CHAPTER 3

Non-Neutral Geometry

This chapter's propositions differ from those of the previous one in that they depend on Postulate 5 for their validity. Their proofs are therefore not valid in the context of non-Euclidean geometry.

1. Parallelism

The first of the non-neutral propositions is the converse of Proposition 2.3.35, the last proposition of the previous chapter.

PROPOSITION 3.1.1(I.29)**.** *A straight line falling on parallel straight lines makes the alternate angles equal to one another, the corresponding angles equal to one another, and the interior angles on the same side equal to two right angles.*

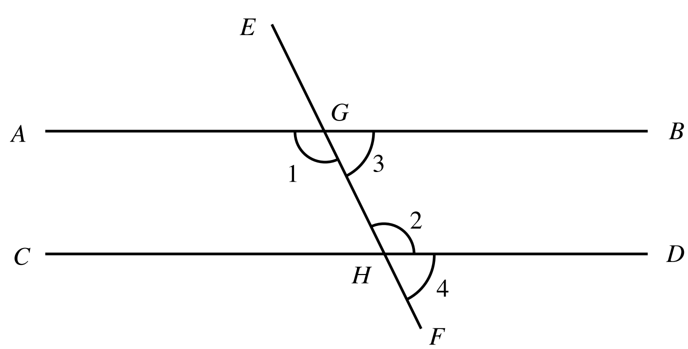


Figure 3.1

GIVEN: Straight lines $AB \parallel CD$, straight line EF intersecting AB and CD at G and H respectively (Fig. 3.1).

TO PROVE: $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, $\angle 2 + \angle 3 = 2$ right angles.

PROOF: By contradiction. Suppose $\angle 1$ and $\angle 2$ are unequal. Then it may be assumed without loss of generality that

	$\angle 1 > \angle 2$	
$\therefore$	$\angle 1 + \angle 3 > \angle 2 + \angle 3$	
but	$\angle 1 + \angle 3 = 2$ right angles	[PN 2.3.17]
$\therefore$	2 right angles $> \angle 2 + \angle 3$	
$\therefore$	AB and CD intersect	[PT 5]

This however, contradicts the fact that $AB \parallel CD$ and so $\angle 1 = \angle 2$. Moreover,

	$\angle 1 + \angle 3 = \angle 2 + \angle 4 = 2$ right angles	[PN 2.3.17]
$\therefore$	$\angle 3 = \angle 4$	[CN 3]
and also	$\angle 2 + \angle 3 = 2$ right angles.	

Q.E.D.

The following proposition provides an alternative and intuitively appealing characterization of parallel lines.

PROPOSITION 3.1.2. *The locus of all points on one side of a straight line that are equidistant from it is a straight line.*

See Exercise 11.

EXERCISES 3.1A

1. Prove that if two parallel straight lines are cut by a third line then the two bisectors of a pair of alternate interior angles are parallel to each other.
2. Prove that if a straight line is perpendicular to one of two parallel straight lines then it is also perpendicular to the other one.
3. Suppose $AB \perp KL$ and $CD \perp MN$ are all straight lines such that $KL \parallel MN$. Prove that $AB \parallel CD$.
4. Suppose $AB \perp KL$ and $CD \perp MN$ are all straight lines such that $KL \nparallel MN$. Prove that $AB \nparallel CD$.
5. Prove that two angles whose sides are respectively parallel are either equal or supplementary
6. Prove that two angles whose sides are respectively perpendicular are either equal or supplementary.
7. In ΔABC, AD is the bisector of $\angle BAC$ and E is a point on AC such that $DE \parallel AB$. Prove that $AE = DE$.
8. For a given ΔABC, $AD \parallel BC$ and $AD = AB$. Prove that BD bisects either the interior angle or the exterior angle at B.
9. Prove that if the points A, B are on the same side of the straight line m and at the same distance from m, then $AB \parallel m$.
10. Prove that if the points A, B are such that $AB \parallel m$, then they are at the same distance from m.
11. Use the above two exercises to prove Proposition 3.1.2.
12. Prove that the internal bisectors of each pair of angles of a triangle intersect.
13. Given two distinct parallel lines, construct a straight line that is parallel to both and also equidistant from both.
14. Comment on Proposition 3.1.1 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
15. Comment on Proposition 3.1.2 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

