CHAPTER 4

Circles and Regular Polygons

Circles and regular polygons are the subject of Books III and IV of *The Elements*. Euclid's abstract exposition of the interrelation of chords, arcs, and tangents lines is augmented with the computation of the circle's circumference and area.

1. The Neutral Geometry of the Circle

Equal circles are circles that have equal radii. A *chord* of a circle is a line segment that joins two of its points. A *diameter* is a chord that contains the center of the circle. An *arc* of a circle is a portion of the circle that joins two of its points. Every chord determines two arcs of the circle. Consequently, it takes at least three letters to denote an arc unambiguously and the two arcs of the circle of Figure 4.1 with endpoints A and B should be denoted, properly speaking, by arc(AEB) and arc(AFB). Nevertheless, it is customary to label both of these arcs arc(AB) and to rely on the context for clarification. A *segment* of a circle is the portion between a chord and either of its arcs. A *sector* of a circle is the portion between two radii. The arcs determined by a diameter are each called a semicircle. That the two semicircles determined by a diameter are equal (in length) is a proposition that Euclid mentions in Definition 17 (Chapter 2). This observation is proved as part of Proposition 4.1.1 below.

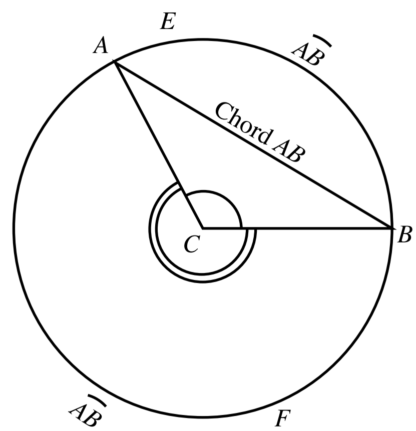

Figure 4.1

A *central angle* of a circle is one both of whose sides are radii. Every arc subtends a central angle that is either greater or less than 180° according as the arc is greater or less than a semicircle. Every chord subtends a central angle that is at most 180°.

The following four propositions of Euclid's are established here with a single unified proof.

PROPOSITION 4.1.1(III.26). *In equal circles equal central angles stand on equal arcs.*

PROPOSITION 4.1.1(III.27). *In equal circles central angles standing on equal arcs are equal to one another.*

PROPOSITION 4.1.1(III.28). *In equal circles equal chords cut off equal arcs, the greater equal to the greater and the less to the less.*

PROPOSITION 4.1.1(III.29). *In equal circles, equal arcs are subtended by equal chords.*

GIVEN: Equal circles centered at E and E' respectively. Points A, B on the first circle and points A', B' on the second (Fig. 4.2).

TO PROVE: The following are equivalent:

1. arc(AB) = arc($A'B'$)
2. $AB = A'B'$
3. $\angle AEB = \angle A'E'B'$

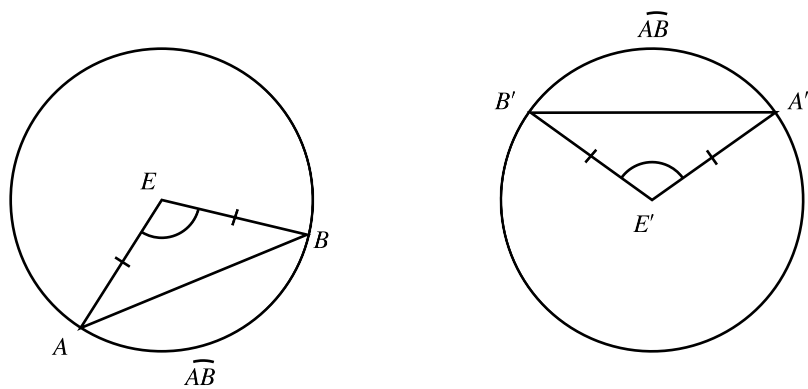

Figure 4.2

PROOF: 1 => 2: Since the given circles are equal, it is possible to apply the circle centered at E to that centered at E' so that E and A fall on E' and A' respectively, and arc AB falls along arc $A'B'$. The two arcs having equal lengths, B falls on B'. It follows from PT 1 that the chord AB falls on the chord $A'B'$ and hence, by CN 4, $AB = A'B'$.

2 => 3: $\Delta AEB \cong \Delta A'E'B'$ by SSS because

AB = $A'B'$		[Given]
AE = $A'E'$		[Given]
BE = $B'E'$		[Given]

$$\therefore \quad \angle AEB = \angle A'E'B'$$

3 => 1: Since $\angle AEB = \angle A'E'B'$ it is possible to apply the first circle to the second so that E falls on E' and these angles coincide. Since the circles have equal radii it follows that arc AB falls on arc $A'B'$. Consequently these arcs have equal lengths.

Q.E.D.

COROLLARY 4.1.2. *In a circle all the semicircles are equal to each other.*
See Exercise 1.

PROPOSITION 4.1.3(III.3). *In a circle, a radius bisects a chord not through the center if and only if the radius and the chord are perpendicular to each other.*
See Exercise 2.

EXERCISES 4.1A

1. Prove Corollary 4.1.2.
2. Prove Proposition 4.1.3.
3. Prove that in a circle, a diameter is greater than any chord which is not a diameter.
4. Prove that two chords of a circle are equal if and only if they are at equal distances from its center. (Exercise 2.3N.2 can be used to produce a neutral proof.)
5. Prove that a circle cannot contain three collinear points (III.2)
6. Prove that in a circle, the radius perpendicular to a chord bisects that chord's central angle and arc.
7. Prove that in a circle two equal intersecting chords cut each other into respectively equal segments.
8. Prove that of two unequal chords in a circle, the greater one is closer to the center. (This is Proposition III.15. It can be easily proved on the basis of the Theorem of Pythagoras, but such a proof is would not be neutral. Euclid's neutral proof is based on Proposition I.24 (Exercise 2.3Q.5).)

9. Let A be a given point and p a circle centered at C. If the point P moves along the circle p prove that the midpoint of AP describes a circle centered at the midpoint of CA. (See Exercises 3.1D.7-8. It is necessary to consider three cases, depending on the relative positions of A and p.)
10. Construct the midpoint of a given arc on a given circle.
11. Given an arc of a circle, construct the center of the circle.
12. Given points A, B, C, D construct a circle through A and B whose center is equidistant from C and D.
13. Given a point A inside a circle, construct a chord that is bisected by A. Prove that this chord is the shortest of all the chords through A.
14. Given an angle α and a line segment a, construct a circle whose center is on one side of α and which cuts a segment equal to a on the other side.
15. Given a circle p and a point A outside it, construct a straight line through A which cuts the circle so that the segment from A to the circle equals the segment in the circle. (See Exercises 3.1.D7-8.)
16. Comment on Proposition 4.1.1 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
17. Comment on Proposition 4.1.3 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

An infinitely extended straight line is said to be *tangent* to a circle if they have exactly one point in common, and that point is called their *point of contact*.

PROPOSITION 4.1.4(III.16, 18). *If a straight line intersects a circle, then they are tangent if and only if the straight line is perpendicular to the radius through the point of contact.*

GIVEN: Circle $(C; CP)$, straight line PT (Fig. 4.3).

TO PROVE: PT is tangent to $(C; CP)$ if and only if $CP \perp PT$.

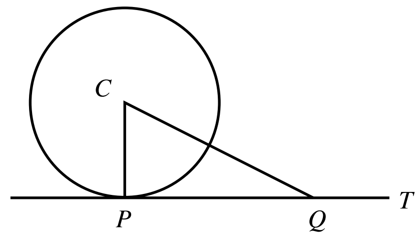

Figure 4.3

PROOF: Suppose first that PT is tangent to $(C; CP)$. Hence, if Q is any point of PT that is distinct from P, it must lie outside the circle so that $CP < CQ$. Consequently, $CP \perp PT$ [PN 2.3.24].

Conversely, suppose that $CP \perp PT$. Then, by Proposition 2.3.24, for any point Q of PT that is distinct from P, $CP < CQ$. Consequently no such point Q can lie on the circle $(C; CP)$ and hence PT is tangent to it.

Q.E.D.

EXERCISES 4.1B

1. Suppose S and T are the contact points of the tangents to a circle from a point P outside it. Prove that $PS = PT$.
2. Prove that in a circle the contact points of two parallel tangents are the endpoints of a diameter.
3. Prove that the straight line that joins the center of a circle to the intersection of two of its tangents bisects the angle between these tangents.
4. Prove that for each side of the triangle there is a circle that is tangent to that side at one of its interior points and tangent to the other two sides at points on their extensions. Construct these circles. (Use Exercise 3 above.)

Two circles are said to be tangent if they intersect in exactly one point. If one circle lies inside the other the tangency is said to be internal; otherwise it is external.

5. Prove that if two circles are externally tangent then the line segment joining their centers contains the point of contact. (Hint: Proceed by contradiction and examine the triangle formed by the centers and the contact point.)

6. Prove that if two circles are internally tangent then the line joining their centers contains the point of contact.
7. Prove that if two circles are tangent to each other then they have a common tangent line at their point of contact.
8. Prove that if two circles lie outside each other then they have four different common tangent lines.
9. Let m and n be common tangents to unequal circles such that both circles lie inside one of the angles formed by these tangents. Prove that the line joining the centers of the circles bisects this angle.
10. Let m and n be common tangents to unequal circles such that the circles lie in vertically opposite angles formed by these tangents. Prove that the line joining the centers of the circles bisects these angles.
11. Given two circles with the same center and unequal radii, prove that all the chords of the larger circle that are tangent to the smaller circle have the same length.
12. Construct a circle with a given radius tangent to a given line.
13. Construct a circle with a given radius, tangent to a given line, and containing a given point.
14. Construct a circle containing a given point and tangent to a given straight line at a given point on the line.
15. Construct a circle that is tangent to two given parallel straight lines.
16. Construct a circle that is tangent to two given parallel straight lines and contains a given point between them. How many solutions are there?
17. Construct a circle that is tangent to two intersecting straight lines.
18. Construct a circle that is tangent to two given intersecting straight lines and contains a given point.
19. Construct a circle that is tangent to two given parallel straight lines as well as to a given third line that intersects them.
20. Given a circle p and a point A construct a straight line containing A such that its segment inside p has a given length. (Hint: See Exercise 11.)
21. Construct a point such that the lengths of the tangents from it to two given circles are given.
22. Comment on Proposition 4.1.4 in the context of the following geometries:

    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

The following proposition was proved by Euclid in its entirety. The proof offered in this text is incomplete in two ways. In the first place, the argument is restricted to rational values of the ratios in question. Moreover, given an angle $\angle ABC$ and a positive integer m, this argument makes use of the angle $(\angle ABC)/n$ even though it has not been demonstrated that such an angle can be constructed within Euclid's system.

