \circ \rho_d$ e) $\rho_m \circ R_{C,180^o}$ f) $\rho_n \circ R_{C,180^o}$

   g) $(R_{C,180^o}) \circ \rho_e$ h) $(R_{C,90^o}) \circ R_{C,180^o}$ i) $\rho_e \circ R_{C,180^o}$

3. Identify the following compositions of the symmetries of the regular hexagon of Figure 6.24. Describe them both geometrically and with permutation representations.

   a) $\rho_m \circ \rho_{14}$ b) $(R_{C,60^o}) \circ \rho_m$ c) $\rho_{14} \circ \rho_{36}$

   d) $\rho_{25} \circ \rho_m$ e) $\rho_m \circ R_{C,180^o}$ f) $\rho_m \circ R_{C,120^o}$

   g) $(R_{C,240^o}) \circ \rho_{25}$ h) $(R_{C,-60^o}) \circ R_{C,180^o}$ i) $\rho_{14} \circ R_{C,180^o}$

6. Frieze Patterns

A *frieze pattern* is a one dimensional repeating figure. More formally, a frieze pattern is generated by a finite figure Φ, called a *block,* and a translation τ. The pattern itself consists of the union of all the figures

$$\ldots, \tau^{-2}(\Phi),\ \tau^{-1}(\Phi),\ \Phi,\ \tau(\Phi),\ \tau^{2}(\Phi),\ \ldots\ .$$

where τ^n denotes n applications of τ and τ^{-n} denotes n applications of τ^{-1}. These frieze patterns are the mathematical idealization of such decorative designs as borders used to accent wallpapers and trim sewn or printed around a cloth (Fig. 6.26). However, unlike their physical manifestations, frieze patterns are understood to extend indefinitely in both directions, just like a straight line.

The frieze pattern created by the repetition of a block Φ is denoted by $\wp(\Phi)$ and it inherits some of the symmetries of Φ (see Exercises 1-3). This observation, however, does not account for <u>all</u> the symmetries of the frieze pattern $\wp(\Phi)$. By definition, every such pattern possesses its generating translation τ as a symmetry, since this translation shifts the infinitely extended pattern onto itself. In the case of the block Φ_1 of Figure 6.27, the frieze pattern has no other symmetries, and so its symmetry group is denoted by $\Gamma_1 = <\tau>$.

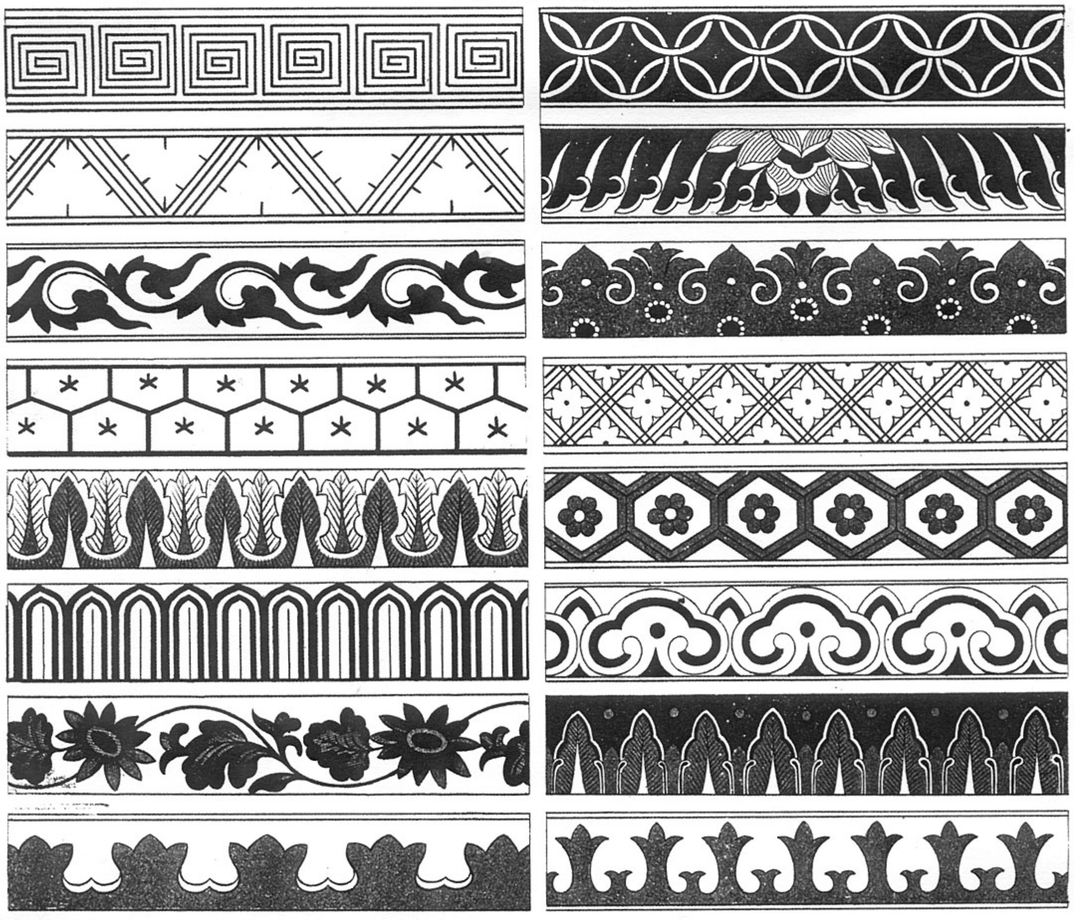


Figure 6.26 Chinese ornamental friezes. Reprinted from *The Grammar of Chinese Ornament*, by Owen Jones, with the permission of Studio Editions.

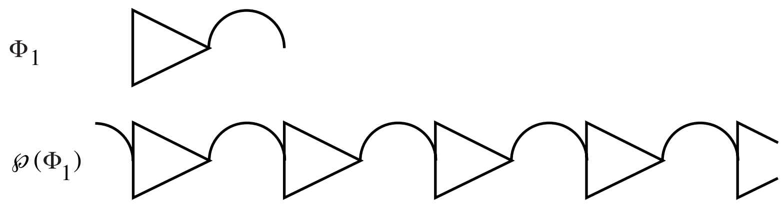


Figure 6.27 A frieze pattern with symmetry group $\Gamma_1 = < \tau >$.

