CHAPTER 5

Towards Projective Geometry

Most mathematical disciplines encounter infinity and find it necessary to incorporate it into their language. Euclidean geometry is no exception to this rule and this process resulted in the beautiful structure known as projective geometry, which was first codified by the Frenchman Gérard Desargues (1591 - 1662).

1. Division of Line Segments

The taming of geometrical infinity begins with a careful examination of geometrical ratios. Let r be a positive real number. If the point D on the line segment AB is such that

$$\frac{AD}{DB} = r \tag{1}$$

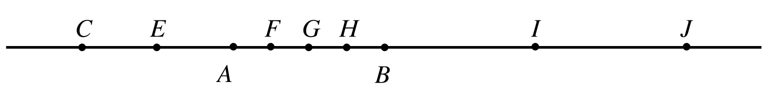


Figure 5.1 Division points.

it is said that D *divides the segment* AB *internally in the ratio* r. In Figure 5.1

$$\frac{AF}{FB} = \frac{1}{3} \qquad \frac{AG}{GB} = 1 \qquad \frac{AH}{HB} = 3$$

On the other hand, if the point D lies on the straight line $\overleftrightarrow{AB}$ but falls <u>outside</u> the line segment AB, and if Equation (1) holds again, then D *divides the line segment* AB *externally in the ratio* r, or

$$\frac{AD}{DB} = -r \qquad (2)$$

In Figure 5.1

$$\frac{AD}{DB} = -\frac{1}{2} \qquad \frac{AE}{EB} = -\frac{1}{3} \qquad \frac{AI}{IB} = -2 \qquad \frac{AJ}{JB} = -\frac{3}{2}$$

Note that when $-1 < r < 0$ in Equation (2), AD must be shorter than DB so that A lies between D and B (see Fig. 5.2). On the other hand, if $r < -1$ AD is longer than DB and so B separates A and D. Exercise 16 contains more detailed information regarding the dependence of the value of AD/DB on the position of D on the line $\overleftrightarrow{AB}$.

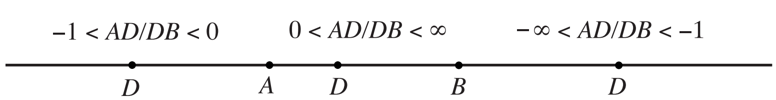


Figure 5.2

An alternative description of the relationship between the sign of a ratio and the relative position of its points is obtained by thinking of the line segments in question as directed segments. In that case the ratio AD/DB is positive or negative according as AD and DB have the same or opposite directions.

It is important to keep in mind that the assignment of signs to ratios applies only in the case where the points A, B, D are collinear. If they are not collinear, then the ratio AD/DB is always taken to be positive. Moreover, while this definition does implicitly assume a choice of a unit of length, the actual value of the ratio AD/DB is independent of the particular choice of unit since changing one's choice has the effect of multiplying the lengths of AD and DB by the same factor which then disappears in the evaluation of the ratio AD/DB.

It should be mentioned that Euclid did not exhibit any interest in external division points. For that reason the correspondence between this chapter's propositions and his is somewhat tenuous.

The next proposition demonstrates that division points are unique.

PROPOSITION 5.1.1. *Given two distinct points A and D and a real number r there exists at most one point D on $\overleftrightarrow{AB}$ such that $AD/DB = r$.*

GIVEN: Two distinct points A and B; two points D, E on AB such that (Fig. 5.3)

$$\frac{AD}{DB} = \frac{AE}{EB}$$

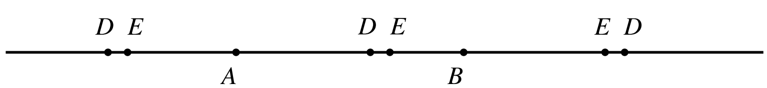


Figure 5.3 Uniqueness of division.

TO PROVE: D and E are identical.

PROOF: Let r be the common value of the ratio AD/DB and AE/EB. Suppose first that $r > 0$. In this case both D and E are in between A and B. It follows from Proposition 3.5.5 and the proportion

$$\frac{AD}{DB} = \frac{AE}{EB}$$

that

$$\frac{AD + DB}{DB} = \frac{AE + EB}{EB} \quad \text{or} \quad \frac{AB}{DB} = \frac{AB}{EB} .$$

Hence $DB = EB$ and so, since D and E are both between A and B, they are identical.

The resolution of the other cases corresponding to $r < -1$, $r = -1$, $-1 < r < 0$, and $r = 0$ are relegated to Exercises 1-3.

Q.E.D.

Before addressing the question of the existence of division points that yield arbitrary ratios, rational divisions are examined.

PROPOSITION 5.1.2(VI.9). *To divide a given segment, both internally and externally in the ratio m/n, where m and n are two distinct positive integers.*

GIVEN: Line segment AB, positive integers m, n (Fig. 5.4).

TO CONSTRUCT: Points D and E on AB such that

$$\frac{AD}{DB} = \frac{m}{n} \qquad \frac{AE}{EB} = -\frac{m}{n} .$$

CONSTRUCTION: Let AC ' AB and let $A_1, A_2, ..., A_{m+n}$ be a sequence of distinct points on AC such that $AA_1 = A_1A_2 = ... = A_{m+n-1}A_{m+n}$. Let D be the intersection

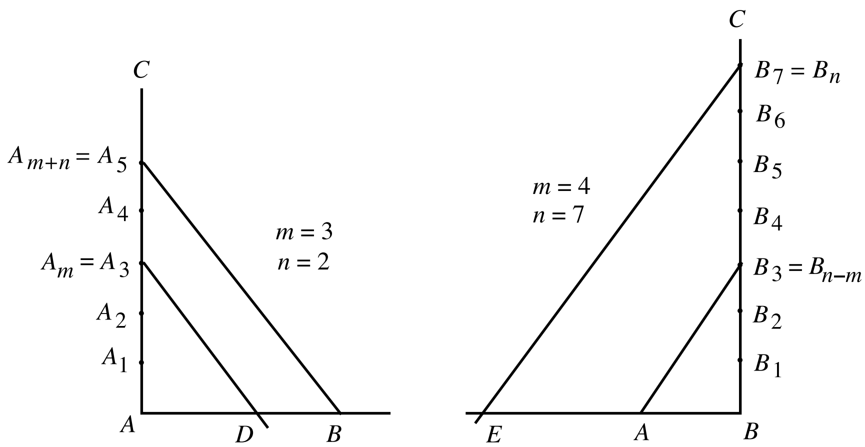


Figure 5.4 Constructing division points.

of AB with the straight line through A_m that is parallel to BA_{m+n}. Turning to E, if $m < n$, then let BC ' AB and let $B_1, B_2, ..., B_n$ be a sequence of distinct points on BC such that $BB_1 = B_1B_2 = ... = B_{n-1}B_n$. Let E be the intersection of $\overleftrightarrow{AB}$ with the straight line through B_n that is parallel to AB_{n-m}. The construction of E in the case $m > n$ is relegated to Exercise 4.