PROPOSITION 4.1.5(VI.33). *In equal circles, central angles are proportional to the arcs on which they stand.*

GIVEN: Equal circles with centers G and H respectively (Fig. 4.4).

TO PROVE: $\frac{\angle BGL}{\angle EHN} = \frac{arc(BL)}{arc(EN)}$

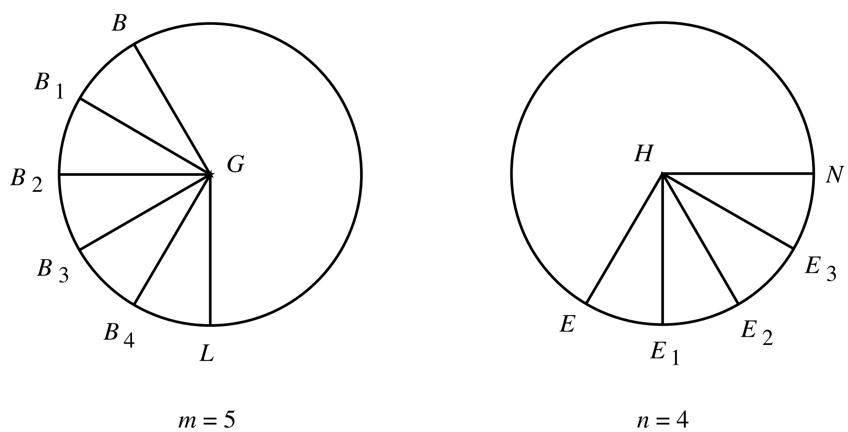

Figure 4.4

SUPPORTING ARGUMENT: The argument is limited to the case where the given ratios are rational. In other words, it is assumed that there exist positive integers m and n such that

$$\frac{\angle BGL}{\angle EHN} = \frac{m}{n} \qquad \text{i.e.,} \qquad \frac{\angle BGL}{m} = \frac{\angle EHN}{n}$$

Let α be an angle such that

$$\alpha = \frac{\angle BGL}{m} = \frac{\angle EHN}{n}.$$

It follows that there exist points $B_1, B_2, ..., B_{m-1}$ on arc(BL) and points $E_1, E_2, ..., E_{n-1}$ on arc(EN)such that

$$\angle BGB_1 = \angle B_1GB_2 = ... = \angle B_{m-1}GL = \angle EHE_1 =$$

$$\angle E_1HE_2 = ... = \angle E_{n-1}HN = \alpha .$$

Hence, by Proposition 4.1.1,

$$\text{arc}(BB_1) = \text{arc}(B_1B_2) = ... = \text{arc}(B_{m-1}L)$$

$$= \text{arc}(EE_1) = \text{arc}(E_1E_2) = ... = \text{arc}(E_{m-1}L).$$

If the common length of these arcs is denoted by β, then

$$\frac{arc(BL)}{arc(EN)} = \frac{m\beta}{n\beta} = \frac{m}{n} = \frac{\angle BGL}{\angle EHN}$$

Q.E.D.

Proposition 4.1.5 was used by Eratosthenes (ca. 275 - 194 BC), director of the Alexandrian library, to obtain a remarkably accurate estimate of the circumference of the earth. He knew that on the summer solstice the sun shone down at mid-day directly into a well in the city of Syene whereas in Alexandria, 5000 stadia to the north, the shadows indicated that the sun formed an angle of 1/50 of 360° (7.2°) with the vertical. Assuming that the sun is so far away that its rays can be considered to be parallel when they reach the earth (Fig. 4.5), he then used Proposition 4.1.5 to obtain the equation

$$\frac{\text{circumference of earth}}{\text{distance from Alexandria to Syene}} = \frac{360^{o}}{7.2^{0}} = 50$$

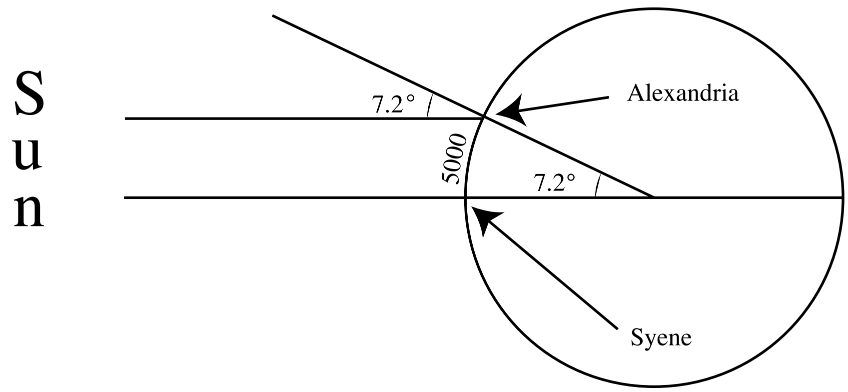

Figure 4.5

from which he concluded that the circumference is $50 \cdot 5000 = 250{,}000$ stadia. In order to make his answer divisible by 60 (probably because of the influence of the Babylonian sexagesimal number system) he adjusted this result to 252,000 stadia. The standard stade of the time had a length of 178.6 meters which converts his rounded estimate to 45,007 km, an overestimate of 12.3%, since the circumference of the earth is actually 40,075 km.

EXERCISES 4.1C

1. A circle has circumference 10 ft. Find the lengths of the arcs that subtend the following angles at the center of the circle:

   a) 10^{o} b) 30^{o} c) 90^{0}

   d) 110^{o} e) 120^{o} f) 180^{o}

2. A location on earth has latitude 25^{o} N. Find its distance from the equator and from the North Pole.

3. A location on earth has latitude 70^{o} N. Find its distance from the equator and from the North Pole.

4. A location on earth has latitude 70^{o} S. Find its distance from the equator and from the North Pole.
5. A location on earth has latitude 70^{o} S. Find its distance from the equator and from the North Pole.
6. A location on earth lies 2000 km north of the equator. Find its latitude.
7. A location on earth lies 1234 km north of the equator. Find its latitude.
8. A location on earth lies 1000 km south of the equator. Find its latitude.
9. A location on earth lies 617 km south of the equator. Find its latitude.
10. Comment on Proposition 4.1.5 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

2. The Non-Neutral Geometry of the Circle

The next proposition is one of the most surprising in *The Elements*. Unlike those appearing the previous section, its implications are quite unexpected.

PROPOSITION 4.2.1(III.20). *In a circle, the angle at the center is double of the angle at the circumference, when the angles have the same arc as base.*

GIVEN: Points *A, B, C* on the circumference of a circle centered at *E* (Fig. 4.6).

TO PROVE: $\angle BEC = 2\angle BAC$.

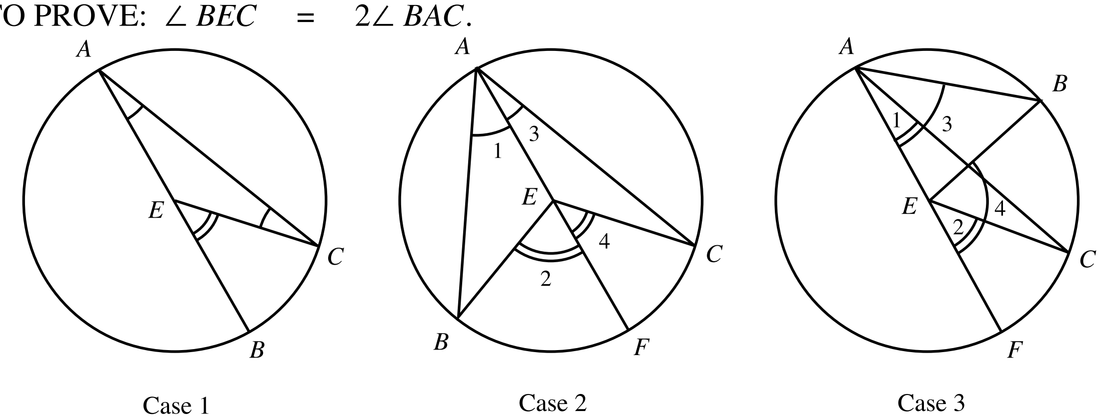

Figure 4.6

PROOF: It is necessary to distinguish three cases.

Case 1: One of the sides of $\angle BAC$ contains the center E.

	$\angle BEC$	$= \angle BAC + \angle ECA$	[Exterior, PN 3.1.6]
	$\angle BAC$	$= \angle ECA$	[$AE = EC$, PN 2.3.5]
$\therefore$	$\angle BEC$	$= 2\angle BAC$	

Case 2: The center E lies in the interior of $\angle BAC$. Let F be the other intersection of $\overrightarrow{AE}$ with the circumference of the given circle. Then,

	$\angle 2$	$= 2\angle 1$	[Case 1]
	$\angle 4$	$= 2\angle 3$	[Case 1]
$\therefore$	$\angle BEC$	$= 2\angle BAC$	[CN 2]

Case 3: The center E lies outside of $\angle BAC$. Let F be the other intersection of $\overrightarrow{AE}$ with the circumference of the given circle. Then,

	$\angle 2$	$= 2\angle 1$	[Case 1]
	$\angle 4$	$= 2\angle 3$	[Case 1]
$\therefore$	$\angle BEC$	$= 2\angle BAC$	[CN 3]

Q.E.D.

Proposition 4.2.1 has several corollaries whose proofs are relegated to the exercises.

PROPOSITION 4.2.2(III.21). *In a circle, the angles in the same segment are equal to one another.*

GIVEN: Points A, B, C, D on the circumference of a circle such that A and D lie on the same side of BC (Fig.4.7).

TO PROVE: $\angle BAC = \angle BDC$.

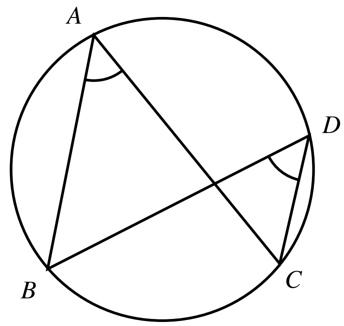

Figure 4.7

PROOF: See Exercise 1.

This proposition is somewhat counterintuitive. Suppose the points A and B in Figure 4.8 are fixed whereas P slides clockwise around the circle occupying positions P_1, P_2, ..., P_5 successively. Proposition 4.2.2 implies that as long as the point P remains in the interior of the upper (or longer) arc(*AB)* the angle *APB* retains a constant (acute) value. When P passes through A or B, *APB* is no longer an angle. Finally, when P is in the interior of the shorter (or lower) arc(*AB)* the angle *APB*

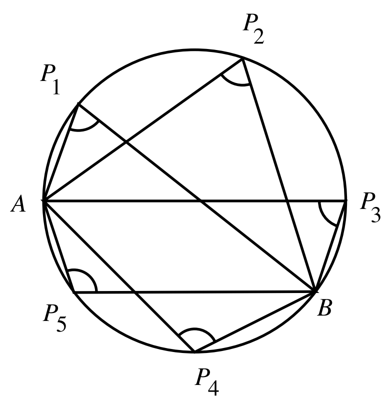

Figure 4.8 A discontinuous function.

assumes a different (obtuse) value. In other words, even though the point P moves in a *continuous* manner, $\angle APB$ varies as a *discontinuous* function of the position of P.