Block Φ_2 of Figure 6.28 possesses the symmetry ρ_h (*h* for horizontal) which is of course also a symmetry of its frieze pattern. In addition, this pattern also necessarily

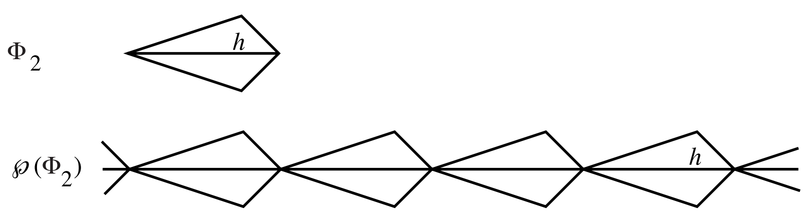


Figure 6.28 A frieze pattern with symmetry group $\Gamma_2 = < \tau, \rho_h, \gamma >$.

possesses the composite glide-reflection $\gamma = \rho_h \circ \tau$ as a symmetry. This frieze's symmetry group is denoted by $\Gamma_2 = < \tau, \rho_h, \gamma >$. The symmetry ρ_v of Φ_3 of Figure 6.29 results in a multitude of symmetries of the frieze $\wp(\Phi_3)$ which are all essentially identical. It

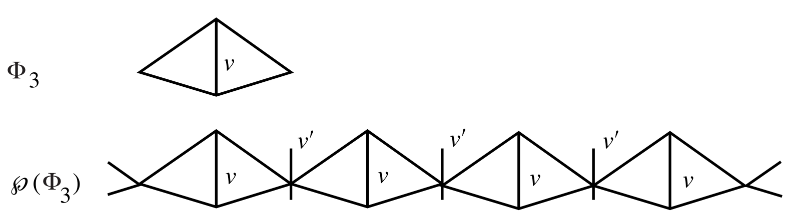


Figure 6.29 A frieze pattern with symmetry group $\Gamma_3 = < \tau, \rho_v >$.

should be noted however, that this frieze pattern possesses an additional symmetry, namely the reflection $\rho_{v'}$, which has no counterpart in the generating block Φ_3. Because of its similarity to ρ_v, the symmetry $\rho_{v'}$ is not listed in the symmetry group $\Gamma_3 = < \tau, \rho_v >$ of this pattern. Such an additional reflection could not have appeared with the horizontal reflection of Φ_2, but similar "accidental" symmetries can arise in other cases, as will be seen below. The symmetry $R_{C,180^o}$ of the block Φ_4 of Figure 6.30 results

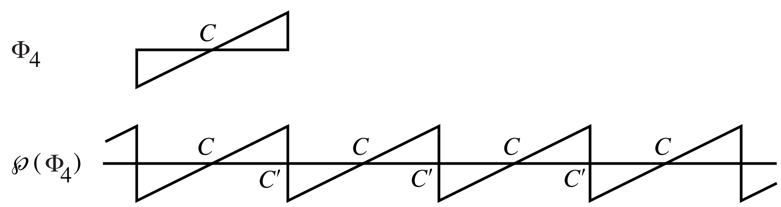


Figure 6.30 A frieze pattern with symmetry group $\Gamma_4 = < \tau, R >$.

in the symmetries $R = R_{C,180^o}$ of the frieze pattern. Once again the frieze pattern has the additional symmetry $R_{C',180^o}$. This frieze pattern's symmetry group is $\Gamma_4 = < \tau, R >$. The block Φ_5 of Figure 6.31 possesses all three of the above symmetries, as does the generated frieze pattern. Its symmetry group is $\Gamma_5 = < \tau, \rho_h, \rho_v, R, \gamma >$. Just like Γ_2,

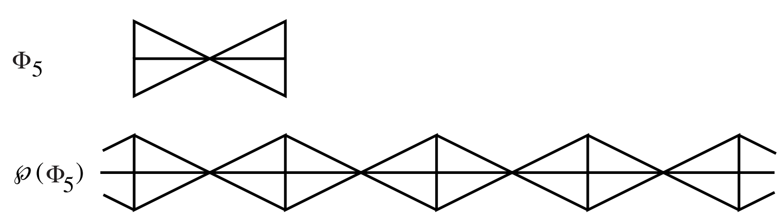


Figure 6.31 A frieze pattern with symmetry group $\Gamma_5 = < \tau, \rho_h, \rho_v, R, \gamma >$.

the next two patterns of Figure 6.32 and Figure 6.33 have symmetry groups $\Gamma_6 = <\tau, \gamma>$ and $\Gamma_7 = <\tau, \gamma, R, \rho_v>$ that contain glide reflections. However, unlike the glide-reflection of Γ_2, those of Γ_6 and Γ_7 do not have their component translation and reflection in the group.

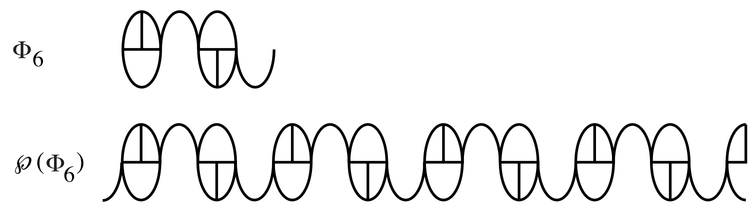


Figure 6.32 A frieze pattern with symmetry group $\Gamma_6 = <\tau, \gamma>$.

Figure 6.33 A frieze pattern with symmetry group $\Gamma_7 = <\tau, \gamma, R, \rho_v>$.

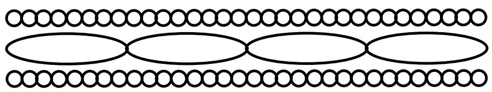
The following theorem, attributed to Paul Niggli (1888 – 1953), states that the foregoing is a complete list of all the possible types of symmetry groups that frieze patterns can possess.

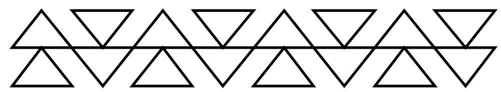
PROPOSITION 6.6.1 (P. Niggli, 1926)**.** *Every frieze pattern has a symmetry group that is identical with one of the groups* $\Gamma_1 = <\tau>$, $\Gamma_2 = <\tau, \rho_h, \gamma>$, $\Gamma_3 = <\tau, \rho_v>$, $\Gamma_4 = <\tau, R>$, $\Gamma_5 = <\tau, \rho_h, \rho_v, R, \gamma>$, $\Gamma_6 = <\tau, \gamma>$, $\Gamma_7 = <\tau, \gamma, R, \rho_v>$.

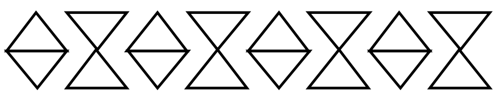
EXERCISES 6.6

1. If the block Φ's frieze pattern $\wp(\Phi)$ has a horizontal symmetry, must Φ necessarily have a horizontal symmetry?
2. If the block Φ's frieze pattern $\wp(\Phi)$ has a vertical symmetry, must Φ necessarily have a vertical symmetry?
3. If the block Φ's frieze pattern $\wp(\Phi)$ has a rotational symmetry, must Φ necessarily have a rotational symmetry?

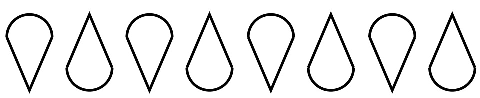
Identify the groups of the following frieze patterns.

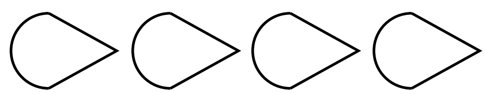
4. 