PROOF: It follows from Proposition 3.5.6 that

$$\frac{AD}{DB} = \frac{AA_m}{A_mA_{m+n}} = \frac{mAA_1}{nAA_1} = \frac{m}{n}$$

and

$$\frac{AE}{EB} = \frac{B_{n-m}B_n}{B_nB} = -\frac{mBB_1}{nBB_1} = -\frac{m}{n} .$$

Q.E.D.

It is intuitively clear that given any positive real number r, there is a point D that divides AB internally in the ratio r. After all, one need simply choose a point D on AB such that

$$AD = \frac{r}{r+1} AB$$

so that

$$BD \quad = \quad AB - AD \quad = \quad AB - \frac{r}{r+1}AB \quad = \quad (1 - \frac{r}{r+1})AB \quad = \quad \frac{1}{r+1}AB$$

and hence

$$\frac{AD}{DB} \quad = \quad \frac{r}{1} \quad = \quad r.$$

This, however, is merely an existential statement. The following proposition shows how this division point can be constructed within the framework of Euclid's *Elements*.

PROPOSITION 5.1.3(VI.10). *To divide a given line segment both internally and externally in any given ratio ≠ 1.*

GIVEN: Line segment AB, line segment PQ of length $a \neq 1$ (Fig. 5.5).

TO CONSTRUCT: Points D and E such that $\frac{AD}{DB} = a$ and $\frac{AE}{EB} = -a$.

CONSTRUCTION: The construction of a point E is described for $a > 1$ only, leaving the other cases to Exercises 5, 6. On $AC \perp AB$ let X, Y be points such that $AX = PQ$ and XY has unit length. Let E be the intersection of $\overleftrightarrow{AB}$ with the straight line through X that is parallel to BY.

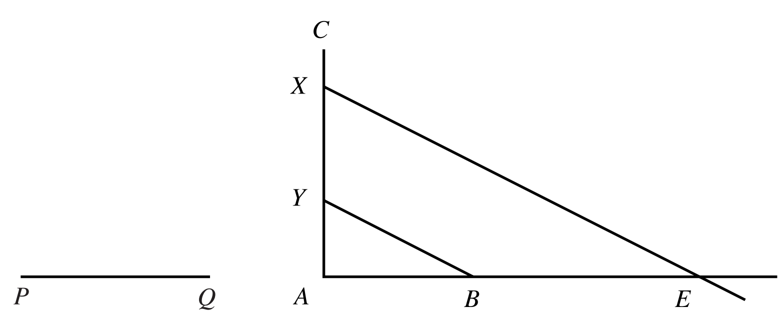


Figure 5.5 An external division point.

PROOF: By Proposition 3.5.6

$$\frac{AE}{EB} = \frac{AX}{XY} = -\frac{a}{1} = -a.$$

Q.E.D.

In the exceptional case where the ratio is 1, the internal division point is the midpoint which was already dealt with in Proposition 2.3.10, whereas the external division point does not exist (see Exercise 15). An alternative method for dividing a line segment in a prespecified signed ratio is described in Exercises 3.5B.6-7. The foregoing discussion is summarized by the following proposition.

PROPOSITION 5.1.4. *Let AB be a straight line segment and r a real number different from -1. Then there is a unique point D on $\overleftrightarrow{AB}$ such that* $\frac{AD}{DB} = r$.

EXAMPLE 5.1.5. To divide a line segment AB of Figure 5.6 in the ratio $-\sqrt{2}$.
CONSTRUCTION: The number $\sqrt{2}$ is represented geometrically by the hypotenuse of an isosceles right triangle whose legs are 1 unit long. The remainder of the construction follows the steps outlined in the proof of Proposition 5.1.1, again with $AX = PQ$.

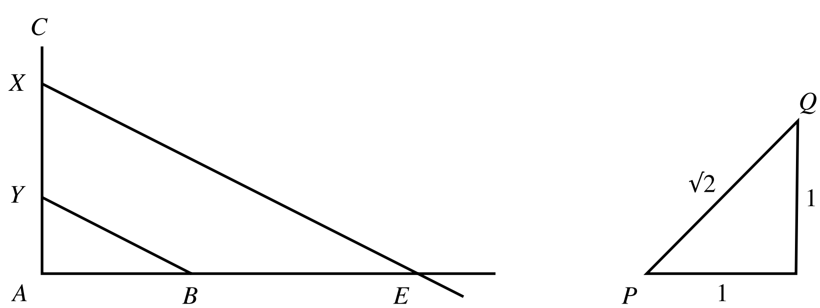


Figure 5.6 An external division.