PROPOSITION 4.2.3(III.31). *In a circle, the angle subtended by a diameter from any point on the circumference is a right angle.*

See Exercise 2.

PROPOSITION 4.2.4(III.32) *Let* AB *be a chord of a circle and let* $\overleftrightarrow{AT}$ *be any straight line at* A*. Then the line* $\overleftrightarrow{AT}$ *is tangent to the circle if and only if* $\angle TAB$ *is equal to the angle at the circumference subtended by the intercepted arc.*

GIVEN: Circle p with chord AB, straight line $\overleftrightarrow{AT}$, arc(AB) (Fig 4.9).

TO PROVE: $\overleftrightarrow{AT}$ is tangent to p if and only if $\angle TAB$ equals the angle at the circumference of p subtended by arc(AB).

PROOF: Let AD be the diameter of the circle containing A, and join BD (Figure 4.9).

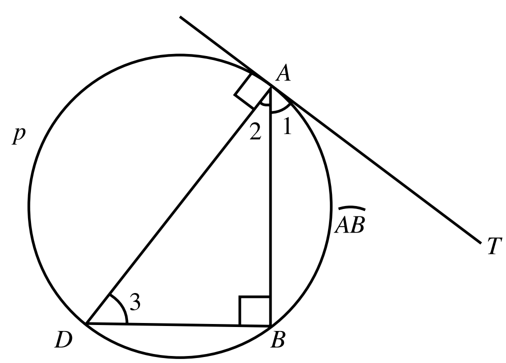

Figure 4.9

By Proposition 4.2.3 $\angle ABD = 90^{\circ}$. Hence the following statements are all equivalent to each other:

$\overleftrightarrow{AT}$ is tangent to the circle

$\angle DAT = 90^{\circ}$

$\angle 1 = 90^{\circ} - \angle 2$

$\angle 1 = \angle 3$.

Q.E.D.

PROPOSITION 4.2.5(III.36-37). *Let P be a point outside a given circle p let T be a point on p, and let PAB be a secant line with chord AB. Then PT is tangent to p if and only if*

$$PA \cdot PB = PT^2 .$$

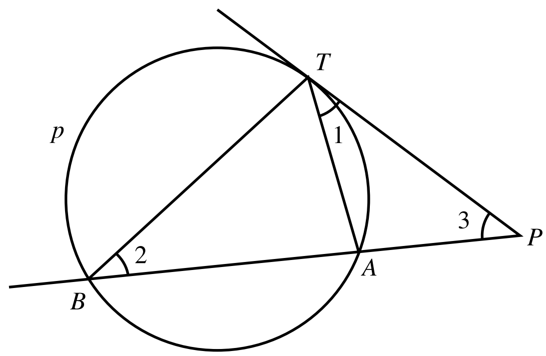

Figure 4.10

GIVEN: Point P outside circle p, straight lines PT and PAB that intersect p in T, A, B (Fig. 4.10).

TO PROVE: *PT is tangent to p if and only if* $PA \cdot PB = PT^2$.

PROOF: The following statements are all equivalent to each other:

The line *PT* is tangent to p

$\angle 1 = \angle 2$; [PN 4.2.4]

$\Delta\, TPA$ and $\Delta\, BPT$ are similar to each other [PN 3.5.7]

$$\therefore \quad \frac{PA}{PT} = \frac{PT}{PB}$$

$$\therefore \quad PA \cdot PB = PT^2 .$$

Q.E.D.

A polygon is said to be *cyclic* if all of its vertices lie on a circle.

PROPOSITION 4.2.6(III.22). *The opposite angles of a cyclic quadrilateral are equal to two right angles*.

See Exercise 3.

EXERCISES 4.2A

1. Prove Proposition 4.2.2.
2. Prove Proposition 4.2.3.
3. Prove Proposition 4.2.6.
4. Prove that in a circle parallel chords enclose equal arcs.
5. Prove that if the quadrilateral $ABCD$ is cyclic, then the exterior angle at A equals the interior angle at C.
6. In a circle the extensions of the chords AB and KL intersect in a point P outside the circle. Prove that $\angle AKP = \angle LBP$ and $\angle BKP = \angle LAP$.
7. In a circle the extensions of the chords AB and CD intersect in a point P outside the circle. Prove that $PA \cdot PB = PC \cdot PD$ (Proposition III.35).
8. In a circle the chords AB and CD intersect in a point P inside the circle. Prove that $PA \cdot PB = PC \cdot PD$ (Proposition III.36).
9. Prove that two equal and parallel chords in a circle constitute the opposite sides of a rectangle.
10. Prove that if the hexagon $ABCDEF$ is cyclic and the interior angles at A and D are equal, then $BC \parallel EF$.
11. In the cyclic quadrilateral $ABCD$, $AD = BC$. Prove that the interior angles at A and B are equal to each other (as are those at C and D).
12. Prove that the sum of the interior angles at A, C and E in the cyclic hexagon $ABCDEF$ is four right angles.

13*. Prove that if the perpendicular chords AB and CD of a circle intersect at the point M (inside the circle) then the straight line through M that is perpendicular to AD bisects the chord BC.

14. Prove that every cyclic rhombus is a square.
15. Prove that if A and B are two distinct points and D is any other point on AB then the locus of all the points P in the plane such that $\frac{AP}{PB} = \frac{AD}{DB}$ is a circle. (This is the circle of Apollonius.)
16. State and prove the converse of Proposition 4.2.6.

17. Given a line segment AB, construct the circle which consists of all the points from which AB subtends an angle of 90^{o}.

18. Given a line segment AB, construct the arc which consists of all the points from which AB subtends an angle of 60^{o}.

19. Given a line segment AB, construct the arc which consists of all the points from which AB subtends an angle of 120^{o}.

20. Given a line segment AB, construct the arc which consists of all the points from which AB subtends an angle equal to a given angle α.

21. Construct a triangle given the data:
    a) a, h_b, h_c b) a, h_a, α c) a, m_a, α
    d) $a + b + c, h_a, \alpha$.

22. Construct a parallelogram given its two diagonals and one of its angles.

23. Given line segment AB and CD and angles α and β, construct a point P such that $\angle APB = \alpha$ and $\angle CPD = \beta$.

24. In a given ΔABC construct a point P such that $\angle APB = \angle BPC = \angle CPA$.

25. Comment on Proposition 4.2.1 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

26. Comment on Proposition 4.2.2 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

27. Comment on Proposition 4.2.3 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

28. Comment on Proposition 4.2.6 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

29(C). Use a computer application to verify the following propositions: a) 4.2.1 b) 4.2.2
    c) 4.2.3b d) 4.2.4.

Three (or more) straight lines are said to be *concurrent* if they all contain the same point.

PROPOSITION 4.2.7. *The three perpendicular bisectors of the sides of a triangle are concurrent.*

GIVEN: ΔABC; DD', EE', FF' are the perpendicular bisectors of AB, AC, and BC respectively (Fig. 4.11).

TO PROVE: DD', EE', FF' are concurrent.

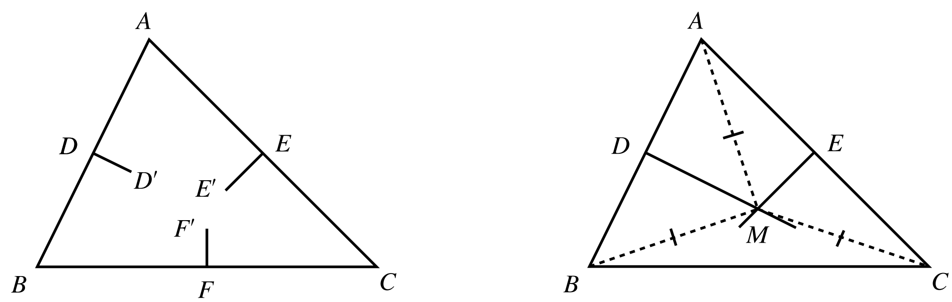

Figure 4.11

PROOF: Exercise 3.1A.4 guarantees that DD' and EE' intersect in some point M. Draw AM, BM, CM. Then

	$AM = BM$	[PN 2.3.12]
	$AM = CM$	[PN 2.3.12]
$\therefore$	$BM = CM$	[CN 1]
$\therefore$	M is on the perpendicular bisector to BC	[PN 2.3.13]

Q.E.D.

A circle is said to *circumscribe* a triangle if all of the triangle's vertices are on the circle. Its center and radius are, respectively, the triangle's *circumcenter* and *circumradius*.

PROPOSITION 4.2.8(IV.5). *About a given triangle to circumscribe a circle.*

See Exercise 1.

PROPOSITION 4.2.9. *The bisectors of the three interior angles of a triangle are concurrent.*

GIVEN: Δ *ABC*, *AA'*, *BB'*, *CC'* are the bisectors of $\angle$ *BAC*, $\angle$ *ACB*, $\angle$ *ABC*, respectively (Fig. 4.12).

TO PROVE: *AA'*, *BB'*, *CC'* are concurrent.

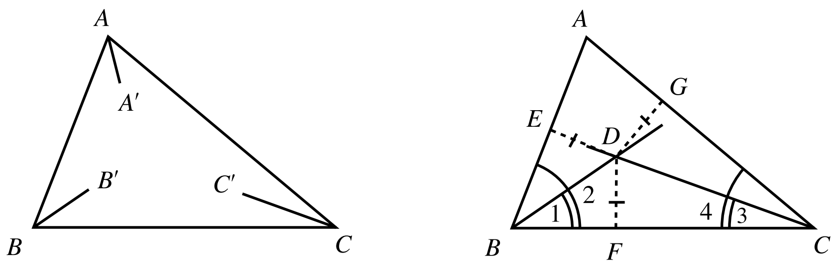

Figure 4.12

PROOF: Since

$$\angle 1 + \angle 3 = \tfrac{1}{2}(\angle 2 + \angle 4) < \tfrac{1}{2}\, 180^{\circ} \qquad \text{[PN 2.3.21]}$$

it follows from Postulate 5 that *BB'* and *CC'* intersect in some point *D*. Let *E, F* and *G* be those points on *AB, BC, CA* respectively such that $DE \perp AB$, $DF \perp BC$, and $DG \perp AC$. Then

	DE = *DF* = *DG*	[PN 2.3.31]
∴	*DE* = *DG*	[CN 1]
∴	*DA* bisects $\angle$ *BAC*	[PN 2.3.32]
		Q.E.D.