PROPOSITION 3.1.3(I.30). *(Distinct) Straight lines parallel to the same straight line are also parallel to one another.*

GIVEN: Distinct straight lines $AB \parallel EF$, $CD \parallel EF$ (Fig. 3.2).

TO PROVE: $AB \parallel CD$

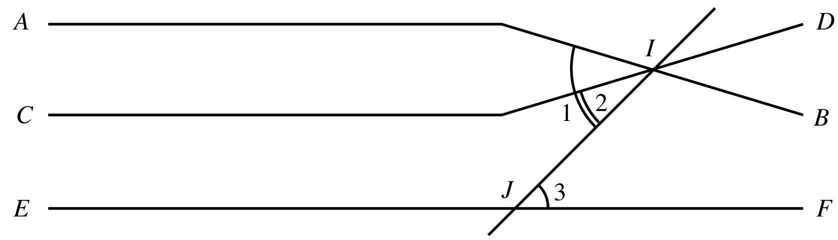


Figure 3.2

PROOF: By contradiction. Suppose AB and CD intersect in some point I. Join I to any point J of EF. Then

	$\angle 1$	$=$	$\angle 3$	[PN 3.1.1, $AB \parallel EF$]
	$\angle 2$	$=$	$\angle 3$	[PN 3.1.1, $CD \parallel EF$]
$\therefore$	$\angle 1$	$=$	$\angle 2$	[CN 1]

but this is impossible since the straight lines AB and CD are distinct. Hence $AB \parallel CD$.

Q.E.D.

Euclid begins his proof of this proposition by drawing a straight line PQ that intersects all the three given lines. While intuitively plausible, the existence of such a line calls for a justification and Euclid's proof is therefore incomplete. The need for such a justification is demonstrated by Figure 3.3 which exhibits three pairwise parallel <u>hyperbolic</u> <u>geodesics</u> such that no single geodesic intersects all three.

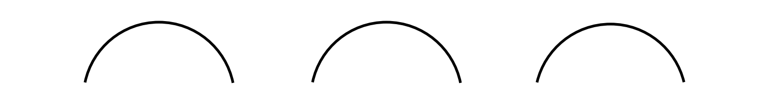


Figure 3.3 Three hyperbolic parallel straight lines that are <u>not</u> intersected by the same hyperbolic straight line.

EXERCISES 3.1B

1. If a straight line intersects one of two parallel straight lines (in only one point) then it also intersects the other one.
2. Comment on Proposition 3.1.3 in the context of the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

PROPOSITION 3.1.4(I.31). *Through a given point to draw a straight line parallel to a given straight line.*

GIVEN: Straight line BC, point A not on $\overleftrightarrow{BC}$ (Fig. 3.4).

TO CONSTRUCT: A straight line AE such that $AE \parallel BC$.

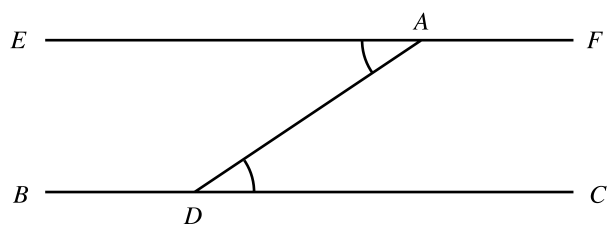


Figure 3.4

CONSTRUCTION: Let D be any point on BC and draw AD. Construct $\angle DAE$ = $\angle ADC$ [PN 2.3.28]. Then AE is the required straight line.