EXERCISES 5.1

1. Prove Proposition 5.1.1 in the case $r < -1$.
2. Prove Proposition 5.1.1 in the case $-1 < r < 0$.
3. Discuss cases $r = 0, -1$ of Proposition 5.1.1.
4. Construct the point E in the case $m > n$ of Proposition 5.1.2.
5. Prove Proposition 5.1.3 in the case $1 > a > 0$.
6. Discuss Proposition 5.1.3 in the case $a = 1$.
7. Let AB be a line segment.
   a) Divide AB internally in the ratio 4;
   b) Divide AB internally in the ratio 1/4;
   c) Divide AB externally in the ratio 4;
   d) Divide AB externally in the ratio 1/4.
8. Let AB be a line segment.
   a) Divide AB internally in the ratio 5/3;
   b) Divide AB internally in the ratio 3/5;
   c) Divide AB externally in the ratio 5/3;
   d) Divide AB externally in the ratio 3/5.
9. Let AB be a line segment and n a positive integer. Divide AB into n equal segments (Proposition VI.9).
10. Let AB and PQ be line segments and X a point on the straight line PQ. Divide AB in the ratio $\frac{PX}{XQ}$ (Proposition VI.10) . (Hint: The construction is similar to that of PN 5.1.2 and PN 5.1.3.)
11. Let AB be a line segment.
    a) Divide AB internally in the ratio $\sqrt{5}$;
    b) Divide AB internally in the ratio $1/\sqrt{5}$;
    c) Divide AB externally in the ratio $\sqrt{5}$;
    d) Divide AB externally in the ratio $1/\sqrt{5}$.
12. Let AB be a line segment.
    a) Divide AB internally in the ratio $\sqrt{3}$;
    b) Divide AB internally in the ratio $1/\sqrt{3}$;
    c) Divide AB externally in the ratio $\sqrt{3}$;
    d) Divide AB externally in the ratio $1/\sqrt{3}$.
13. Let AB be a line segment.
    a) Divide AB internally in the ratio $\frac{1+\sqrt{5}}{2+\sqrt{3}}$;
    b) Divide AB externally in the ratio $\frac{1+\sqrt{5}}{2+\sqrt{3}}$;

c) Divide AB externally in the ratio $\frac{2+\sqrt{3}}{1+\sqrt{5}}$;

14. Supply the details needed to complete the proof of Proposition 5.1.4.
15. Let AB be a straight line segment. Prove that there is no point E such that $AE/EB = -1$.
16. Prove that if $A = (0, 0)$, $B = (1, 0)$, and $D = (x, 0)$ in some Cartesian coordinate system, then $AD/DB = x/(1 - x)$.
17. Comment on Proposition 5.1.1 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
18. Comment on Proposition 5.1.4 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

2. Collinearity and Concurrence

This section presents some key theorems that were added to the geometric lore over the centuries that followed the writing of *The Elements*. They were selected for this text because they provide a natural transition to projective geometry.

The set of points $\{P, Q, R\}$ is said to be a *transversal* of ΔABC if these points are distinct from A, B, C and they fall on the straight lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$ respectively (see Fig. 5.7).

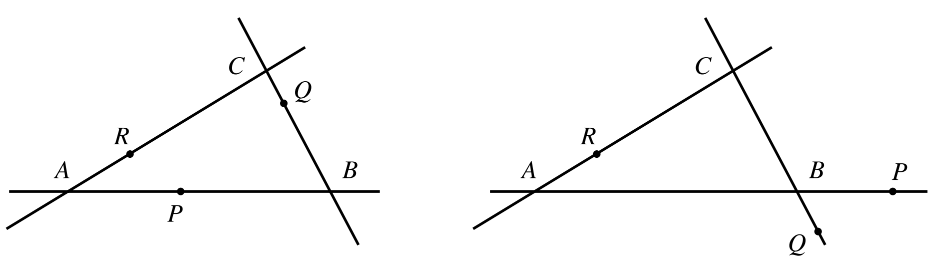


Figure 5.7 Transversals.

PROPOSITION 5.2.1 (The Theorem of Menelaus). *Let* $\{P, Q, R\}$ *be a transversal of* ΔABC. *Then* P, Q, R *are collinear if and only if*

$$\frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = -1 \qquad (1).$$

GIVEN: ΔABC, P on $\overleftrightarrow{AB}$, Q on $\overleftrightarrow{BC}$, R on $\overleftrightarrow{CA}$. (Fig. 5.8).

TO PROVE: P, Q, R are collinear if and only if $\frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = -1$.

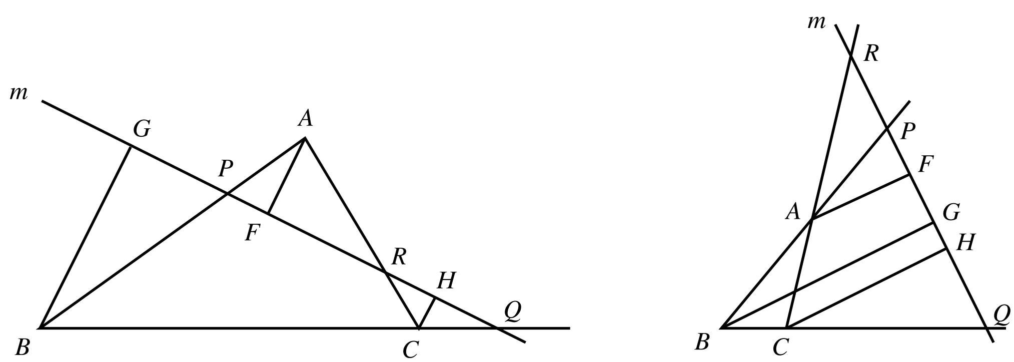


Figure 5.8

PROOF: Assume first that P, Q, R are collinear and let m be the straight line containing them. Let F, G, H be points of the line m such that $AF, BG,$ and CH are all perpendicular to m. Then each of the similarities below is justified by the observation that the triangles in question are all right-angled and each pair either shares an acute angle or else has vertically opposite acute angles:

$$\Delta APF \sim \Delta BPG, \qquad \Delta BQG \sim \Delta CQH, \qquad \Delta CRH \sim \Delta ARF$$

$$\therefore \quad \frac{AP}{PB} = \pm\frac{AF}{BG}, \qquad \frac{BQ}{QC} = \pm\frac{BG}{CH}, \qquad \frac{CR}{RA} = \pm\frac{CH}{AF}$$

Since m does not pass through any of the vertices of ΔABC, it cuts either 1 or 3 of its sides externally. Consequently the product

$$\frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = \left(\pm\frac{AF}{BG}\right)\left(\pm\frac{BG}{CH}\right)\left(\pm\frac{CH}{AF}\right) \qquad (2)$$

contains an odd number of negative factors. After the obvious cancellations are carried out only -1 remains in the right hand side of Equation (2).