A circle that lies in the interior of a triangle and is tangent to all of its sides is said to be *inscribed* in the triangle. Its center and radius are, respectively, the triangle's *incenter* and *inradius*.

PROPOSITION 4.2.10(IV.4). *In a given triangle to inscribe a circle.*

See Exercise 2.

EXERCISES 4.2B

1. Prove Proposition 4.2.8.
2. Prove that similar triangles have circumradii that are proportional to their sides.
3. Prove Proposition 4.2.10.
4. Prove that similar triangles have inradii that are proportional to their sides.
5. Prove that the circumcenter of a right triangle is the midpoint of its hypotenuse.
6. Prove that if the circumcenter of ΔABC lies inside the triangle then the triangle is acute, if the center is on a side the triangle is right, and if the center is outside the triangle then the triangle is obtuse.
7. Prove that the circumcenter of an acute triangle lies inside the triangle.
8. Prove that the circumcenter of an obtuse triangle lies outside it.
9. Prove that the area of the triangle equals the product of half its perimeter with the inradius. (Hint: Examine the three triangles formed by the center of the circle with the triangle's three sides.)
10. In a given circle inscribe a triangle similar to a given triangle.
11. Prove that the altitudes of the triangle are concurrent. (Hint: Through each vertex of the triangle draw a line parallel to the opposite side. Then show that the altitudes in question are the perpendicular bisectors of the triangle formed by these parallels.)
12. Comment on Proposition 4.2.7 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
13. Comment on Proposition 4.2.8 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
14. Comment on Proposition 4.2.9 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
15. Comment on Proposition 4.2.10 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

16(C). Use a computer application to verify the following propositions: a) 4.2.5 b) 4.2.7.

3. Regular Polygons

A polygon is *regular* if all of its sides and all of its interior angles are equal. The equilateral triangles are the regular triangles and they are the subject matter of Proposition 1 of Book I. Squares are the regular quadrilaterals and they are constructed in Proposition 3.3.1. Book IV of *The Elements* is mostly concerned with the constructibility of other regular polygons and their inscription in circles. Regular hexagons are also easily constructed.

PROPOSITION 4.3.1(IV.15). *In a given circle to inscribe a regular hexagon.*

GIVEN: Circle $p = (O; r)$ (Fig. 4.13).

TO CONSTRUCT: Points A, B, C, D, E, F on p such that $ABCDEF$ is a regular hexagon.

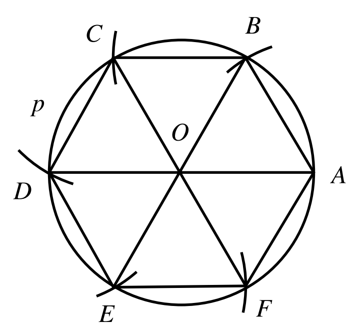

Figure 4.13

CONSTRUCTION: Let A be an arbitrary point on the circle p. Let B be the intersection of an arc of radius r and center A with p. Let C be the intersection of an arc of radius r and center B with p, and let D, E, F be constructed in a similar manner. Then $ABCDEF$ is a regular hexagon.

PROOF: By construction $\Delta\ AOB$, $\Delta\ BOC$, $\Delta\ COD$, $\Delta\ DOE$, and $\Delta\ EOF$ are all equilateral so that $\angle\ AOB = \angle\ BOC = \angle\ COD = \angle\ DOE = \angle\ EOF = 60^{\circ}$. It follows that $\angle\ FOA = 360^{\circ} - 5\cdot 60^{\circ} = 60^{\circ}$ and hence the isosceles $\Delta\ FOA$ is also equilateral. Thus, $FA = OA$ and so each of the sides of $ABCDEF$ has length r. It also follows that each of the interior angles of $ABCDEF$ equals 120°. Thus, $ABCDEF$ is a regular hexagon.

Q.E.D.

A slight variation on the construction of the regular hexagon yields the flower-like configuration of Figure 4.14.

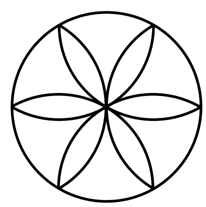

Figure 4.14

The construction of the regular pentagon is a considerably more difficult matter. Some of the technically demanding details are isolated in the following lemma. Others were listed as exercises above.

PROPOSITION 4.3.2(IV.10). *To construct an isosceles triangle having each of the angles at the base equal to double of the remaining one.*

TO CONSTRUCT: $\Delta\ ABC$ such that $\angle\ ABC = \angle\ ACB = 2\angle\ BAC$ (Figure 4.15).

CONSTRUCTION: Let AB be an arbitrary line segment and let D be a point such that $AB\cdot BD = AD^2$ [PN 3.4.1]. Then the required $\Delta\ ABC$ is that triangle such that $AC = AB$ and $BC = AD$ [PN 2.3.27].

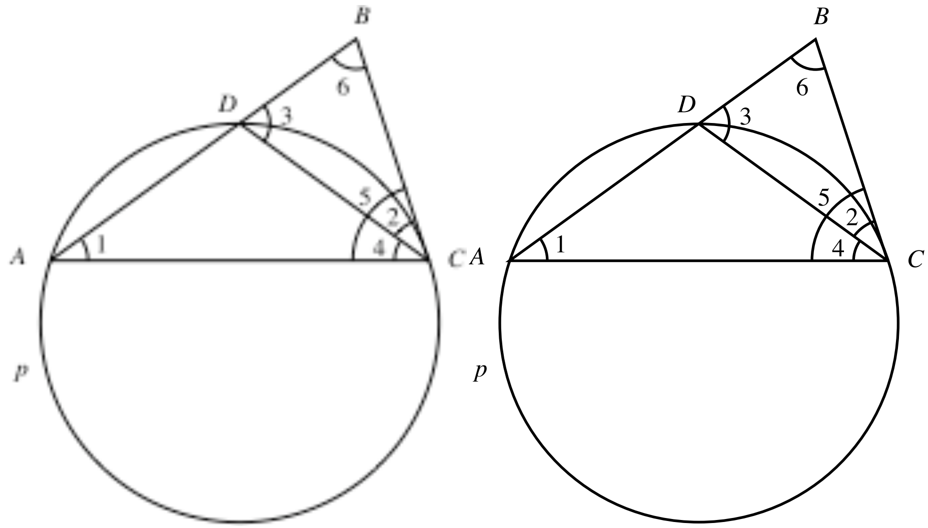

Figure 4.15

PROOF: By construction, $BC^2 = AD^2 = AB \cdot BD$. It therefore follows that from Proposition 4.2.5 that BC is tangent to the circle p that circumscribes ΔACD. Hence,

$$\angle 1 = \angle 2 \qquad \text{[PN 4.2.4]}$$

$$\therefore \quad \angle 3 = \angle 1 + \angle 4 = \angle 2 + \angle 4 = \angle 5 = \angle 6$$

$$\therefore \quad DC = BC = AD$$

$$\therefore \quad \angle 1 = \angle 4 = \frac{1}{2}\angle 3 = \frac{1}{2}\angle 6 = \frac{1}{2}\angle 5$$

Q.E.D.

If the smallest of the angles of the triangle of Proposition 4.3.2 is denoted by x, then the other two angles are each $2x$ and so, by Proposition 3.1.6,

$$180^0 = x + 2x + 2x = 5x$$

from which it follows that $x = 36^o$. Hence the following corollaries hold.

PROPOSITION 4.3.3. *To construct angles of* 36^o *and* 72^o.

We are now ready to construct the regular pentagon.

PROPOSITION 4.3.4(IV.11). *In a given circle to inscribe a regular pentagon.*

GIVEN: Circle $p = (O; r)$ (Fig. 4.16).

TO CONSTRUCT: Points A, B, C, D, E on p such that $ABCDE$ is a regular pentagon.

CONSTRUCTION: At the center O of the circle construct five non-overlapping central angles of 72^{o} [PN 4.3.3]. Label the successive intersections of their sides with the circle A, B, C, D, E. Then $ABCDE$ is the required pentagon.

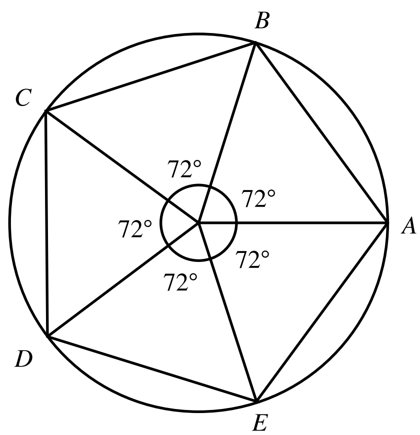

Figure 4.16

PROOF: The five constructed isosceles triangles are all congruent by SAS. It follows that the five sides $AB, BC, CD, DE,$ *and* EA are all equal. Moreover, the base angles of these triangles are $\frac{1}{2}(180^{o} - 72^{o}) = 54^{o}$ each and hence all of the pentagon's interior angles are equal (to 108^{o} each).

Q.E.D.

Euclid took the trouble to prove that the regular 15-sided polygon is constructible (Exercise 3). It is reasonable to suppose that this was his way of pointing out that there is an interesting question to be pondered here. Namely, for which integers n can the regular n-sided polygon be constructed? It has already been shown above that this is possible for $n = 3, 4, 5, 6$. Some more such n can be easily produced by simply doubling the number of sides of any constructible regular polygon (see Exercises 1, 2, 4). However, this does not answer the question for such numbers as 7, 9, 11, 13, 14, 17, ... The surprising intricacy of the construction of the pentagon indicates that such polygons might pose an even greater challenge. This problem continued to excite the interest of mathematicians after Euclid, but no progress was made for over 2000 years until the young Gauss demonstrated the constructibility of the 17-sided polygon in 1796. Actually, he did much more. Using the newly emergent theory of complex numbers Gauss proved that a regular p-sided polygon can be constructed for every prime integer p that has the form $2^{2^n} + 1$ for some nonnegative integer n. These include the values

$$2^{2^0} + 1 = 2^1 + 1 = 3, \qquad 2^{2^1} + 1 = 2^2 + 1 = 5,$$

$$2^{2^2} + 1 = 2^4 + 1 = 17, \qquad 2^{2^3} + 1 = 2^8 + 1 = 257,$$

$$2^{2^4} + 1 = 2^{16} + 1 = 65{,}537.$$

Curiously, the next number in this sequence, namely $2^{2^5} + 1 = 2^{32} + 1 = 4{,}294{,}967{,}297$ fails to be a prime since it can be factored as $641 \cdot 6{,}700{,}417$, a fact that had already been noted by Euler over fifty years earlier. The same is true for all the numbers of this form for $n = 6, 7, ..., 16$ and several other values including $n = 1945$. In fact, it is not known whether there are any more primes p that can be expressed in this form above and beyond the five listed above.