PROOF: $\angle EAD = \angle CDA$ [Construction]

$\therefore$ $AE \parallel BC$ [PN 2.3.35]

Q.E.D.

The following proposition has supplanted Euclid's Postulate 5 in many texts where it is known as *Playfair's Postulate*. Although this will not be demonstrated here, the two are in fact logically equivalent.

PROPOSITION 3.1.5(Playfair's Postulate). *Through a point not on a given straight line there exists exactly one straight line that is parallel to the given line.*
See Exercises 1 and 2.

Just like Postulate 5, Playfair's postulate does not hold in hyperbolic geometry. Figure 3.5 exhibits three distinct geodesics p, q, r, all of which contain the same point P and all of which are parallel to the same geodesic m.

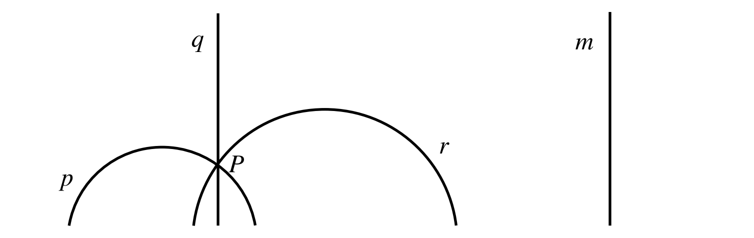


Figure 3.5 A hyperbolic counterexample to Playfair's Postulate.

It is now possible to give a more precise definition of hyperbolic geometry. This calls for negating Playfair's postulate, which is equivalent to Postulate 5. In view of Proposition 2.3.7 the following postulate is the proper negation of Playfair's postulate.

H (Hyperbolic). *There exists a straight line that is parallel to two intersecting distinct straight lines.*

Hyperbolic geometry is the geometry based on Euclid's Postulates 1, 2, 3, 4, A, S and Postulate H.

PROPOSITION 3.1.6(I.32). *In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.*

GIVEN: ΔABC, side BC extended to D (Fig. 3.6).

TO PROVE: $\angle ACD = \angle ABC + \angle CAB$,

$$\angle ABC + \angle BCA + \angle CAB = 2 \text{ right angles}$$

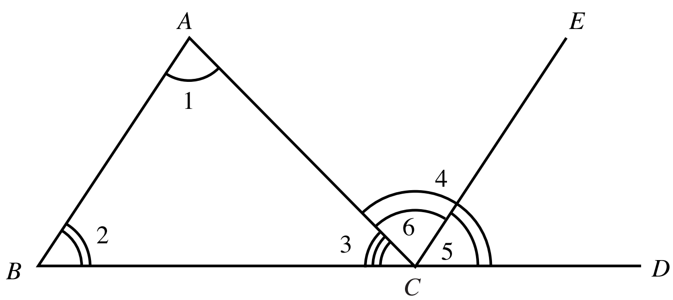


Figure 3.6

PROOF: Draw $CE \parallel AB$ [PN 3.1.4]. Then

	$\angle 5 = \angle 2$	[PN 3.1.1]
	$\angle 6 = \angle 1$	[PN 3.1.1]
$\therefore$	$\angle 4 = \angle 2 + \angle 1$	[CN 2]
$\therefore$	$\angle 4 + \angle 3 = \angle 1 + \angle 2 + \angle 3$	[CN 2]
$\therefore$	$2 \text{ right angles} = \angle 1 + \angle 2 + \angle 3$	[PN 2.3.17]

Q.E.D.

Recall that according to Chapter 1, the sum of the angles of every spherical triangle is greater than 180° [PN 1.1.5] whereas the sum of the angles of every hyperbolic triangle is less than 180° [PN 1.2.6].

EXERCISES 3.1C

1. Prove Proposition 3.1.5.

2. Prove that in ΔABC the bisector of the exterior angle at A is parallel to BC if and only if $AB = AC$.

3. Prove that a straight line that is parallel to one side of an isosceles triangle cuts off another isosceles triangle. (Note: There are two distinct cases to be considered here.)