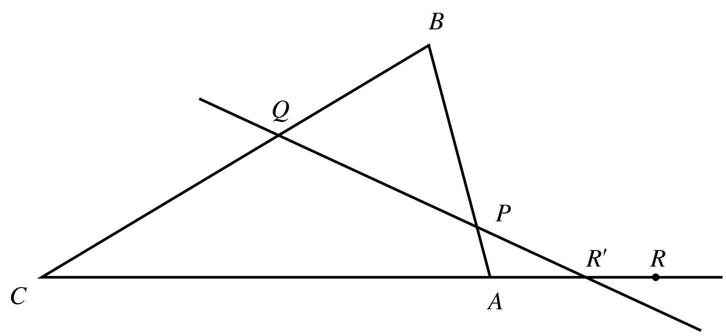


Figure 5.9

Conversely, suppose P, Q, R are such that Equation (1) holds (Fig. 5.9). Set $R' = PQ \cap AC$ (see Exercise 17). It follows from the first part of the proof that

$$\frac{AP}{PB}\frac{BQ}{QC}\frac{CR'}{R'A} = -1 .$$

In combination with Equation (1) this yields

$$\frac{CR}{RA} = \frac{CR'}{R'A} .$$

By Proposition 5.1.4 $R = R'$ and so the points P, Q, R are collinear.

Q.E.D.

EXERCISES 5.2A

1. Suppose a straight line m bisects side AB of ΔABC, and cuts BC internally into two segments one of which is double the other. Describe the two possible points where it intersects the (extended) third side.
2. Use the Theorem of Menelaus to prove that the straight line joining the midpoints of two sides of a triangle is parallel to the third side.
3. Prove that if each of the bisectors of a triangle's exterior angles intersects the opposite side, then the three intersection points are collinear. (Hint: Use Exercise 3.5B.6-7)
4. Prove that if the bisector of one of the triangle's exterior angles intersects the opposite side then this intersection is collinear with the intersections of the bisectors of the interior angles at the other two vertices with the opposite sides. (Hint: Use Exercises 3.5B.6-7.)
5. Let $ABCD$ be a trapezoid in which the non-parallel sides AB and CD intersect in the point M and the diagonals intersect in the point N. Prove that the straight line MN bisects both of the sides BC and AD. (Hint: Apply the Theorem of Menelaus to two different triangles.)
6. Let $ABCD$ be a trapezoid in which the non-parallel sides AB and CD intersect in the point M and let N be the midpoint of AD. Prove that if $P = BD \cap CN$ and $Q = AD \cap MP$, then $AQ = 2QD$.
7. Let $ABCD$ be a trapezoid in which the non-parallel sides AB and CD intersect in the point M and Q divide AD internally in the ratio of 2. Prove that if $R = BD \cap CQ$ and $S = AD \cap MR$, then $AS = 3SD$.
8. Let $ABCD$ be a trapezoid in which the non-parallel sides AB and CD intersect in the point M. (Figure 5.10). Define $A_1 = A$, $B_1 = B$, and, for each positive integer n, let $B_{n+1} = CA_n \cap BD$, and let $A_{n+1} = MB_{n+1} \cap AD$. Prove that, $DA = nDA_n$ for $n \geq 1$.

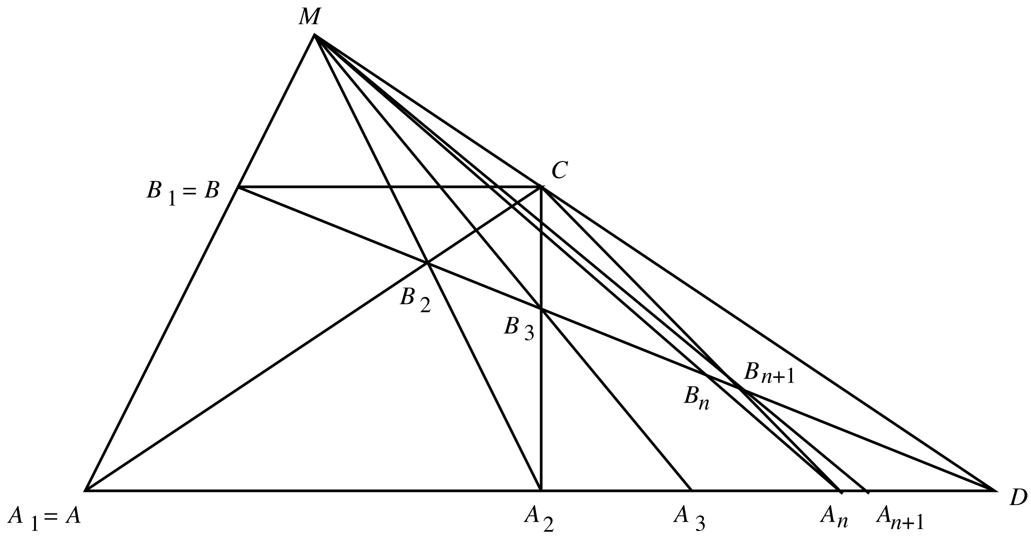


Figure 5.10

9. Let A', B', C' be the respective midpoints of the sides BC, CA, AB of ΔABC. If $P = AA' \cap B'C'$ and $Q = CP \cap AB$ prove that $AB = 3\,AQ$.

10. Two distinct straight lines intersect the sides of ΔABC in the transversals P, Q, R and P', Q', R' respectively. Show that the points $X = BC \cap RP'$, $Y = CA \cap PQ'$, $Z = AB \cap QR'$, are collinear, provided they exist. (Hint: Apply the Theorem of Menelaus to ΔABC with each of the transversals $\{P, Q, R\}$, $\{P', Q', R'\}$, $\{Z, Q, R'\}$, $\{P, Q', Y\}$, $\{P', X, R,\}$.)

11. Two equal segments AE and AF are taken on the sides AB and AC of ΔABC, and M is the midpoint of BC. Show that if $G = AM \cap EF$ then $AG/GF = AB/AC$. (Hint: Let $X = BC \cap EF$. Apply the Theorem of Menelaus to both ΔBEX and ΔCFX with the transversal $\{A, G, M\}$.)