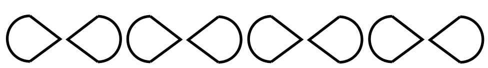
5. 

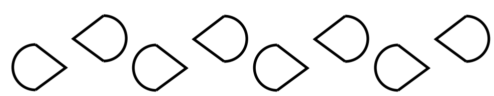
6. 

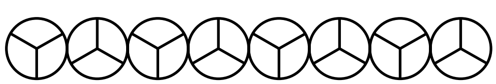
7. 

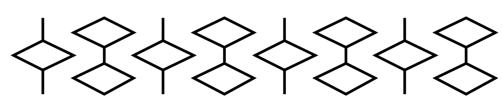
8. 

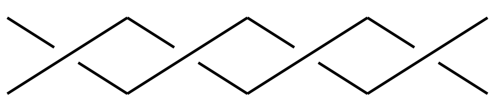
9. 

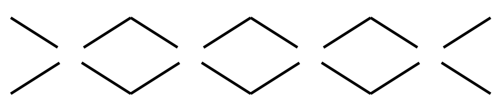
10. 

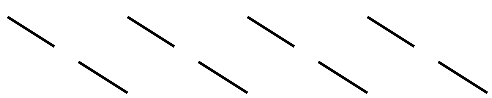
11. 

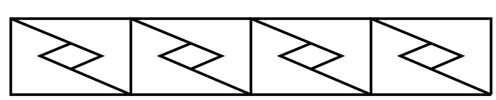
12. 

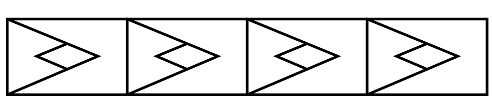
13. 

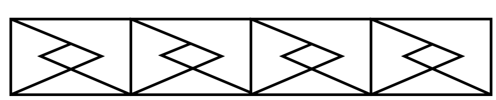
14. 

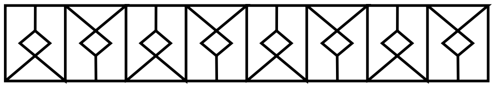
15. 

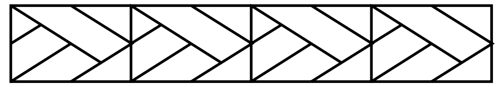
16. 

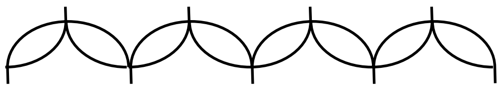
17. 

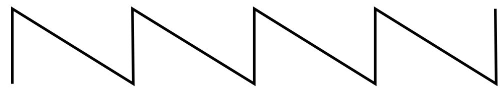
18. 

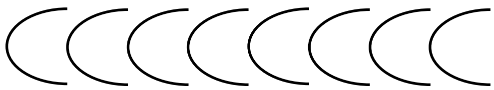
19. 

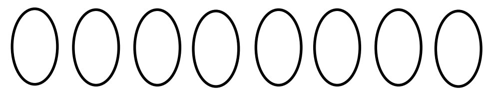
20. 

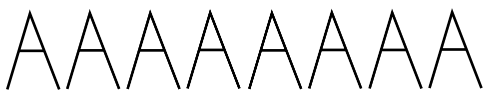
21. 

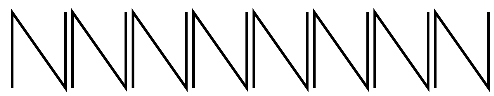
22. 

23. 

24. 

25. 

26. 

27. 

7. Wallpaper Designs

Wallpaper designs are the two dimensional analogs of frieze patterns. More technically, let $\wp$ be the frieze pattern generated by a block Φ and a translation τ. If τ^* is another translation whose direction is not parallel to that of τ, then the union of the figures

$$\ldots, \tau^{*-2}(\wp), \tau^{*-1}(\wp), \ \wp, \ \tau^*(\wp), \ \tau^{*2}(\wp), \ \ldots$$

is the *wallpaper design* $W(\Phi)$ generated by Φ, τ, and τ^* (see Figures 6.34-35). As their name implies, such designs are the mathematical patterns that underlie the repeating decorative artworks illustrated in Figure 6.36. Unlike the carpets and walls that carry these artworks, the mathematical wallpaper designs extend ad infinitum in all the directions of the plane.

It is clear that both τ and τ^* are symmetries of the wallpaper design they generate. As was the case for frieze patterns, the generated design $W(\Phi)$ may possess further symmetries that are not present in Φ. In contrast with the seven different groups of symmetries of frieze patterns, there are seventeen different possibilities for the symmetry groups of wallpaper designs. These, together with their labels are exhibited below. The presence of reflectional and glide-reflectional symmetries is denoted by a dashed line with a label of either ρ (for reflection) or γ (for glide-reflection). The centers of rotational symmetries are denoted by the symbols ◊ (180^o), Δ (120^o), h (90^o), ⬡ (60^o). The following table lists the salient symmetry characteristics of each design. A rotation through an angle of $360^o/n$ is said to have *order* n. A glide-reflection is said to be *non-trivial* if its component translation and reflection are not symmetries of the pattern.

The symbols *p1, pgg, p31m, ...* are used to denote both a type of wallpaper design and its symmetry group. This is known as the crystallographic notation for the symmetry groups. If the second character in this symbol is an integer, it is the highest order of all the rotations in that group. The significance of the other characters is too complicated to explain here.