Gauss completely resolved the issue of the constructibility of regular polygons as follows.

PROPOSITION 4.3.5. *It is possible to construct (in the sense defined by Euclid) a regular g-gon ($g \geq 3$) if and only if the factorization of g into primes has the form*

$$g = 2^k p_1 p_2 ... p_m$$

where $m \geq 0$ and $p_1, p_2, ..., p_m$ are distinct primes each of which has the form $2^{2^n} + 1$.

Thus, the regular 2040-gon is constructible because $2040 = 2^3 \cdot 3 \cdot 5 \cdot 17$ whereas the regular 28-sided and 100-sided polygons are not constructible because $28 = 2^2 \cdot 7$ and $100 = 2^2 \cdot 5^2$.

EXERCISES 4.3

1. Prove that the regular octagon is constructible .
2. Prove that the regular decagon is constructible.
3. Prove that the regular 15-sided polygon is constructible (Proposition IV.16).
4. Let g be a positive integer. Prove that if the regular g-sided polygon is constructible, so is the regular $2g$-sided polygon.
5. Let p and q be two prime integers. Prove that if the regular p-sided and q-sided polygons are constructible so is the regular pq-sided polygon.
6. Let g and h be relatively prime integers such that the regular g-sided and h-sided polygons can be constructed. Prove that the regular gh-sided polygon can be constructed.
7. Let $g, h > 1$ be integers such that h is an integer multiple of g. Prove that if the regular h-sided polygon is constructible so is the regular g-sided polygon.
8. Use a calculating device to prove that $2^{2^6} + 1$ is not a prime integer.
9. For which $n = 3, 4, ..., 100$ is the regular n-gon constructible?
10. For which $n = 101, 102, ..., 200$ is the regular n-gon constructible?
11. Comment on Proposition 4.3.1 in the context of the following geometries:

    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
12. Comment on Proposition 4.3.2 in the context of the following geometries:

a) spherical; b) hyperbolic; c) taxicab; d) maxi.

13. Comment on Proposition 4.3.4 in the context of the following geometries:

a) spherical; b) hyperbolic; c) taxicab; d) maxi.

14. Show that in taxicab geometry equilateral triangles are not necessarily equiangular. Can all three of the angles of an equilateral taxicab triangle be distinct?

15(C). Perform the construction of Proposition 4.3.1 using a computer application.

4. Circle Circumference and Area

The fundamental observation that the circumference of a circle is proportional to its diameter (and hence also its radius) goes back several millennia BC. Surprisingly, Euclid says nothing on this topic in *The Elements*.

PROPOSITION 4.4.1. *Circumferences of circles are proportional to their radii.*

GIVEN: Circles $p_1 = (O_1; r_1)$ and $p_2 = (O_2; r_2)$ of circumferences c_1 and c_2 respectively (Fig. 4.17).

TO PROVE: $\frac{c_1}{c_2} = \frac{r_1}{r_2}$

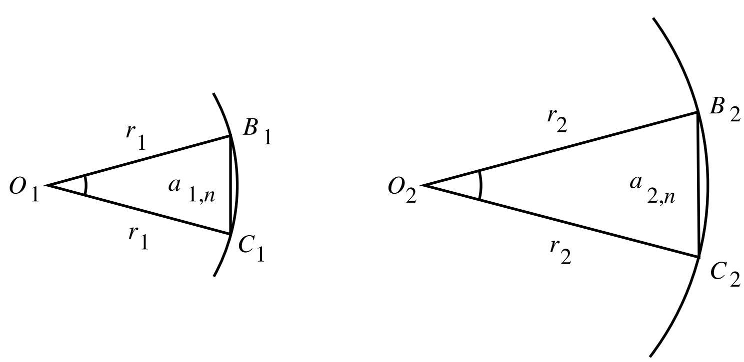

Figure 4.17

SUPPORTING ARGUMENT: It follows from Proposition 4.3.1 that it is possible to inscribe a regular hexagon in each of the given circles. By repeatedly bisecting the central angles subtended by the sides of the polygons it is possible to inscribe in each circle p_i, $i = 1, 2$, a regular n-sided polygon of side, say, $a_{i,n}$ where n is an integer of the form $3{\cdot}2^m$. It is clear that $\angle B_1O_1C_1 = \angle B_2O_2C_2 = 360^\circ/n$. Since $\Delta\, O_1B_1C_1$ and $\Delta\, O_2B_2C_2$ are isosceles they must be equiangular [PN 3.1.6] and hence they are similar [PN 3.5.7]. In other words, the sides of the inscribed polygons are proportional to the radii. Making the reasonable assumption that for large n the difference between the circumferences of each circle and that of its inscribed polygon is negligible, it follows that

$$\frac{c_1}{c_2} = \frac{na_{1,n}}{na_{2,n}} = \frac{a_{1,n}}{a_{2,n}} = \frac{r_1}{r_2} \quad ,$$

Q.E.D.

An alternate supporting argument that makes use of calculus is described in Exercise 1.

It follows from the above proposition that if c and r denote the circumference and radius of an arbitrary circle, then the ratio

$$\frac{c}{r}$$

has a constant value, say α. This constant number can be used to restate the above proposition in the following form.

PROPOSITION 4.4.2. *There is a number* α *such that if* c *and* r *are respectively the circumference and radius of any circle, then* $c = \alpha r$.

[]

The numerical value of α is, of course, of interest, and will be estimated at the end of this section. Next, the area of the circle is examined. The following proposition was proved by Euclid using the *Method of Exhaustion* which was the Greeks' version of the integral calculus. This method was developed by Euclid's predecessor Eudoxus whom Archimedes (287 – 212 B.C.) credits with this and other similar propositions.

PROPOSITION 4.4.3(XII.2). *The areas of circles are proportional to the squares of their radii.*

GIVEN: Circles $p_1 = (O_1; r_1)$ and $p_2 = (O_2; r_2)$ of areas A_1 and A_2 respectively (Fig 4.17).

TO PROVE: $\frac{A_1}{A_2} = \frac{r_1^2}{r_2^2}$.

SUPPORTING ARGUMENT: It follows from Proposition VI.19 (Exercise 3.5E.10) that the areas of $\Delta\, O_1B_1C_1$ and $\Delta\, O_2B_2C_2$ are proportional to the squares of the radii r_1 and r_2. Making the reasonable assumption that for large n the difference between the areas of each circle and that of its inscribed polygon is negligible, it follows that

$$\frac{A_1}{A_2} = \frac{n(\Delta\, O_1B_1C_1)}{n(\Delta\, O_2B_2C_2)} = \frac{\Delta\, O_1B_1C_1}{\Delta\, O_2B_2C_2} = \frac{r_1^2}{r_2^2}$$

Q.E.D.

An alternate supporting argument makes use of calculus (see Exercise 2).

It follows from the above proposition that if A and r denote the area and radius of an arbitrary circle, then the ratio

$$\frac{A}{r^2}$$

has a constant value, say π. This number can be used to restate the above proposition in the following form.

PROPOSITION 4.4.4. *There is a number π such that if A and r are the area and radius respectively, of any circle, then $A = \pi r^2$.*

The numerical value of π is, of course, of interest, but the relationship between α and π needs to be addressed first. The discovery of this relationship is attributed by Proclus to Archimedes.

PROPOSITION 4.4.5. *The proportionality constants of the circumference and area of a circle are related by the equation $\alpha = 2\pi$.*

SUPPORTING ARGUMENT: Suppose the circle p is divided into n equal sectors

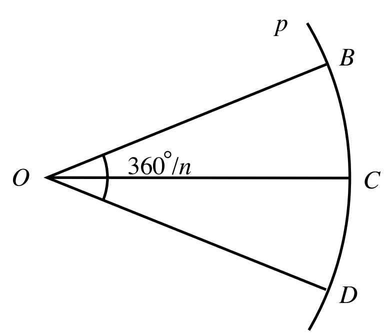

Figure 4.18

each of which has a central angle of $360^0/n$, and let OBD be a typical sector (Fig. 4.18). If n is large it may be assumed that OBD is a triangle with altitude $OC = r$. Applying Proposition 3.2.5 it follows that this triangle has area $r \cdot \text{arc}(BD)/2$. Hence the circle p has area

$$A = n\frac{r}{2}arc(BD) = c\frac{r}{2} = (\alpha r)\frac{r}{2} = \frac{\alpha}{2}r^2.$$

It now follows from Proposition 4.4.4 that $\pi = \frac{\alpha}{2}$ or $\alpha = 2\pi$.

Q.E.D.

COROLLARY 4.4.6. *The circumference of a circle of radius r is $2\pi r$.*

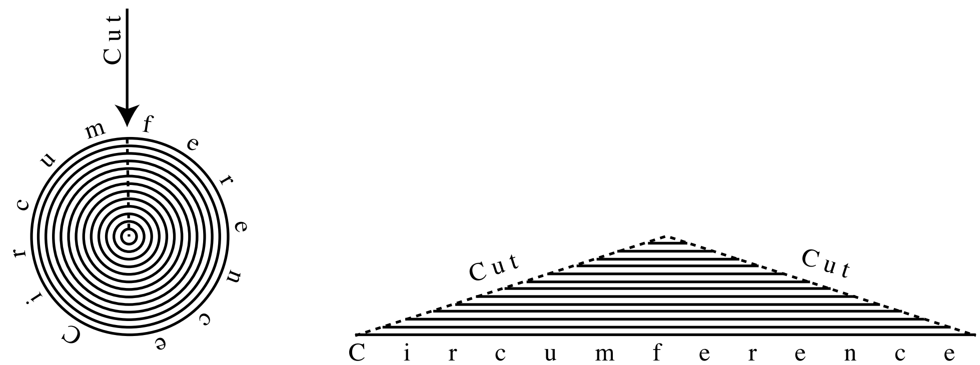

Figure 4.19

An alternate supporting argument for Proposition 4.4.5 can be based on Figure 4.19. Imagine that the circle of radius r on the left is filled with circular strands. Cut the circle along the vertical dashed radius and straighten out all the strands as indicated until they form an isosceles triangle (that its sides are straight follows from Proposition 4.4.2). It follows from Proposition 3.2.5 that the area of this triangle, and hence also the area of the circle, is

$$A = (\text{circumference} \cdot r)/2 \quad = \quad cr/2 = \alpha r \cdot r/2 = (\alpha/2)r^2,$$

and the rest of the argument proceeds as before. Appealing as this argument is, it is fraught with logical perils which are discussed in Exercise 21.