12. The points A, B, C, D on the straight line m and A', B', C', D' on the straight line n are such that AA', BB', CC', DD' are concurrent. Prove that $\dfrac{AB/BC}{AD/DC} = \dfrac{A'B'/B'C'}{A'D'/D'C'}$.

13. Show that if each of the tangents to the circumcircle of a triangle at the vertices of the triangle intersects the extended opposite side of the triangle, then the points of intersection are collinear.

14. What happens to the theorem of Menelaus if P, Q, R are not distinct from A, B, C?

15. Let p, q, r be three circles of unequal radii each of which lies in the others' exterior. Prove that the three intersections of the common external tangents of each pair of circles are collinear.

16. Formulate and prove an analog of Exercise 15 that involves the intersections of common <u>internal</u> tangents.

17. Explain why the point R' in the proof of part 2 of the Theorem of Menelaus exists.

18. Let P, Q, R be a spherical transversal of the spherical ΔABC. Prove that P, Q, R are spherically collinear if and only if

$$\frac{\sin AP}{\sin PB}\ \frac{\sin BQ}{\sin QB}\ \frac{\sin CR}{\sin RA} = -1$$

(Note: Here AP denotes the length of the geodesic joining A and P, etc.)

19. Let P, Q, R be a hyperbolic transversal of the hyperbolic $\Delta\, ABC$. Prove that P, Q, R are hyperbolically collinear if and only if

$$\frac{\sinh AP}{\sinh PB}\ \frac{\sinh BQ}{\sinh QB}\ \frac{\sinh CR}{\sinh RA} = -1\ .$$

20. Comment on Proposition 5.2.1 in the context of taxicab geometry.

21. Comment on Proposition 5.2.1 in the context of maxi geometry.

22(C). Use a computer application to verify the Theorem of Menelaus.

A *Cevian* of $\Delta\, ABC$ is a straight line that joins a vertex of the triangle to a point on the extended opposite side that is not a vertex.

PROPOSITION 5.2.2 (The Theorem of Ceva). *The three Cevians AQ, BR, CP of $\Delta\, ABC$ are concurrent if and only if*

$$\frac{AP}{PB}\ \frac{BQ}{QC}\ \frac{CR}{RA} = 1\ .$$

See Exercise 1.

EXERCISES 5.2B

1. Prove the Theorem of Ceva. (Hint: A Cevian forms two triangles with the sides of the given triangle. Apply the Theorem of Menelaus to these two. Prove the converse in the same indirect manner used to prove the converse part of the Theorem of Menelaus).

Use the Theorem of Ceva to prove the statements below.

2. Prove that the three medians of the triangle are concurrent.

3. Prove that the bisectors of the three interior angles of a triangle are concurrent. (See Exercise 3.5B.6.)

4. Prove that the bisector of an interior angle of $\Delta\, ABC$ and the bisectors of the exterior angles at the other two vertices are concurrent. (See Exercise 3.5B.7.)

5. Prove that in a triangle, the Cevians through the points of contact of the inscribed circle are concurrent.

6. Prove that the three altitudes of every triangle are concurrent. Be sure that your proof also works for obtuse triangles. (Hint: Each of the three altitude "divides" the triangle into two triangles and some of these six triangles are similar.)

7. Suppose AD, BE, CF are concurrent Cevians of ΔABC and the circle through D, E, F intersects the sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AB}$ again in the points D', E', F'. Prove that the Cevians AD', BE', CF' are also concurrent.

8. Let AD, BE, CF be three concurrent Cevians of ΔABC. Then the points $\overleftrightarrow{BC} \cap \overleftrightarrow{EF}$, $\overleftrightarrow{CA} \cap \overleftrightarrow{DF}$, $\overleftrightarrow{AB} \cap \overleftrightarrow{ED}$ are collinear.

9. Formulate and prove a converse to Exercise 8.

10. Two parallelograms $ABCD$ and $AB'C'D'$ have a common angle at A. Prove that the lines $\overleftrightarrow{BD'}$, $\overleftrightarrow{B'D}$, $\overleftrightarrow{C'C}$ are concurrent.

11. If equilateral triangles BCA', CAB', ABC' are described externally on the sides of ΔABC, then the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ are concurrent.

12. If A'', B'', C'' are the centers of the equilateral triangles of the previous exercise, then the lines $\overleftrightarrow{AA''}$, $\overleftrightarrow{BB''}$, $\overleftrightarrow{CC''}$ are concurrent.

13. In the quadrilateral $ABCD$, $E = \overleftrightarrow{AC} \cap \overleftrightarrow{BD}$, $F = \overleftrightarrow{AD} \cap \overleftrightarrow{BC}$, $G = \overleftrightarrow{AB} \cap \overleftrightarrow{CD}$, and $H = \overleftrightarrow{AB} \cap \overleftrightarrow{EF}$. Prove that $AH/HB = -AG/GB$. Does your proof depend on whether E is inside or outside $ABCD$? Does it remain valid even if the cyclic ordering of the vertices of the given quadrilateral is not A, B, C, D?

14. State and prove (using spherical trigonometry) a spherical version of the Theorem of Ceva. (Hint: See Exercise 5.2A.18.)

15. Use Exercise 14 to prove that the spherical medians of a spherical triangle are concurrent.

16. Use Exercise 14 to prove that the spherical angle bisectors of a spherical triangle are concurrent.

17. State and prove (using hyperbolic trigonometry) a hyperbolic version of the Theorem of Ceva. (Hint: See Exercise 5.2A.19.)

18. Use Exercise 17 to prove that the hyperbolic medians of a hyperbolic triangle are concurrent.

19. Use Exercise 17 to prove that the hyperbolic angle bisectors of a hyperbolic triangle are concurrent.

20. Comment on Proposition 5.2.2 in the context of taxicab geometry.

21. Comment on Proposition 5.2.2 in the context of maxi geometry.

22(C). Use a computer application to verify the Theorem of Ceva.

PROPOSITION 5.2.3 (The Theorem of Pappus). *If* $\{A, B, C\}$ *and* $\{A', B', C'\}$ *are two sets of collinear points, then the points* $\overleftrightarrow{AB'} \cap \overleftrightarrow{A'B}$, $\overleftrightarrow{AC'} \cap \overleftrightarrow{A'C}$, $\overleftrightarrow{BC'} \cap \overleftrightarrow{B'C}$, *are also collinear (provided these intersection points all exist).*

See Exercise 1.