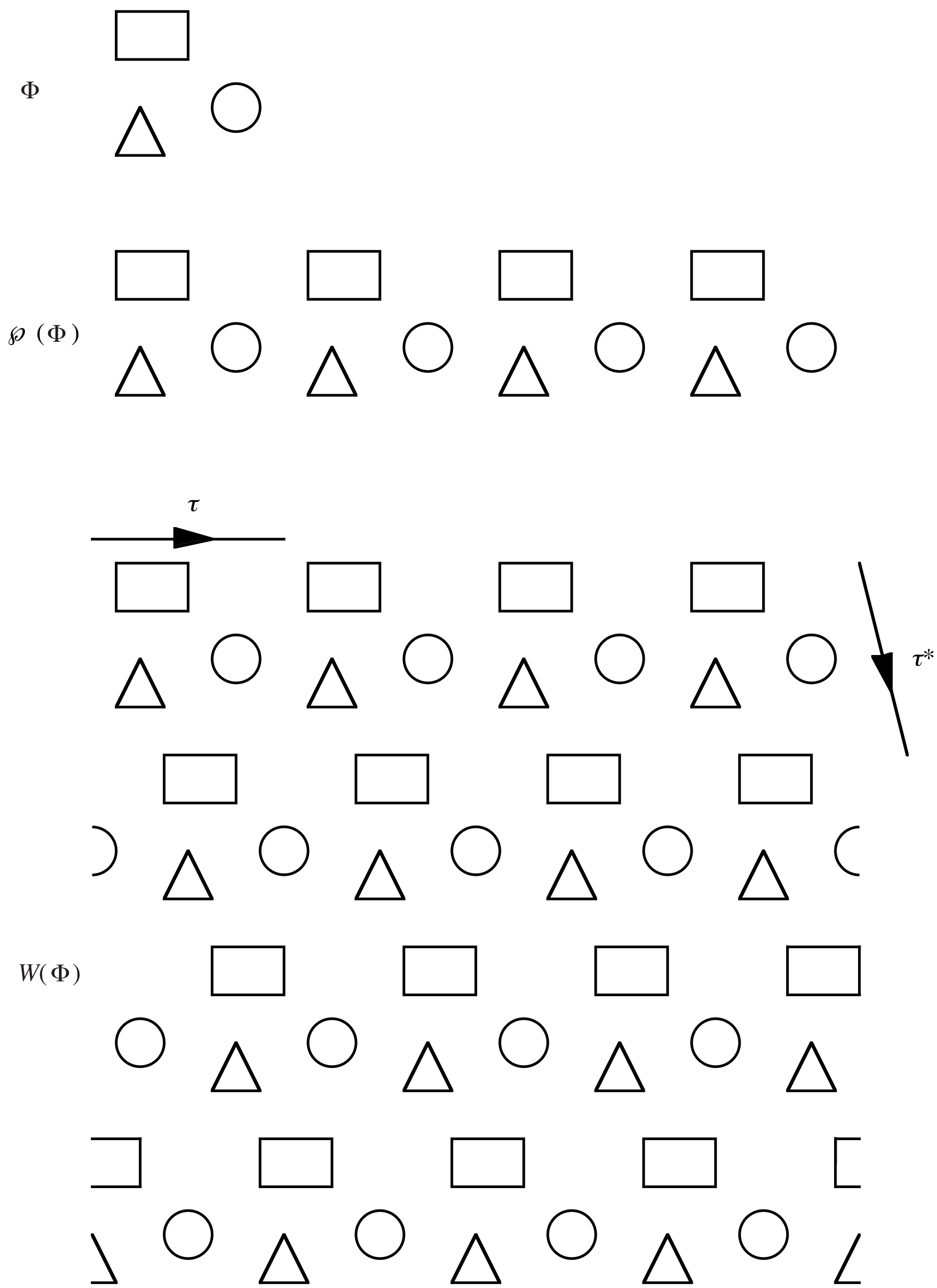


Figure 6.34 A wallpaper design

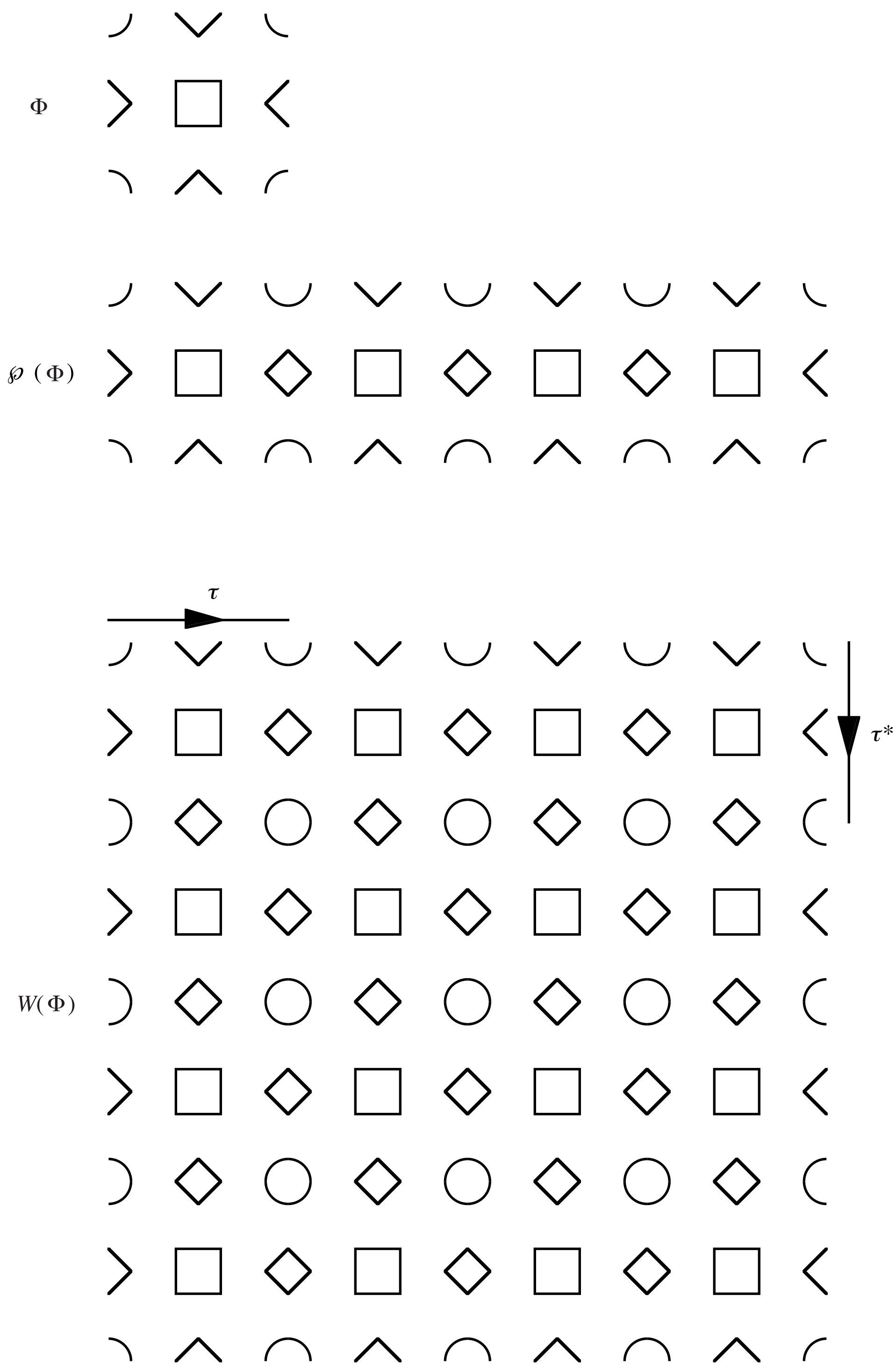


Figure 6.35

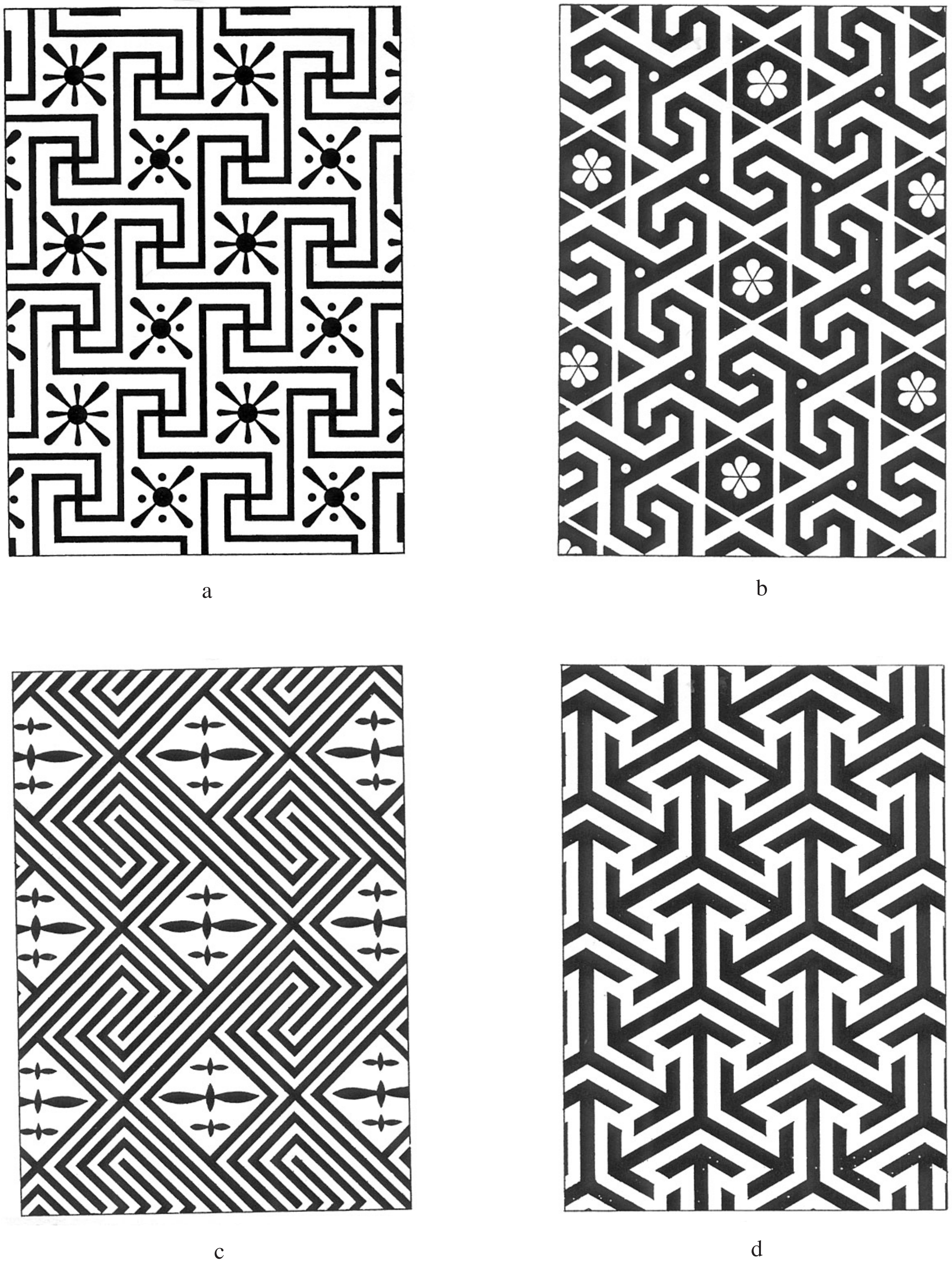


Figure 6.36 Middle-eastern ornamental designs.

The following is the two dimensional analog of Proposition 6.6.1.

PROPOSITION 6.7.1. *There are exactly seventeen wallpaper symmetry groups.*

The characteristics of the wallpaper designs that correspond to these groups are displayed in the table below. The table itself is then followed by Figures 6.37, 6.38, 6.39 which display one illustration for each of these groups.

Proposition 6.7.1 was first discovered in 1891 by Evgraf Stepanovich Fedorov (1853-1919), thirty five years before Niggli stated and proved its 1-dimensional analog on frieze groups (Proposition 6.6.1). Curiously, this work had been preceded by Fedorov's and Arthur Schönflies's (1853-1928) independent classifications of the 230 crystallographic groups, these being the 3-dimensional analogs of the wallpaper groups. It has since then been established that there are exactly 4783 classes of such groups in 4-dimensional space. For spaces of more than four dimensions it is only known that the number of such symmetry groups is finite.

Recognition Chart for Wallpaper Symmetry Groups
(From Doris Schattschneider's article)

Type	Highest order of rotation	Reflections	Non-Trivial Glide Reflections	Helpful Distinguishing Properties
p1	1	no	no	
p2	2	no	no	
pm	1	yes	no	
pg	1	no	yes	
cm	1	yes	yes	
pmm	2	yes	no	
pmg	2	yes	yes	parallel reflection axes
pgg	2	no	yes	
cmm	2	yes	yes	perpendicular reflection axes
p4	4	no	no	
p4m	4	yes	yes	4-fold centers on reflection axes
p4g	4	yes	yes	4-fold centers not on reflection axes
p3	3	no	no	
p3m1	3	yes	yes	all 3-fold centers on reflection axes
p31m	3	yes	yes	not all 3-fold centers on reflection axes
p6	6	no	no	
p6m	6	yes	yes	