Yet another alternative argument in support Proposition 4.4.5 calls for slicing the circle into an even number of equal sectors and rearranging these to form the near-parallelogram at the top of Figure 4.20. As the number of sectors increases to infinity the

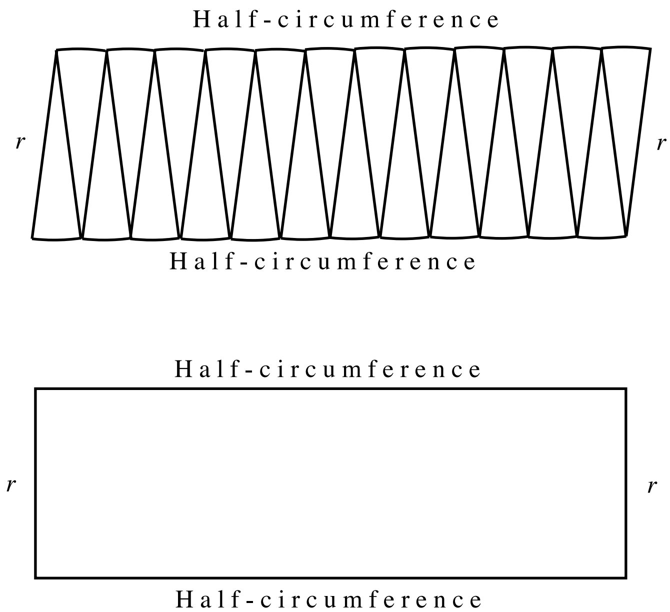

Figure 4.20

near-parallelogram converges to the rectangle below it, whose area is clearly

$$A = (\text{half circumference})r = cr/2 = \alpha r{\cdot}r/2 = (\alpha/2)r^2 .$$

The procedure of successive approximations used in Proposition 4.4.1 also yields a method for obtaining numerical estimates of the constant of proportion π. This was first carried out by Archimedes and constitutes the first of a long (and still ongoing) series of scientific estimations of π. Let a_n denote the length of the chord AB which is one side of the regular n-gon inscribed in a circle of radius 1 (Fig. 4.21). If C is the midpoint of the corresponding arc(AB), then the chord AC has length a_{2n}. Two

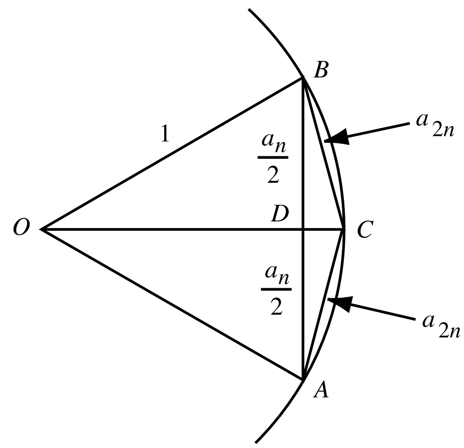

Figure 4.21

applications of the Theorem of Pythagoras then yield

$$a_{2n}^2 = AC^2 = AD^2 + DC^2 = (\frac{a_n}{2})^2 + (OC - OD)^2$$

$$= (\frac{a_n}{2})^2 + (1 - \sqrt{OA^2 - AD^2}\,)^2$$

$$= (\frac{a_n}{2})^2 + (1 - \sqrt{1 - (a_n/2)^2}\,)^2$$

$$= (\frac{a_n}{2})^2 + (1 - 2\sqrt{1 - (\frac{a_n}{2})^2} + 1 - (\frac{a_n}{2})^2)$$

$$= 2 - \sqrt{4 - a_n^2}\ .$$

Hence,

PROPOSITION 4.4.7. *If* a_n *denotes the length of the regular polygon with* n *sides that is inscribed in a circle of radius* 1, *then*

$$a_{2n} = \sqrt{2 - \sqrt{4 - a_n^2}}$$

Since the length of the side of the inscribed regular hexagon equals the radius, it follows that $a_6 = 1$ and so

$$a_{12} = \sqrt{2 - \sqrt{4 - 1}} = \sqrt{2 - \sqrt{3}} = .5176380902...$$

$$a_{24} = \sqrt{2 - \sqrt{4 - (2 - \sqrt{3})}} = \sqrt{2 - \sqrt{2 + \sqrt{3}}}$$

$$= .2610523844...$$

$$a_{48} = \sqrt{2 - \sqrt{4 - (2 - \sqrt{2 + \sqrt{3}})}}$$

$$= \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}} = .1308062585...$$

$$a_{96} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}} = .0654381656... .$$

Since the length of any arc exceeds that of its chord (another one of those reasonable assumptions), the circumference of the circle exceeds that of any inscribed polygon. As the circle of radius 1 has circumference 2π it follows that for each positive integer n,

$$\pi > \frac{na_n}{2}$$

and in particular

$$\pi > 48a_{96} = 3.14103195089... .$$

To obtain upper bounds for the value of π Archimedes examined regular n-gons whose sides were tangent to the circle. Suppose a regular n-gon is inscribed in a circle of radius 1 and at each of the vertices a straight line tangent to the circle is constructed. It is easily verified that the resulting polygon surrounding the circle is also a regular n-gon of

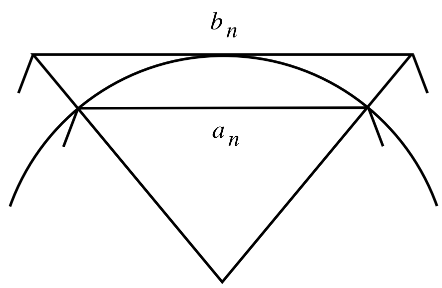

Figure 4.22 Comparing a circumscribed polygon with an inscribed one.

side, say, b_n (Fig. 4.22). The side b_n can be estimated by showing that (see Exercise 9)

$$b_6 = \frac{2}{\sqrt{3}} \qquad \text{and} \qquad b_{2n} = \frac{2(\sqrt{4 + b_n^2} - 2)}{b_n} \qquad (1) .$$

Alternately, the figure above can be used to show that

$$b_n = \frac{2a_n}{\sqrt{4 - a_n^2}} .$$

This gives us $b_{96} = .0654732208...$ and hence

$$\pi < \frac{96b_{96}}{2} = 3.1427145996... .$$

Archimedes did not have the decimal number system at his disposal and he had to work within the much more cumbersome systems that were then current. Using some of the quite complicated methods for the estimation of square roots by means of fractions that were then known he was able to show that

$$48a_{96} > \frac{6336}{2017\frac{1}{4}} > 3\tfrac{10}{71} \quad (= \quad 3.1408...)$$

and

$$48b_{96} < \frac{14688}{4673\frac{1}{2}} < 3\tfrac{1}{7} \quad (= \quad 3.1428...) \quad .$$

Thus, Archimedes proved that

PROPOSITION 4.4.8. $3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}$.

This section and chapter conclude with a discussion of some paradoxes and problems regarding the areas of circles. Consider Figure 4.23 where $\Delta\, ABC$ is both right and

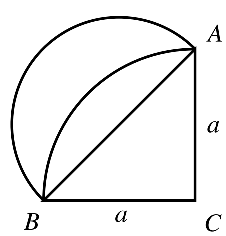

Figure 4.23

isosceles with legs of length a and hypotenuse of length $a\sqrt{2}$. The outside arc is a semicircle with diameter AB and radius $a\sqrt{2}/2$ whereas the inside arc is a quartercircle centered at C with radius a. Regions bounded between two such arcs are called *lunes* (but these are different from the spherical lunes). Note that the area of this lune is the difference between the entire figure and the quartercircle centered at C. Thus,

$$\text{area of lune} = \text{area of semicircle} + \text{area of } \Delta ABC - \text{area of quartercircle}$$

$$= \frac{1}{2}\pi\left(\frac{a\sqrt{2}}{2}\right)^2 + \frac{a^2}{2} - \frac{\pi a^2}{4} = \frac{\pi a^2}{4} + \frac{a^2}{2} - \frac{\pi a^2}{4} = \frac{a^2}{2}$$

$$= \text{area of } \Delta\, ABC.$$

This surprising equation is due to Hippocrates of Chios. A similar equation appears in Exercise 10. There are two unexpected aspects to this equation. First, the area of the lune, whose boundary consists of circular arcs, turns out to have an expression that is free of π. Second, this curvilinear figure has the same area as a triangle. This leads naturally to the question of whether it is possible to construct a triangle, or a square, for that matter, whose area equals that of a given circle (the simplest of all the curvilinear regions). The operative word here is *construct*. It is clear that any circle of radius r has the same area as a square of side $r\sqrt{\pi}$. The difficulty lies in constructing $r\sqrt{\pi}$ within the framework of Euclidean geometry. This problem drew the attention of many mathematicians and non-mathematicians both in classical times and during the subsequent two and a half millennia. Although many individuals dedicated their lives to the solution of this problem, and some even deluded themselves into believing they had discovered the construction, all their efforts were in fact wasted. In 1882 the German mathematician C. L. Ferdinand von Lindemann (1852-1939) proved a theorem which had the following corollary amongst many others:

Given a line segment a it is impossible to construct (in the sense of The Elements) a square whose area equals that of the circle of radius a.

EXERCISES 4.4A

1. The length of the graph of the function $y = f(x)$ for $a \le x \le b$ is $\int_a^b \sqrt{1 + f'^2}\, dx$. Use this formula to prove Proposition 4.4.1.
2. Use calculus to prove Proposition 4.4.3.
3. Using 3.14 for the value of π, in a circle of radius 10" find the length of the arc and the area of a sector determined by a central angle of
   a) 60^o b) 20^o c) 90^o d) 100^o
   e) 180^o f) 230^o.
4. Compute the radical expressions for a_{192} and b_{192} and use them (and a calculator) to obtain decimal bounds of the value of π.
5. Compute the radical expressions for a_{384} and b_{384} and use them (and a calculator) to obtain decimal bounds of the value of π.
6. Compute the radical expressions for a_{768} and b_{768} and use them (and a calculator) to obtain decimal bounds of the value of π.
7. Compute the radical expressions for a_{1536} and b_{1536} and use them (and a calculator) to obtain decimal bounds of the value of π.
8. Prove Equations (1).
9. Prove that of two circular arcs joining two given points, the one with the longer radius has shorter length.
10. Semicircles are constructed on the sides of a right $\Delta\, ABC$ (Fig. 4.24). Prove that the sum of the areas of the two lunes I and II equals the area of $\Delta\, ABC$.

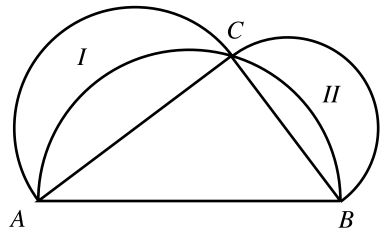

Figure 4.24

11. Show that the circumference of a circle of spherical radius r on a sphere of radius R is $2\pi R \sin (r/R)$.

12. Show that the area of a spherical circle of spherical radius r on a sphere of radius R is $2\pi R^2[1 - \cos (r/R)]$.