PROPOSITION 5.2.4 (The Theorem of Desargues). *For any* ΔABC *and* $\Delta A'B'C'$, *the lines* $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ *are concurrent if and only if the points* $\overleftrightarrow{AB} \cap \overleftrightarrow{A'B'}$, $\overleftrightarrow{BC} \cap \overleftrightarrow{B'C'}$, $\overleftrightarrow{AC} \cap \overleftrightarrow{A'C'}$ are collinear. *(provided the intersections* $\overleftrightarrow{AA'} \cap \overleftrightarrow{BB'}$, $\overleftrightarrow{BB'} \cap \overleftrightarrow{CC'}$, $\overleftrightarrow{CC'} \cap \overleftrightarrow{AA'}$, $\overleftrightarrow{AB} \cap \overleftrightarrow{A'B'}$, $\overleftrightarrow{BC} \cap \overleftrightarrow{B'C'}$, $\overleftrightarrow{AC} \cap \overleftrightarrow{A'C'}$ *all exist).*

See Exercises 2, 3.

PROPOSITION 5.2.5 (The Theorem of Pascal). *The intersections of the three pairs of opposite sides of a cyclic hexagon are collinear* (*provided these intersections all exist).*

See Exercise 4.

Each of the above three propositions ends with an annoying parenthetical qualification that, at a higher level, turns out to be unnecessary. An elegant reinterpretation of the elements of geometry will be offered in the next section which indicates how such nuisances can be evaded.

EXERCISES 5.2C

1. Prove the Theorem of Pappus.
2. Prove the first half of the Theorem of Desargues: For any ΔABC and $\Delta A'B'C'$, if the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ are concurrent then the points $P = \overleftrightarrow{AB} \cap \overleftrightarrow{A'B'}$, $Q = \overleftrightarrow{BC} \cap \overleftrightarrow{B'C'}$, $R = \overleftrightarrow{AC} \cap \overleftrightarrow{A'C'}$ are collinear.
3. Prove the second half of the Theorem of Desargues: For any ΔABC and $\Delta A'B'C'$, the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ are concurrent if the points $\overleftrightarrow{AB} \cap \overleftrightarrow{A'B'}$, $\overleftrightarrow{BC} \cap \overleftrightarrow{B'C'}$, $\overleftrightarrow{AC} \cap \overleftrightarrow{A'C'}$ are collinear.
4. Prove the Theorem of Pascal.

5. Comment on Proposition 5.2.3 in the context of the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

6. Comment on Proposition 5.2.4 in the context of the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

7. Comment on Proposition 5.2.5 in the context of the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

8(C). Use a computer application to verify a) the Theorem of Menelaus

   b) the Theorem of Desargues c) the Theorem of Pascal.

9. Draw nine points in the plane so that ten triples of these points are collinear.

3. The Projective Plane

It is well known that parallel lines look as though they meet in a "vanishing point" or at a "point at infinity". The edges of a long straight road look like they meet at a point on the horizon, as do adjacent railway tracks. This illusion is further supported by the fact that in the representation of such a scene on a canvas or in a photograph, the said edges do indeed meet in the plane of the representation. These suggestive and informal observations were turned by Desargues into a formal and fertile geometrical discipline, called *projective geometry,* in the mid-seventeenth century. Over the centuries this direct descendent of Euclidean geometry acquired great depth and applicability and became an integral part of the mainstream of mathematical evolution. We will now explain how mathematicians converted the informal phrase "meet at infinity" into a formally correct statement.

An *ordinary point* is a point of the Euclidean plane. An *ordinary line* is a straight line of the Euclidean plane. The set of all the ordinary lines parallel to the ordinary line m is the *ideal point* or *point at infinity* or the *vanishing point* $\mathfrak{I}_m$ (Fig. 5.11). If m is any ordinary line then the *extended line* m^* consists of all the points of m together with

$\Im_m$; In other words, $m^* = m \cup \{\Im_m\}$. The *ideal line* or *line at infinity* Λ consists of the set of all the ideal points. A *projective point* is either an ordinary or an ideal point. The *projective plane* consists of all the projective points. A *projective line* is either an extended line or the ideal line.

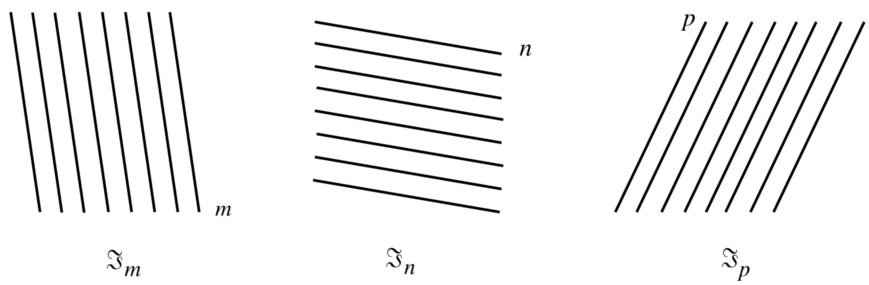


Figure 5.11 Three ideal points.

The assignment of only a single ideal point to a straight line may seem counterintuitive and it is commonly argued that since every line extends to infinity in two directions, each of those directions should receive its own vanishing, or ideal, point. This misconception is reinforced by the observation that when one looks along the aforementioned railroad tracks first in one direction and then in the opposite, the tracks seem to meet in two "different" ideal points. It is important to remember, however, that the vanishing point depends on the observer's point of view. In other words, the above "two" vanishing points are merely two different manifestations of the same ideal point. Figure 5.12 indicates that there are in fact not only two but infinitely many such manifestations.

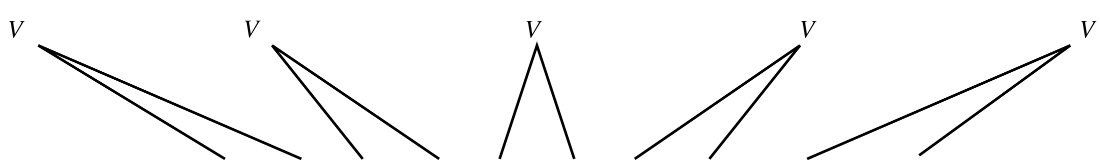


Figure 5.12 Five views of the same vanishing point.