4. A straight line cuts off an isosceles triangle from a given isosceles triangle. Prove that the straight line is parallel to one of the sides of the given isosceles triangle.

5. In an isosceles ΔABC, a line perpendicular to the base BC intersects AB and AC in the points D and E respectively. Prove that ΔADE is also isosceles.

6. In ΔABC, $\angle BAC = 90^o$ and $\angle ACB = 30^o$. Prove that $BC = 2AB$.

7. In ΔABC, $\angle ABC = 60^o$ and $BC = 2AB$. Prove that ΔABC is a right triangle.

8. Prove that in a right triangle the angle between the altitude to the hypotenuse and one of the legs equals the angle opposite that leg.

9. Let D be that point on side BC of ΔABC such that AD is the bisector of $\angle BAC$. Prove that $\angle ADC$ is half the sum of the interior angle at B and the exterior angle at C.

10. Prove that in ΔABC the bisectors of the interior angle at B and the exterior angle at A form an angle that is half the interior angle at C.

11. Prove that in ΔABC the angle bisector and the altitude at A form an angle that is half the difference between the interior angles at B and C.

12. Prove that in a right ΔABC the bisector of $\angle ABC$, the altitude to the hypotenuse BC, and the side AC form an isosceles triangle.

13. The point D on the hypotenuse BC of the right isosceles ΔABC is such that $BD = AB$. Prove that $\angle BAD = 67.5^o$.

14. Prove that if the diagonals of quadrilateral $ABCD$ are equal and the sides $AB = CD$, then $AD \parallel BC$.

15. Prove that the sum of the interior angles of a quadrilateral is 360^o.

16. Prove that in quadrilateral $ABCD$ the bisectors of the interior angles at A and B form an angle that is half the sum of the interior angles at C and D, and, if the bisectors of the interior angles at A and C intersect, they form an angle that is half the difference between the angles at B and D.

A polygon is said to be convex if all of its diagonals fall in its interior.

17. Prove that the sum of the interior angles of a convex n-sided polygon is $(n - 2)180^o$.

18. Prove that the sum of the interior angles of an arbitrary n-sided polygon is $(n - 2)180^o$. (Go ahead and use the difficult to prove fact that every polygon has a diagonal that lies completely

inside it.)

19. Prove that the number of acute interior angles of a convex polygon cannot exceed 3.

20. Prove that the sum of the exterior angles of a convex polygon is 360°. (Try to prove this without making use of Exercise 17 above).

21. Construct ΔABC given the data a, A, B.

22. Construct an isosceles triangle given one of its angles and one of its sides.

23. Construct ΔABC given the data $a, b + c, A$.

24. Construct ΔABC given the data $b + c, A, B$.

25. Comment on Proposition 3.1.4 in the context of the following geometries:
a) spherical; b) hyperbolic; c) taxicab; d) maxi.

26. Comment on Proposition 3.1.5 in the context of the following geometries:
a) spherical; b) hyperbolic; c) taxicab; d) maxi.

27. Comment on Proposition 3.1.6 in the context of the following geometries:
a) spherical; b) hyperbolic; c) taxicab; d) maxi.

28. Explain why there are no rectangles in spherical geometry.

29. Explain why there are no rectangles in hyperbolic geometry.

30. Are there rectangles in taxicab geometry?

31. Are there rectangles in maxi geometry?

32. The following method for trisecting an arbitrary angle is credited to Archimedes. If that attribution is correct he must have been aware of its shortcomings as a construction in the sense of Euclid.