13. Comment on Proposition 4.4.1 in the context of the following geometries:
a) spherical; b) hyperbolic; c) taxicab; d) maxi.

14. Comment on Proposition 4.4.2 in the context of the following geometries:
a) spherical; b) hyperbolic; c) taxicab; d) maxi.

15. Comment on Proposition 4.4.3 in the context of the following geometries:
a) spherical; b) hyperbolic; c) taxicab; d) maxi.

16. Comment on Proposition 4.4.4 in the context of the following geometries:
a) spherical; b) hyperbolic; c) taxicab; d) maxi.

17. Comment on Proposition 4.4.5 in the context of the following geometries:
a) spherical; b) hyperbolic; c) taxicab; d) maxi.

18. Comment on Proposition 4.4.6 in the context of the following geometries:
a) spherical; b) hyperbolic; c) taxicab; d) maxi.

19. Comment on Proposition 4.4.7 in the context of the following geometries:
a) spherical; b) hyperbolic; c) taxicab; d) maxi.

20. Comment on Proposition 4.4.8 in the context of the following geometries:
a) spherical; b) hyperbolic; c) taxicab; d) maxi.

21. Explain the following paradox. Suppose the method that was used to convert a circle into a triangle (see paragraph following Proposition 4.4.6) is applied to the same square *ABCD* in two different manners - first by slicing from a corner to the center and second by slicing from the middle of a side to the center (Fig. 4.25). The two triangles so obtained have their bases equal to the perimeter of the square but their altitudes are clearly different. Why are two triangles of different areas obtained?

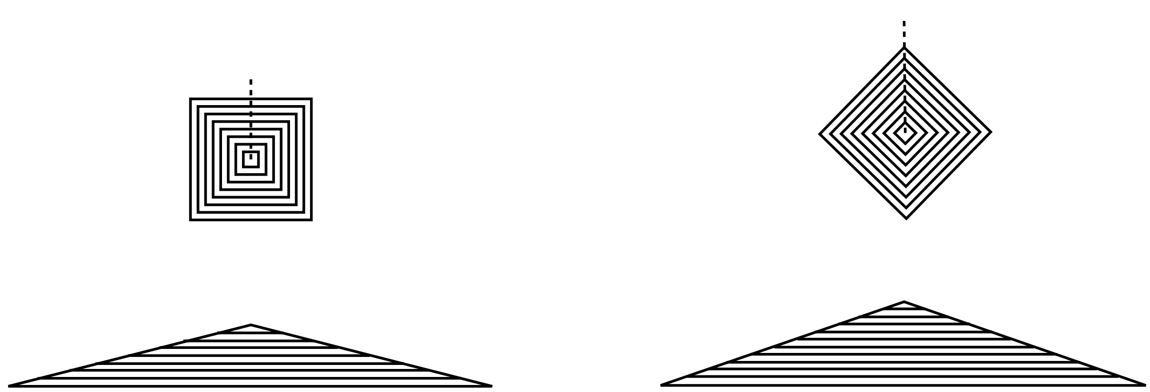

Figure 4.25

4.5 Impossible Constructions (Mostly)

Part of the legacy that the Greek mathematicians passed on to their successors was a collection of construction problems they could not resolve by ruler and compass alone. While most of these problems have already been discussed (Sections 2.3 and 4.3), it might be a good idea to reexamine this topic here in order to provide a better perspective on its outcome. We begin by listing the specific construction problems in question.

1. *To divide a given angle into three equal parts.*
2. *To construct a regular n-gon for each integer* $n \geq 3$.
3. *To construct a square whose area equals that of a given circle.*
4. *To construct a cube whose volume is double that of a given cube.*

The reader will recall that Cartesian coordinates were invented for the purpose of expressing geometrical problems in the language of algebra. Since construction problems are geometrical, this applies to them as well. Some of this relation between geometry and algebra has already been pointed out. If a and b are the lengths of two given line segments, then it is possible to construct line segments of lengths $a + b$ (Exercise 2.3A.3) and $a - b$ (Proposition 2.3.3). Assuming a to be a unit length, Exercise 3.5B.13 shows how, given segments of lengths b and c, it is possible to construct a segment of length bc. Assuming c to be a unit length, the same exercise can be used to construct, for any given segments of lengths a and b, a segment of length a/b. Finally, assuming b to have unit length, Exercise 3.5B.14 can be used to construct, for any given segment of length a, a line segment of length $\sqrt{a}$. Thus, the four arithmetical operations, as well as the taking of square roots, can be mimicked by ruler and compass constructions.

The Cartesian coordinate system can be used to argue that the power of ruler and compass constructions cannot be extended beyond these five algebraic operations. To do this, it is first necessary to formalize some notions. A *configuration* is a set of points, straight lines and circles. An *elementary ruler and compass construction* is any of the following five operations.

1. Draw the line joining two given points;
2. Draw a circle with a given center and radius;
3. Find the intersection of two given straight lines;
4. Find the intersection of a given circle and a given straight line;
5. Find the intersection of two given circles.

A configuration T is said to be constructible from configuration S provided every element of T can be obtained from the elements of S by a succession of elementary ruler and compass constructions. In particular, note that every construction problem stipulates a given configuration and aims at the derivation of a desired configuration.

Assume now that a Cartesian coordinate system has been chosen to which all the configurations below are referred. The *numerical aspects* of the point (x, y) are x and

y. The *numerical aspects* of the straight line with equation $ax + by + c = 0$ are $a, b,$ and, c. The *numerical aspects* of the circle with equation $x^2 + y^2 + ax + by + c = 0$ are $a, b,$ and c.

A real number r is said to be a *Hippasian function* of the set S provided it is obtainable from the elements of S and the rational numbers by rational operations and extractions of real square roots (this terminology honors Hippasus of Metapontum, the discoverer of the irrationality of $\sqrt{2}$). For example, the numbers below are all Hippasian functions of the set $S = \{\pi, \sqrt[3]{2}, e\}$.

$$1, \frac{3}{5}, \frac{\sqrt[3]{2}}{e}, \frac{\pi + 3e}{2 - \sqrt[3]{2}}, \frac{3\sqrt{\pi} - 4\sqrt[3]{e}}{\sqrt{10 + \sqrt[3]{2}}}$$

The *Hippasian numbers* are those that are obtainable from the rational numbers alone by the rational operations and the extractions of real square roots. In fact, this is tantamount to saying that they are obtainable from the number 1 by the said operations. The numbers below are all Hippasian numbers.

$$1, \frac{3}{5}, -\sqrt{2}, \sqrt{2 + \sqrt{2 + \sqrt{3}}}, \frac{\sqrt{1 + \sqrt{2}}}{5 + \sqrt{35 - \sqrt{13}}}.$$

The following theorem formalizes the intuitively plausible connection between constructibility and Hippasian functions.

THEOREM 4.5.1. *If configuration T is constructible from configuration S then the numerical aspects of T are Hippasian functions of the numerical aspects of S.*

OUTLINE OF PROOF: Suppose configuration T is obtained from configuration S by the elementary construction (i), where $i = 1, 2, 3, 4, 5$. If $i = 5$, for example, let the two given circles have equations

$$x^2 + y^2 + ax + by + c = 0 \qquad \text{and} \qquad x^2 + y^2 + a'x + b'y + c' = 0$$

By Exercise 5, the intersection point of these two circles, if it exists, has coordinates which are Hippasian functions of a, b, c, a', b', c'. The proof of the cases $i = 1, 2, 3, 4$ is similar (see Exercises 1-4) and we conclude that if T is constructible from S by any ruler and compass operations, then the numerical aspects of T are obtainable from those of S in the desired manner. Q.E.D.

We note in passing that the converse of Theorem 4.5.1 is also valid, albeit somewhat harder to prove. As this converse is not needed for the proof of the impossibilities below, it is relegated to Exercise 6.

The strategy for demonstrating the non-feasibility of a ruler and compass construction calls for demonstrating that the numerical aspects of the desired configuration are not Hippasian functions of those of the given configuration. Matters can, and will be, simplified below by setting things up so that the numerical aspects of the given data are either integers or Hippasian numbers, and hence it will suffice to show that the desired configuration has a non-Hippasian number as one of its numerical aspects

The following unproven proposition provides an easily applied criterion for recognizing non-Hippasian numbers. It is found, in a more general form, in many undergraduate modern algebra texts.

PROPOSITION 4.5.2. *Let x be a real solution of the equation*

$$ax^3 + bx^2 + cx + d = 0, \qquad (1)$$

where a, b, c, d are integers. Then x is Hippasian if and only if this equation has a rational solution. u

Unlike the above proposition, the next one is found in many precalculus texts and is easily proven (Exercise 7).

PROPOSITION 4.5.3 (The Rational Zeros Theorem). *Let*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \; + \ldots + a_1 x + a_0 ,$$

where all the a_i's are integers. If p/q is a rational number in lowest terms such that

$$P(p/q) = 0,$$

then p is a factor of a_0 and q is a factor of a_n. u

The four construction problems of antiquity are now reexamined one at a time. The simplest of these turns out to be the doubling of the cube.

DOUBLING THE CUBE: If it were possible to double a given cube by ruler and compass constructions, then it would certainly be possible to construct a cube of volume 2. The length of the side of such a cube would be $\sqrt[3]{2}$ and it would have to be a Hippasian number. However, $\sqrt[3]{2}$ is clearly a solution of the equation

$$x^3 - 2 = 0 \qquad (2)$$

and hence, by Proposition 4.5.2, this equation would have to have a rational solution, say p/q in lowest terms. By Proposition 4.5.3 p must be a factor of 2 and q a factor of 1. Hence p/q must be one of the numbers $\pm 1/1, \pm 2/1$ none of which, by Exercise 9, is a solution of the Equation (2). Thus, the supposed feasibility of a ruler and compass doubling of the cube has lead to a contradiction and we conclude that

The cube cannot be doubled by ruler and compass alone.