The geometry of the projective plane is very rich and elegant. Some of this elegance can be seen in the next two basic propositions.

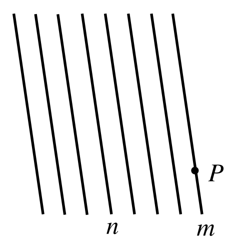
PROPOSITION 5.3.1. *Every two distinct projective points are contained in exactly one projective line.*

GIVEN: Projective points $P \neq Q$.

TO PROVE: There exists exactly one projective line that contains both P and Q.

PROOF: Case 1: P and Q are both ordinary points. In this case P and Q are contained in exactly one ordinary line $m = \overleftrightarrow{PQ}$. Hence they are contained in exactly one extended line m^*. Since both are ordinary points they are not contained in the ideal line. Thus P and Q are contained in exactly one projective line.

Case 2: P is ordinary and Q is ideal, say $Q = \mathfrak{I}_n$ (Fig. 5.13). Let m be that ordinary line that belongs to Q and contains P. Then the extended line m^* contains both P



$Q = \mathfrak{I}_m = \mathfrak{I}_n$

Figure 5.13

and $\mathfrak{I}_m = \mathfrak{I}_n = Q$. If p^* is any extended line that contains both P and $Q = \mathfrak{I}_n$, then, by definition, p is an ordinary line that contains P and is parallel to n. Thus, by Playfair's Postulate, $p = m$ and so $p^* = m^*$. Since the ideal line Λ consists of ideal points only, it cannot contain P. Thus, the points P and Q are contained in exactly one projective line.

Case 3: P and Q are both ideal points. Both P and Q are in the ideal line Λ. Since each extended line contains only one ideal point no extended line contains both P and Q. Hence there is exactly one projective line that contains both P and Q.

Q.E.D.

PROPOSITION 5.3.2. *Every two distinct projective lines intersect in exactly one projective point.*

GIVEN: Two distinct projective lines.

TO PROVE: There is exactly one projective point P on both of these lines.

PROOF: It follows from Proposition 5.3.1 that any two distinct projective lines can intersect in at most one point. Hence it suffices to show that every two projective lines intersect.

Case 1: The two lines are both extended Euclidean lines, say m^* and n^*. If $m \parallel n$ then $\Im_m = \Im_n$ and so m^* and n^* intersect in this common ideal point. Otherwise, m^* and n^* intersect in the ordinary point $m \cap n$

Case 2: One of the straight lines is the ideal line Λ and the other is an extended line m^*. In this case both of the projective lines contain the point $\Im_m$.

Q.E.D.

Despite the fact that the projective plane incorporates points that are seemingly infinitely far away, it is possible to extend the notion of the ratio of lengths of ordinary segments to some cases that involve ideal points in a very fruitful way. The following conventions are needed to accomplish this task.

RATIO CONVENTIONS. *Let A, B, C, D, E, F, G, H be projective points. The equations below hold in the sense that if any term has a numerical value, then all the others have the same numerical value.*

1. *If A, B are ordinary points and C is any ideal point,, then* $\frac{AC}{CB} = -1$.

2. $\frac{AB}{CD} = \frac{BA}{DC}$

3. $\frac{AB}{CD}\frac{EF}{GH} = \frac{AB}{GH}\frac{EF}{CD} = \frac{EF}{CD}\frac{AB}{GH} = \frac{EF}{GH}\frac{AB}{CD}$

4. *If A, B, C, D are ordinary points such that AB//CD, then* $\frac{AB}{CD}$ *is assigned a negative or positive value according as the line segments AC and BD do or do not intersect* (see Fig. 5.14).

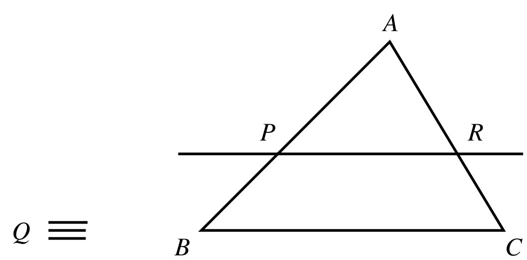


Figure 5.14

It turns out that many interesting theorems of Euclidean geometry are valid in the projective plane as well. Moreover the projective point of view has the advantage of converting annoying exceptions to Euclidean theorems into interesting propositions. This is illustrated by a reexamination of the powerful Theorem of Menelaus in the projective plane. In the figures below ideal points are represented by three short parallel line segments that indicate the entire family of parallel lines that constitute that ideal point.

PROPOSITION 5.3.3. *The Theorem of Menelaus is also valid when one of the transversal points is ideal.*

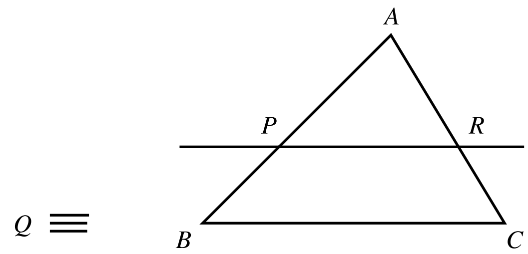


Figure 5.15

GIVEN: Ordinary ΔABC with transversal $\{P, Q, R\}$ where P and R are ordinary and Q is ideal (Fig. 5.15).

TO PROVE: The points P, Q, R are collinear if and only if $\frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = -1$.

PROOF: Suppose P, Q, R are collinear, then $PR \parallel BC$ and so it follows from Proposition 3.5.6 that

$$\frac{AP}{PB} = \frac{AR}{RC} = \frac{RA}{CR}$$

$$\therefore \quad \frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = \frac{BQ}{QC} = -1.$$

Conversely, if $\frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = -1$, then $\frac{AP}{PB}\frac{CR}{RA} = 1$ so that $\frac{AP}{PB} = \frac{AR}{RC}$. It follows from Proposition 3.5.6 that $PR \parallel BC$ and hence P, Q, R are collinear.