Let α be a given angle with vertex A. (Fig. 3.7) Draw a circle of radius $AB = AC$. On a ruler mark two points D and E such that $DE = AB = AC$ and place the ruler on the page so that the point E falls on the extension of AB, the point D falls on the circle $(A; AB)$ and the ruler also passes through the point C. Prove the following assertions

a) $\angle ADC = \angle ACD\ (= \beta)$

b) $\angle AED = \angle EAD\ (= \gamma)$

c) $\alpha = 3\gamma$, or $\gamma = \alpha/3$

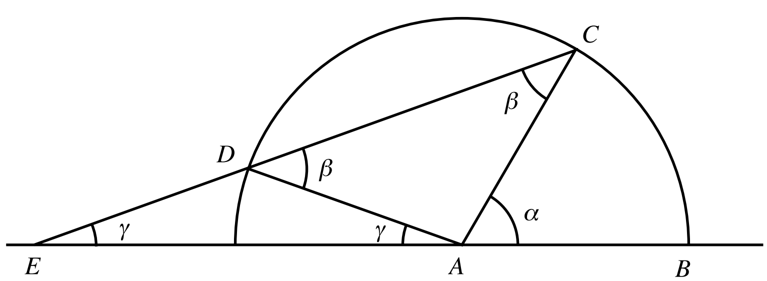


Figure 3.7 An angle "trisection".

Explain why this "trisection" of α does not meet Euclid's standards for a construction.

33. Criticize the following "neutral proof" of Playfair's Postulate, offered by Proclus (410-485): "I say that if any straight line cuts one of two parallels, it will cut the other also. For let AB, CD be parallel and let EFG cut AB [at F, with G between AB and CD]; I say that it will cut CD also. For, since BF, FG are two straight lines from one point F, they have, when produced indefinitely, a distance greater than any magnitude, so that it will be greater than the interval between the parallels. Whenever, therefore, they are at a distance from one another greater than the distance between the parallels, FG will cut CD."

34. Criticize the following "proof" of the fact that the sum of the interior angles of a triangle is 180°. Let ABC be a given triangle let d be a line segment that lies on the straight line AB with its center at A. Slide d along AB until its center falls on B and then rotate it through the exterior of the triangle, about B as a pivot, until it falls along side BC. Next slide d along BC until its center reaches C and rotate it about C as a pivot through the exterior of the triangle until it falls along CA. Finally, slide d along CA until its center reaches A and rotate it about A as pivot so that it comes into its initial position. If the triangle's interior angles are $\alpha, \beta, \gamma,$ then the segment d has been rotated successively by the angles $180^{\circ} - \beta$, $180^{\circ} - \gamma$, and $180^{\circ} - \alpha$ before it returned to its original position. Consequently $(180^{\circ} - \beta) + (180^{\circ} - \gamma) + (180^{\circ} - \alpha) = 360^{\circ}$ from which it follows that $\alpha + \beta + \gamma = 180^{\circ}$.

35(C). Perform the construction of Proposition 3.1.4 using a computer application.

36(C). Use a computer application to verify Proposition 3.1.6.

Euclid's statement of the following proposition is awkward and so it appears here in a paraphrased form.

PROPOSITION 3.1.7(I.33). *A quadrilateral in which two opposite sides are both equal and parallel to each other is a parallelogram.*

GIVEN: Straight line segments $AB \parallel CD$, $AB = CD$ (Fig. 3.8).

TO PROVE: $AC \parallel BD$, $AC = BD$.

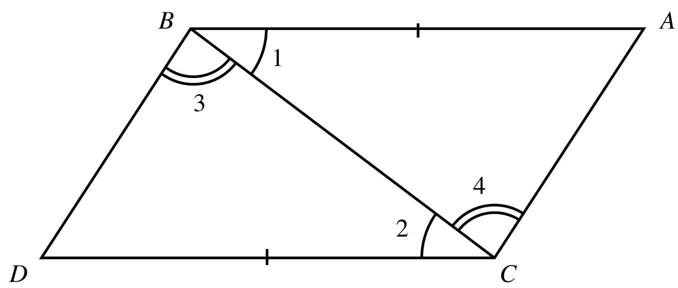


Figure 3.8

PROOF: Draw BD, BC, AC. Then $\Delta ABC \cong \Delta DCB$ by SAS because

$AB = DC$ [Given]

$\angle 1 = \angle 2$ [Alternating angles, $AB \parallel CD$, PN 3.1.1]

$BC = CB$

$\therefore$ $AC = DB$

and $\angle 4 = \angle 3$

$\therefore$ $AC \parallel BD$ [Equal alternating angles, PN 2.3.34]

Q.E.D.