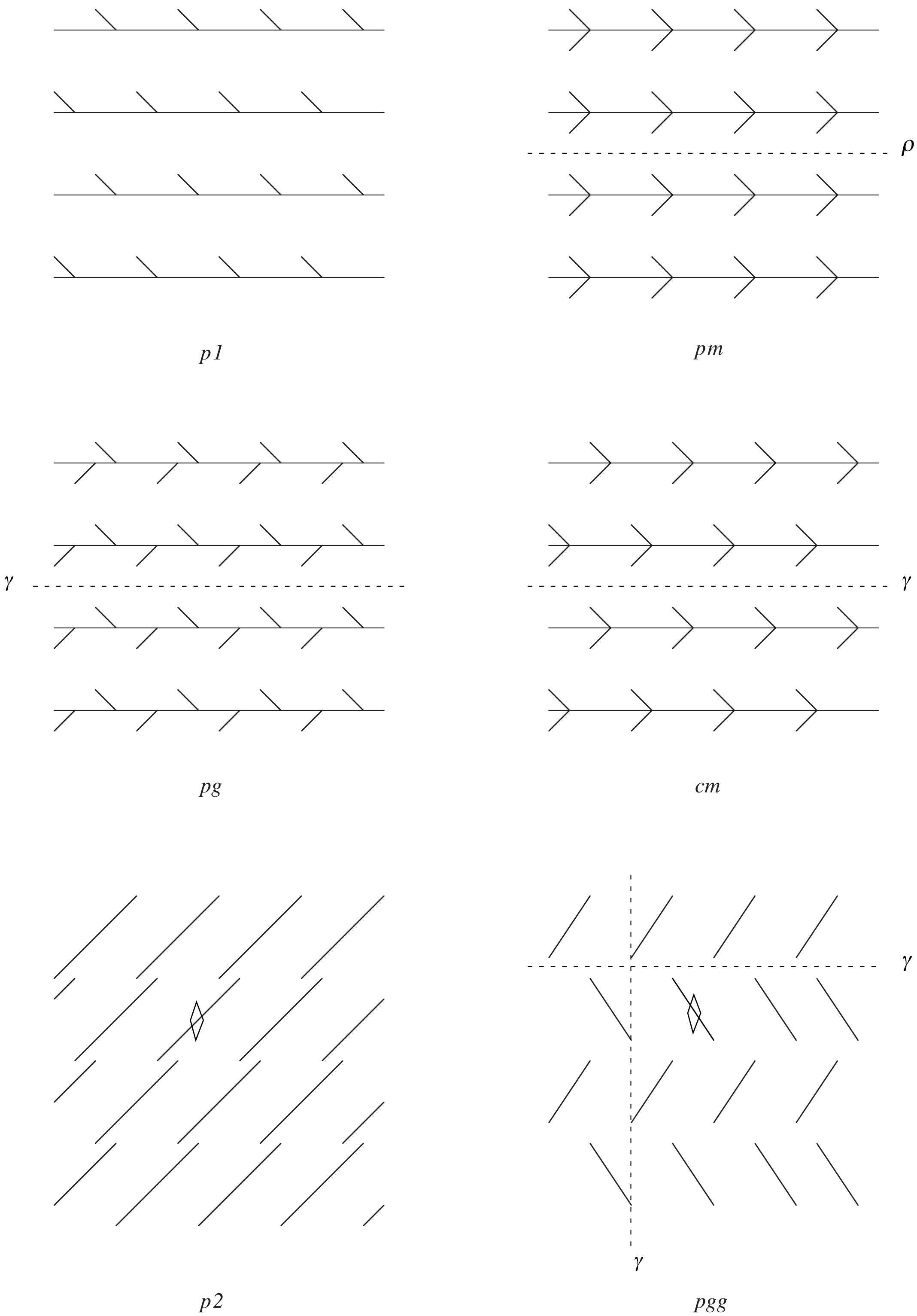


Figure 6.37

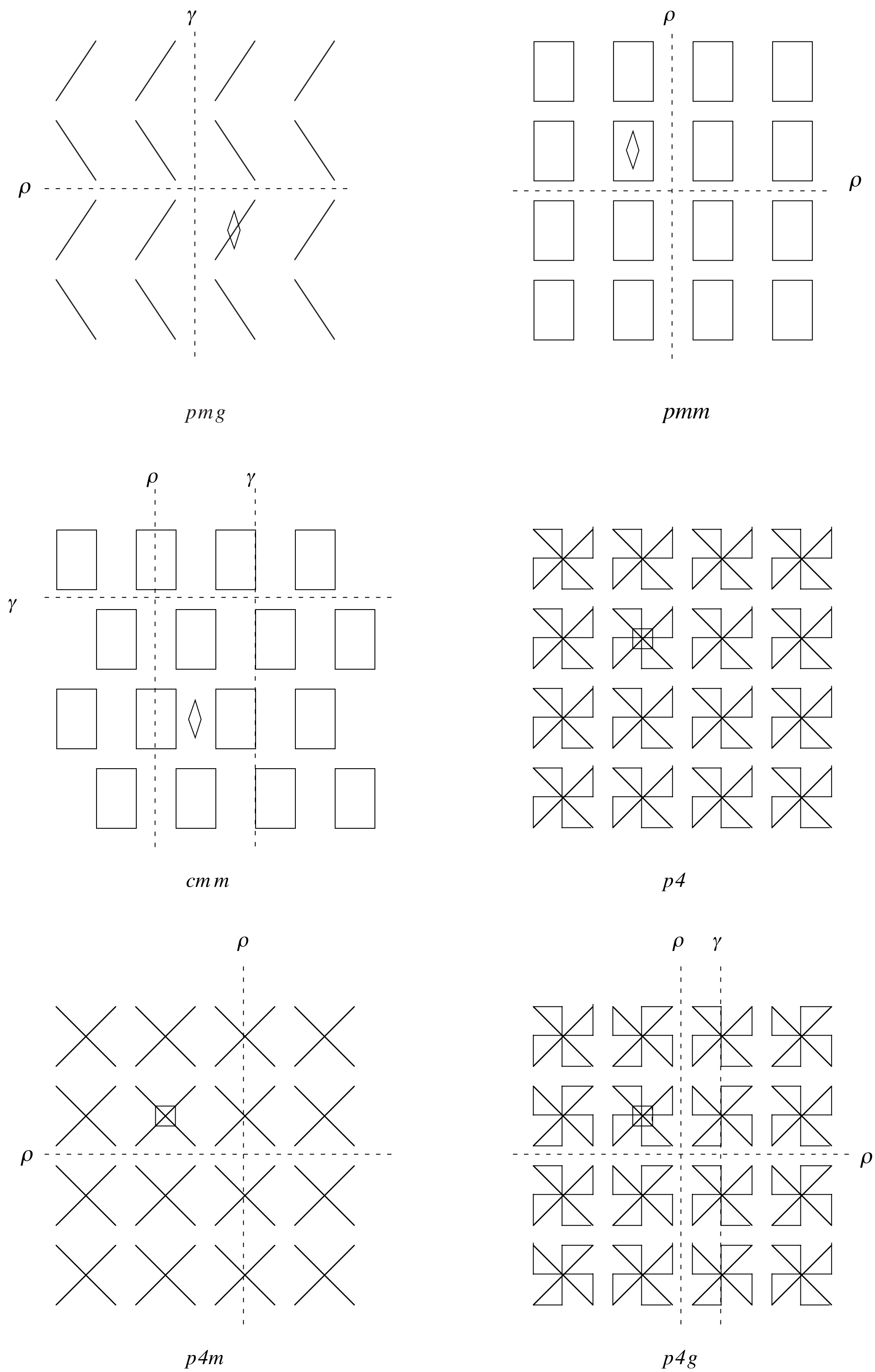


Figure 6.38

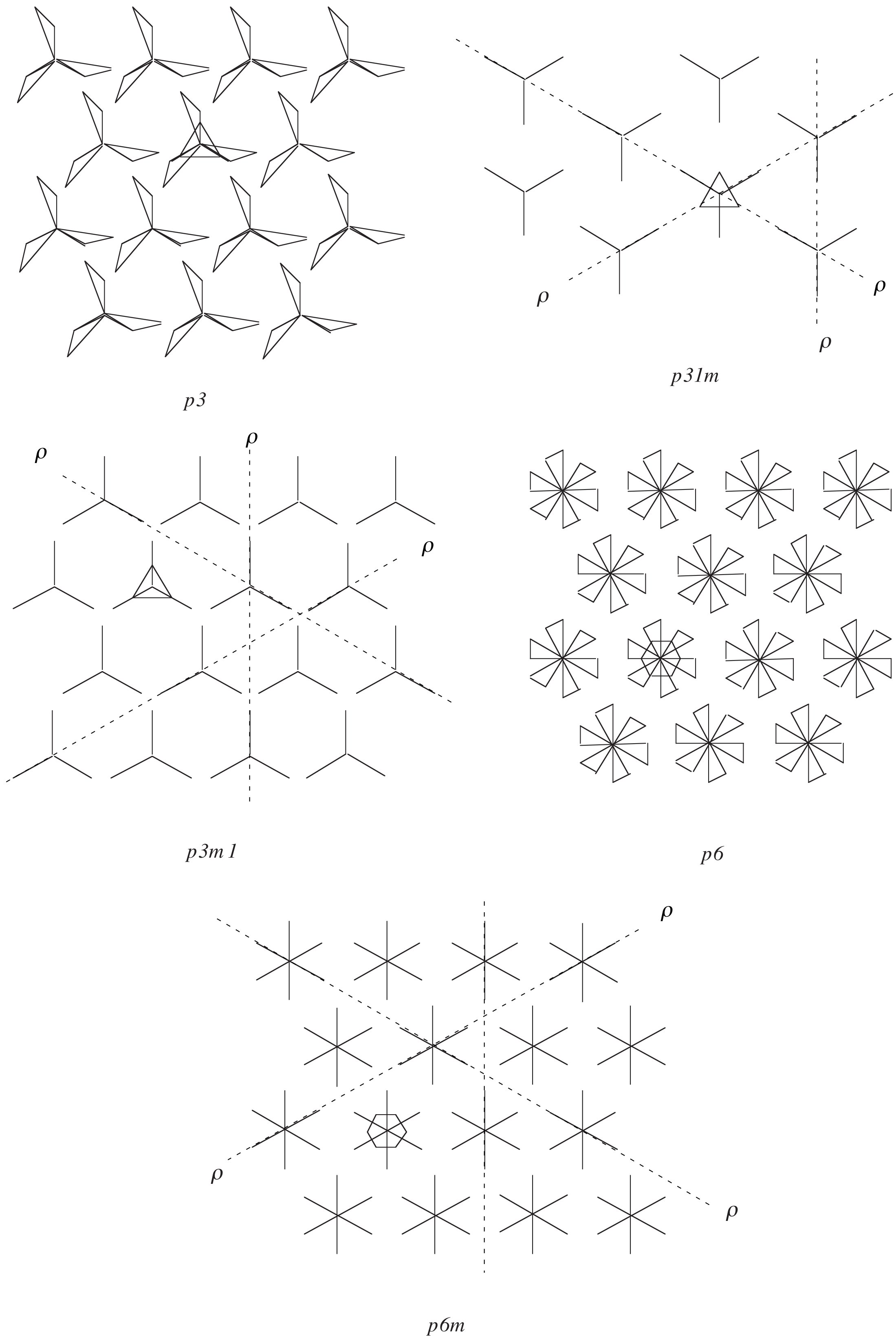


Figure 6.39

EXERCISES 6.7

Determine the crystallographic symbol of each of the wallpaper designs of Exercises 1-34. In each case

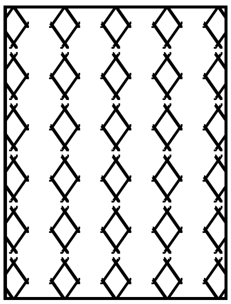
a) display a rotation of the highest order;

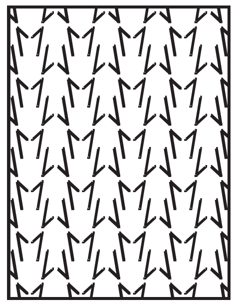
b) denote the presence of a glide-reflection by drawing its axis with an accompanying γ;

c) denote the presence of a reflection by drawing its axis with an accompanying ρ;

d) avoid redundancy by only drawing only one axis in any direction;

e) in case you have to choose between a ρ and a γ, display the γ.

1.


2.


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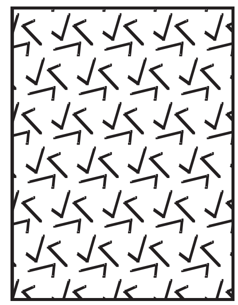

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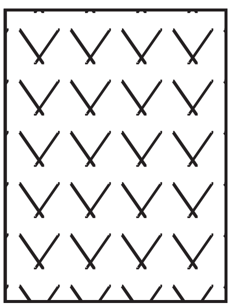