Q.E.D.

PROPOSITION 5.3.4. *The Theorem of Menelaus is also valid when one of the triangle's vertices is ideal.*

GIVEN: Projective ΔABC with A ideal and B and C ordinary, and ordinary transversal $\{P, Q, R\}$ (Fig. 5.16).

TO PROVE: The points P, Q, R are collinear if and only if $\frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = -1$.

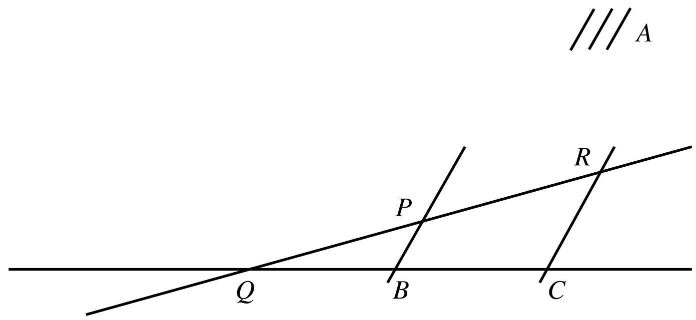


Figure 5.16

Since BP and CR intersect in the ideal point A, it follows that $BP \parallel CR$. Hence, *if* P, Q, R are collinear, then it follows from the similarity of $\Delta\, QBP$ and $\Delta\, QCR$ and the above ratio conventions that

$$\frac{BQ}{QC} = \frac{PB}{CR}$$

$$\therefore \quad \frac{AP}{PB}\,\frac{BQ}{QC}\,\frac{CR}{RA} = \frac{AP}{RA}\,\frac{BQ}{QC}\,\frac{CR}{PB} = (-1)1 = -1.$$

Conversely, if $\frac{AP}{PB}\,\frac{BQ}{QC}\,\frac{CR}{RA} = -1$, then it follows from the value $\frac{AP}{RA} = -1$ that

$$\frac{BQ}{QC}\,\frac{CR}{PB} = 1$$

or $\quad \frac{BQ}{QC} = \frac{PB}{CR}$

But $\angle\, QBP = \angle\, QCR$ [PN 3.1.1]

$\therefore$ $\Delta\, QBP \sim \Delta\, QCR$ [PN 3.5.9]

$\therefore$ $\angle\, BQP = \angle\, CQR$

$\therefore$ P, Q, R are collinear.

Q.E.D.

EXERCISES 5.3

1. Interpret and prove the Theorem of Ceva in the case where the vertex A is ideal and all the other points are ordinary.
2. Interpret and prove the Theorem of Ceva in the case where the transversal point P is ideal and all the other points are ordinary.
3. Interpret and prove the Theorem of Ceva in the case where the transversal points P and Q are ideal and all the other points are ordinary.
4. Interpret and prove the Theorem of Pappus in the case where exactly one of the given intersection points is ideal and all the other points are ordinary.
5. Interpret and prove the Theorem of Pappus in case where two or more of the given intersections are ideal and all the other points are ordinary.
6. Interpret and prove the Theorem of Pascal in the case where exactly one of the given intersection points is ideal and all the other points are ordinary.
7. Interpret and prove the Theorem of Pascal in the case where two or more of the given intersection points are ideal and all the other points are ordinary.
8. Interpret and prove the first half of the Theorem of Desargues in the case where one of the given intersection points is ideal and all the other points are ordinary.
9. Interpret and prove the first half of the Theorem of Desargues in the case where two or more of the given intersection points are ideal and all the other points are ordinary.
10. Discuss ideal points in the context of the following geometries:

    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

CHAPTER REVIEW EXERCISES

In Exercises 1-5 all the points and lines are ordinary.

1. Prove that if the straight line m intersects the sides $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{DA}$ of quadrilateral $ABCD$ in the points P, Q, R, S respectively, then $\frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RD}\frac{DS}{SA} = 1$.
2. Show that the converse of Exercise 1 is false.

3. Generalize the Theorem of Menelaus to arbitrary polygons.

4*. If all the sides of hexagon $ABCDEF$ are tangent to the same circle in its interior, then the three diagonals joining its opposite vertices are concurrent. (Theorem of Brianchon).

5. A parallel to the side BC of $\Delta\, ABC$ meets AB in B' and AC in C'. Prove that BC' and $B'C$ intersect on the median to BC.

6. Divide a given line segment in the ratios $\pm\sqrt{2}/\sqrt{3}$ and $\pm\sqrt{3}/\sqrt{2}$

7. Interpret and prove Exercise 1 if P is ideal and all the other points are ordinary.

8. Interpret and prove Exercise 1 if A is ideal and all the other points are ordinary.

9. Interpret and prove Exercise 5 if the vertex A is ideal.

10. Are the following statements true or false? Justify your answers.

    a) Given two distinct points C and D, there exists exactly one point X on $\overleftrightarrow{CD}$ such that $CX/XD = 3$.

    b) Given two distinct points C and D, there exists exactly one point X on $\overleftrightarrow{CD}$ such that $CX/XD = -3$.

    c) Given two distinct points C and D, there exists exactly one point X on $\overleftrightarrow{CD}$ such that $CX/XD = -1$.

    d) Given two distinct points C and D, there exists exactly one point X on $\overleftrightarrow{CD}$ such that $CX/XD = \pi$.

    e) Given two distinct points C and D, it is possible to construct (in the sense of *The Elements)* a point X on $\overleftrightarrow{CD}$ such that $CX/XD = \pi$.

    f) Given any three distinct points $P, Q, R,$ there exists $\Delta\, ABC$ such that AQ, BR, CP are concurrent Cevians for that triangle.

    g) In the projective plane, every three ideal points are collinear.

    h) In the projective plane, every two ideal lines intersect.

    i) In the projective plane, every two projective lines intersect.

    j) In the projective plane every ideal point lies on some extended line.

    k) Playfair's postulate holds in the projective plane.

    l) If C and D are two ordinary points of the projective plane and r is a real number, then there is exactly one point X on $\overset{\times}{CD}$ such that $CX/XD = r$.