PROPOSITION 3.1.8(I.34). *If both pairs of opposite sides of a quadrilateral are parallel to one another, then they as well as the opposite angles are equal to one another, and the diameter bisects the area.*

GIVEN: Quadrilateral $ACDB$, $AB \parallel CD$, $AC \parallel BD$.

TO PROVE: $AB = CD$, $AC = BD$, $\angle CAB = \angle BDC$, $\angle ABD = \angle DCA$,

$\Delta ABC = \Delta DCB = \frac{1}{2}$ ▱ $ABDC$.

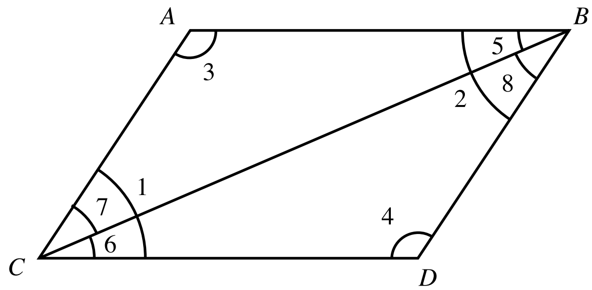


Figure 3.9

PROOF: $\Delta\, ABC \cong \Delta\, DCB$ by ASA because

$\angle 5 = \angle 6$ [Alternating angles, $AB \parallel CD$, PN 3.1.1]

$BC = CB$

$\angle 7 = \angle 8$ [Alternating angles, $AC \parallel DB$, PN 3.1.1]

$\therefore$ $AB = CD$, $AC = BD$, $\angle 3 = \angle 4$

$\Delta\, ABC = \Delta\, DCB = \frac{1}{2}$ ▱ $ABDC$

Also $\angle 1 = \angle 2$ [CN 2]

Q.E.D.

EXERCISES 3.1D

1. Both pairs of opposite sides of a quadrilateral are equal to each other. Prove that the quadrilateral is a parallelogram.
2. Both pairs of opposite angles of a quadrilateral are equal to each other. Prove that the quadrilateral is a parallelogram.
3. Prove that the diagonals of a parallelogram bisect each other.
4. Prove that if the diagonals of a quadrilateral bisect each other then it is a parallelogram.
5. Prove that a parallelogram is a rectangle if and only if its diagonals are equal to each other.
6. Prove that a parallelogram is a rhombus if and only if its diagonals are perpendicular to each other.
7. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and equals half its length.

8. The midpoint of side AB of ΔABC is D and E is a point of AC such that $DE \parallel BC$. Prove that $AE = EC$ and $DE = BC/2$.
9. Prove that the midpoints of the four sides of a quadrilateral are the vertices of a parallelogram.
10. Prove that each of two medians of a triangle is divided by their intersections into two segments one of which is double the other.
11. Prove that the three medians of a triangle all pass through one point.
12*. Point E in the interior of square $ABCD$ is such that $\angle ABE = \angle BAE = 15^{o}$. Prove that ΔCDE is equilateral.
13. In ΔABC, $AB = AC$, and D, E, F are points on the interiors of sides BC, AB, AC respectively, such that $DE \perp AB$ and $DF \perp AC$. Prove that the value of $DE + DF$ is independent of the location of D.
14. Prove that the three segments joining the midpoints of the three sides of a triangle divide it into four congruent triangles.
15. Prove that three parallel straight lines that cut off equal line segments on one straight lines also cut off equal line segments on every straight line that intersects them (Hint: Through the middle intersection point on one straight line draw a line parallel to the other straight line.)
16. A *trapezoid* is a quadrilateral two of whose sides are parallel. Prove that the line segment joining the midpoints of the non-parallel sides of a trapezoid is parallel to the other two sides and equals half the sum of their lengths
17. Construct angles of the following magnitudes