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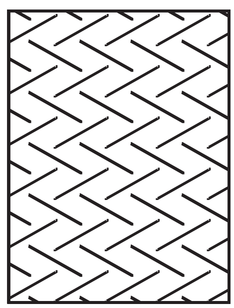

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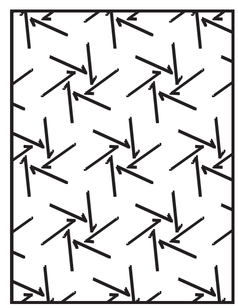

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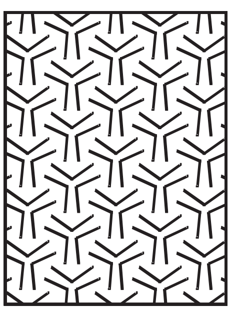

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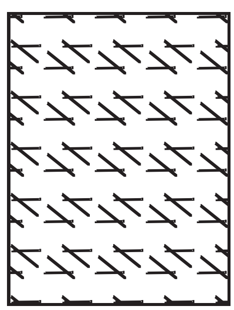

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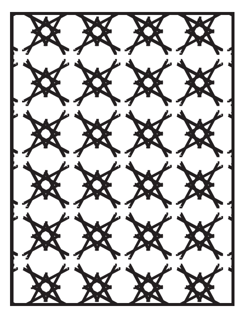

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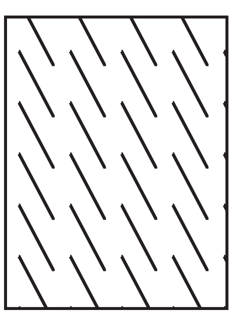

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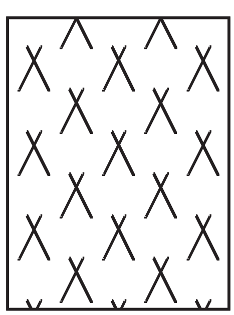

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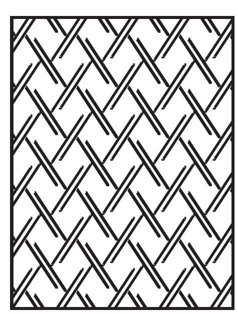
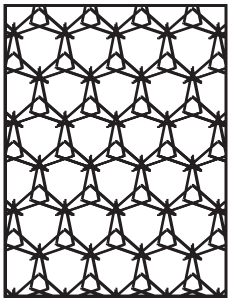
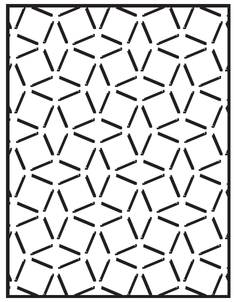
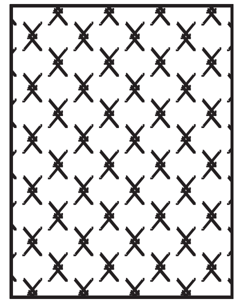
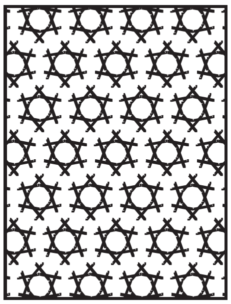
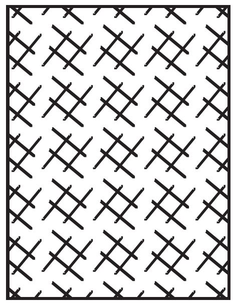


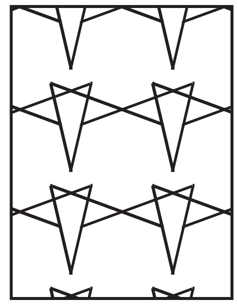
13. 