ANGLE TRISECTION: We next argue that there is no method for trisecting angles by ruler and compass alone. Suppose, to the contrary, that such a method exists and is used to trisect the 60° angle of an equilateral triangle whose side has unit length. Here it may be supposed that the given configuration consists of the three points $O(0, 0)$, $A(0, 1)$,

$B(1/2,\ \sqrt{3}/2)$ (Fig. 4.26) all of whose numerical aspects are Hippasian numbers. The hypothetical construction yields an angle of 20^o that may be placed at the origin with one side on the x-axis (Fig. 4.26). The (constructible) intersection P of this angle's other side with the circle $(O; 1)$ has coordinates $(\cos 20^o, \sin 20^o)$ and hence it follows from Theorem 4.5.1 that $\cos 20^o$ is a Hippasian function of Hippasian numbers. Consequently, $\cos 20^o$ is a Hippasian number. We now go on to obtain an analog of Equation (2). If $x = \cos 20^o$, then, by Exercise 8,

$$1/2 = \cos 60^o = 4 \cos^3 20^o - 3 \cos 20^o = 4x^3 - 3x$$

and hence

$$8x^3 - 6x - 1 = 0 . \qquad (3)$$

By Proposition 4.5.2, this equation has a rational solution, say p/q, where p and q are integers. By Proposition 4.5.3, p/q must be one of the fractions

$$\pm 1, \pm 1/2, \pm 1/4, \pm 1/8$$

none of which, by Exercise 10, is a root of Equation (3). Thus, the assumption of the feasibility of a ruler and compass method for trisecting angles has resulted in a contradiction, and hence

There is no method for trisecting angles by ruler and compass alone.

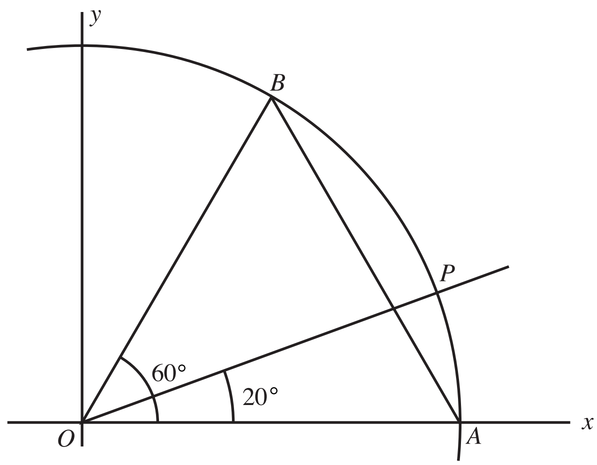

Figure 4.26

REGULAR *n*-GONS: Whether or not a regular n-gon is constructible by ruler and compass turns out to depend on the value of n. In general, such a polygon can be constructed if and only if an angle of $360^o/n$ can be constructed. When this angle is placed at the origin with one side on the x-axis, the other side intersects the circle $(O;\, 1)$ in the point $(\cos 360^o/n, \sin 360^o/n)$. By Theorem 4.5.1 and Exercise 18, this general ruler and compass construction is feasible if and only if $\cos 360^o/n$ is a Hippasian number. We now show that there is no ruler and compass construction for the regular 7-gon. Set $A = 360^o/7$ and $x = \cos 360^o/7$. If such a method existed, then x would be a Hippasian number. However, by Exercises 8 and 12,

$$\cos 3A = 4\cos^3 A - 3\cos A = 4x^3 - 3x$$

$$\cos 4A = 8\cos^4 A - 8\cos^2 A + 1 = 8x^4 - 8x^2 + 1.$$

Since $3A + 4A = 360^{o}$ it follows that $\cos 3A = \cos 4A$ and hence

$$4x^3 - 3x = 8x^4 - 8x^2 + 1$$

$$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0$$

$$(x - 1)(8x^3 + 4x^2 - 4x - 1) = 0 .$$

However, $\cos 360^{o}/7 \neq 1$ and hence

$$8x^3 + 4x^2 - 4x - 1 = 0 .$$

It follows from Proposition 4.5.3 that the only possible rational solutions of this equation are again ±1, ±1/2, ±1/4, ±1/8. Since, by Exercise 10, none of these is a solution, it follows from Proposition 4.5.2 that $\cos 360^{o}/7$ is not a Hippasian number and hence no such method for constructing regular 7-gons can exist.

According to Exercise 11, $\cos 360^{o}/5$ is a Hippasian number and so the regular pentagon is indeed constructible, as demonstrated in Proposition 4.3.4. As was mentioned above, the regular 17-gon is also constructible and hence $\cos 360^{o}/17$ must also be a Hippasian number. In fact, it is known to equal

$$-1/16 + \sqrt{17}/16 + (1/16)\sqrt{34 - 2\sqrt{17}}$$

$$-(1/8)\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}\,.$$

SQUARING THE CIRCLE: To square a circle of unit radius is tantamount to constructing a line segment of length $a = \sqrt{\pi}$. As it happens, neither π nor $\sqrt{\pi}$ are the solutions of any equation of form (1). In fact, it was proven by Lindemann that there exists no polynomial $P(x)$ with integer coefficients of any degree such that either π or $\sqrt{\pi}$ are solutions of $P(x) = 0$. Since every Hippasian number is known to be the solution of such an equation, it follows that $\sqrt{\pi}$ is not a Hippasian number and hence

It is impossible to square a circle by ruler and compass alone.

EXERCISES 4.5

1. Show that the straight line joining the points (a, b) and (a', b') has equation

   $(b' - b)x + (a - a')y + (a'b - ab') = 0.$

2. Show that the circle with center (a, b) and radius r has equation

   $x^2 + y^2 + (-2a)x + (-2b)y + (a^2 + b^2 - r^2) = 0.$

3. Show that if the two lines with equations $ax + by + c = 0$ and $a'x + b'y + c' = 0$ intersect, then their point of intersection has coordinates $((bc' - b'c)/(ab' - a'b), (a'c - ac')/(ab' - a'b))$.

4. Show that if the line with equation $ax + by + c = 0$ and the circle with equation $x^2 + y^2 + a'x + b'y + c' = 0$ intersect, then their points of intersection have coordinates $x = (-B \pm \sqrt{B^2 - 4AC})/(2A)$ and $y = (-ax - c)/b$ where $A = a^2 + b^2$, $B = 2ac + a'b^2 - abb'$, and $C = c^2 - bb'c + c'b^2$.

5. Show that if the two circles with equations $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + a'x + b'y + c' = 0$ intersect, then their points of intersection have coordinates $x = (-B \pm \sqrt{B^2 - 4AC})/(2A)$ and

$y = (-a''x - c'')/b''$ where $A = a''^2 + b''^2$, $B = 2a''c'' + a'b''^2 - a''b''b'$, $C = c''^2 - b''b'c'' + c'b''^2$

and $a'' = a - a'$, $b'' = b - b'$, $c'' = c - c'$.

6*. State and prove the converse of Theorem 4.5.1.

7. Prove Proposition 4.5.3.

8. Prove that $\cos 3A = 4\cos^3 A - 3\cos A$.

9. Verify that none of the numbers $\pm 1/1, \pm 2/1$, is a solution of the equation $x^3 - 2 = 0$.

10. Verify that none of the numbers $\pm 1/1, \pm 1/2, \pm 1/4, \pm 1/8$ is a solution of the equation

$8x^3 - 6x - 1 = 0$.

11. Prove that if $x = \cos 72^\circ$ then $4x^2 + 2x - 1 = 0$.

12. Prove that $\cos 4A = 8\cos^4 A - 8\cos^2 A + 1$.

13. Prove that it is impossible to construct a regular 9-gon by ruler and compass alone.

14. Prove that it is impossible to triple a cube by ruler and compass alone.

15. Prove that it is impossible to halve a cube by ruler and compass alone.

16. Prove that the following numbers are not Hippasian:

a) $\sqrt[3]{5}$ b) $2 + \sqrt[3]{5}$ c) $1/(2 - \sqrt[3]{7})$

17. Is it possible to construct an angle of 1°?

18. Let A be any number. Explain why $\sin A$ is a Hippasian number if and only if $\cos A$ is a Hippasian number.

CHAPTER REVIEW EXERCISES

1. Circle p intersects two concentric circles. Prove that the arcs of p cut off by the two circles are equal.
2. Prove that if each of the sides of a square is extended in both directions by the length of the radius of the circle that circumscribes the square, one obtains the vertices of a regular octagon.
3. Two circles, centered at C and D respectively, intersect at a point A. Prove that if PAQ is the "double chord" that is parallel to CD, then $PQ = 2CD$.
4. The area of the annular region bounded by two concentric circles equals that of a circle whose diameter is a chord of the greater circle that is tangent to the smalller one.
5. In a circle, a diameter bisects the angle formed by two intersecting chords. Prove that the chords are equal.
6. Prove that every equiangular polygon all of whose sides are tangent to the same circle is regular.
7. Through the center of a circle passes a second circle of greater radius and their common tangents are drawn. Prove that the chord joining the contact points of the greater circle is tangent to the smaller circle.
8. Prove that every cyclic equilateral pentagon is regular.

9*. Three circles through the point O and of radius r intersect pairwise in the additional points A, B, C. Prove that the circle circumscribed about ΔABC also has radius r.

10. Prove that in the regular hexagon $ABCDEF$ the diagonals AC and AE cut the diagonal BF into three equal segments.
11. Each of the sides of a cyclic quadrilateral is the chord of a new circle. Prove that the other four intersection points of these new circles also form a cyclic quadrilateral.
12. A circle of radius r is inscribed in ΔABC in which $\angle ACB$ is a right angle. Prove that $a + b = c + 2r$.
13. The chord AB of a circle of radius 1 has the property that if the circle is folded along AB so as to bring AB's arc into the circle, then the arc passes through the center of the circle. Compute the lengths of the chord AB and its arc.
14. In a given ΔABC construct a point whose distances from the sides of the triangle are proportional to three given line segments.

15*. Through the midpoint M of a chord PQ of a circle, any other chords AB and CD are drawn; chords AD and BC meet PQ at points X and Y. Prove that M is the midpoint of XY. (This is the notorious Butterfly Problem.)

16*. The points of intersection of the adjacent trisectors of any triangle are the vertices of an equilateral triangle. (This is known as Morley's Theorem.)

17. Are the following statements true or false? Justify your answers.

a) The Greeks believed that the world is flat.

b) The area of a plane Euclidean figure whose perimeter is composed of circular arcs must involve π in its expression.

c) The Greeks knew that $\pi = 3.14$.

d) If two sectors have equal angles, then their arcs are proportional to their radii.

e) Given a line segment a, it is impossible to construct (in the sense of *The Elements*) a square whose area equals that of the circle of radius a.

f) Given a line segment a, it is impossible to construct (in the sense of *The Elements*) an equilateral triangle whose area equals that of the circle of diameter a.

g) Of two equal chords in unequal circles, the one in the larger circle lies further from the center.

h) The diameter is the circle's longest chord.

i) It is possible to construct a regular 340-sided polygon (in the sense of *The Elements)*.

j) It is possible to construct a regular 140-sided polygon (in the sense of *The Elements)*.

k) In a circle, all the angles subtended by a chord are equal to each other.

l) In a circle, arcs are proportional to their chords.

m) Every circle has only one center.

n) Every circle has only one tangent line.

o) If two chords of a circle bisect each other, then they are both diameters.