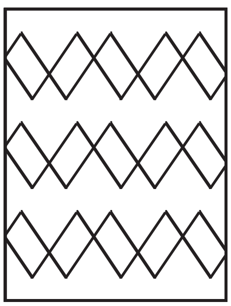
14. 

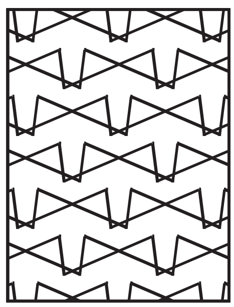
15. 

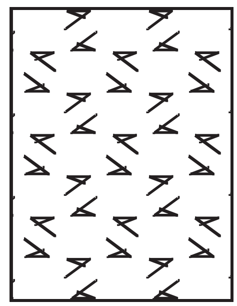
16. 

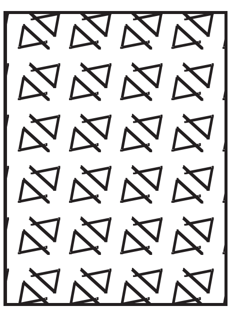
17. 

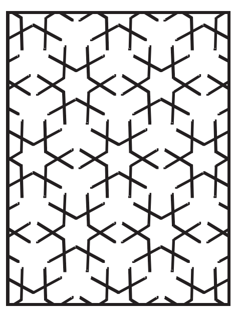
18. 

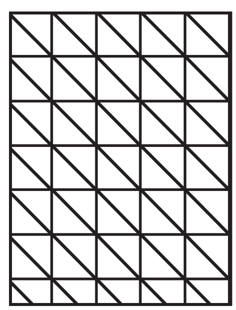
19. 

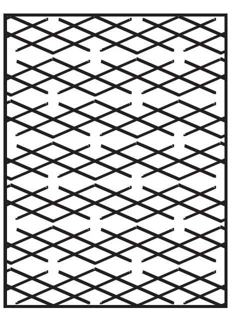
20. 

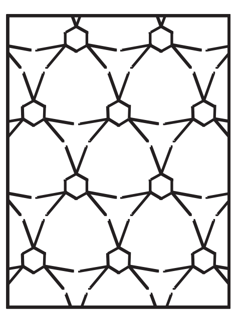
21. 

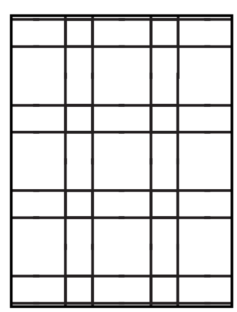
22. 

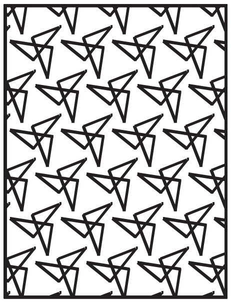
23. 

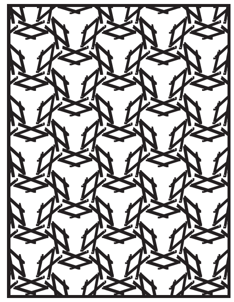
24. 

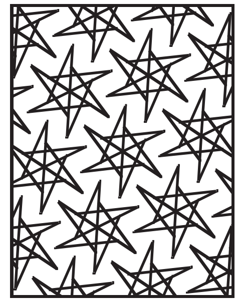
25. 

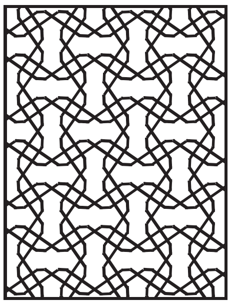
26. 

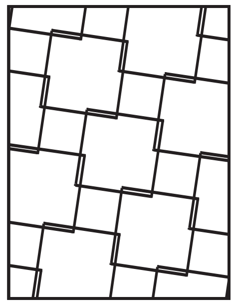
27. 

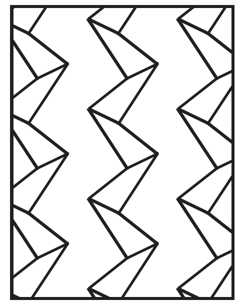
28. 

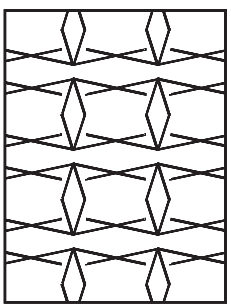
29. 

30. 

31. 

32. 

33. 

34. 

CHAPTER REVIEW EXERCISES

1. If f is any rigid motion and τ is any translation, identify $\tau \circ f \circ \tau^{-1}$.
2. If f is any rigid motion and R is any rotation, identify $R \circ f \circ R^{-1}$.
3. If f is any rigid motion and ρ is any reflection, identify $\rho \circ f \circ \rho^{-1}$.
4. Identify the symmetry groups of the frieze patterns in Figure 6.26.
5. Identify the symmetry groups of the wallpaper designs in Figure 6.36.
6. Are the following statements true or false in Euclidean geometry? Justify your answers.
   a) The composition of two rigid motions is a rigid motion.
   b) The composition of two translations is a translation.
   c) The composition of two rotations is a rotation.
   d) The composition of two reflections is a reflection.

e) The composition of two glide-reflections is never a glide-reflection.

f) The only rigid motion that fixes three distinct points is the identity.

g) Every geometrical figure has at least one symmetry.

h) There exist only seven frieze patterns.

i) The frieze patterns have only seven pairwise distinct frieze pattern symmetry groups.

j) There exist only seventeen wallpaper designs.

k) There exist only seventeen distinct wallpaper symmetry groups.

l) Every frieze pattern has a translation in its symmetry group.

m) For each $n = 1, 2, 3, 4, 6$ there is a wallpaper design which has a rotation of order n and a non-trivial glide-reflectional symmetry.

n) For each $n = 1, 2, 3, 4, 6$ there is a wallpaper design which has a rotation of order n and no non-trivial glide-reflectional symmetries.

o) The inverse of every rigid motion is a rigid motion of the same type.