    a) 60^{o} b) 30^{o} c) 120^{o} d) 75^{o}.
18. Through a given point construct a straight line such that its portion between two given parallel straight lines is equal to a given line segment.
19. Let A be a point in the interior of an angle. Construct a straight line whose segment between the sides of the angle has A as its midpoint.
20. A pair of parallel straight lines is intersected by another pair of parallel straight lines. Through a given point construct another straight line on which the two given pairs cut off equal line segments.
21. Given an angle, determine the locus of all the points the sums of whose distances from the sides of the angle equals a given magnitude.
22. In a given ΔABC construct points M on AB and N on BC such that $BM + NC = MN$ and $MN \parallel BC$.
23. Construct ΔABC given the data

    a) a, h_a, β b) a, h_a, h_b c) a, h_b, α

    d) h_b, h_c, α e) h_b, m_c, α f) $\alpha, h_c, b + c$

    g) $a + b + c, \beta, \gamma$ h) $a + b + c, \beta, h_a$.

24. Construct a parallelogram given:
    a) two adjacent sides and the included angle;
    b) two adjacent sides and a diagonal;
    c) two adjacent sides and the distance between two opposite sides;
    d) a side and the two diagonals;
    e) the diagonals and the angles between them.
25. Construct a rectangle given one side and the diagonal.
26. Construct a rhombus given:
    a) its side and one of its angles;
    b) its side and one diagonal.
    c) both diagonals.
27. Construct a square given:
    a) its side; b) its diagonals.
28. Construct ΔABC given the data:
    a) a, c, m_b; b) a, h_a, m_b .
29. Comment on Proposition 3.1.7 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
30. Comment on Proposition 3.1.8 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

2. Area

Euclid defined the concept of area by means of axioms that he called *Common Notions*. This axiomatic approach is customary today as well, although the specific axioms are different from those used by Euclid. The modern approach to area stipulates that a certain unit of length, called *unit* has been chosen. The square the length of whose side is

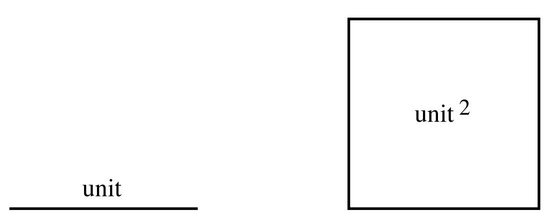


Figure 3.10

1 unit (Fig. 3.10) is denoted by *unit square* or *unit*^{2} and serves as the unit for measuring areas. It is then assumed that area is a measurement of figures that satisfies the three properties (or axioms) listed below:

UNIT: *The unit square has area 1 unit*^{2}.

ADDITIVITY: *If a figure is divided by a line into two subfigures, then the area of the figure equals the sum of the areas of the subfigures.*

INVARIANCE: *Congruent figures have equal areas.*

Loosely speaking, the additivity and invariance axioms were stated by Euclid as Common Notions 2 and 4 respectively. The unit axiom, however, has no analog in Euclid's system. As a consequence, Euclid's *Elements* contains no proposition that computes areas explicitly. Instead, Euclid made comparative statements such as

parallelograms on equal bases and between the same parallels are equal

and

if a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double the triangle.

This has the theoretical advantage of dispensing with units and the practical disadvantage of not answering the reasonable question of *what is the area of a rectangle of dimensions*

3 and 5 stadia? Greek mathematicians did of course make use of units and could resolve such questions with ease. It is just that Euclid, for reasons that can only be guessed at, and that in the author's opinion were probably esthetic, decided to develop his geometry without any units whatsoever.

Propositions 3.2.1 and 3.2.2 below are the explicit modern day analogs of Euclid's I.35 (PN 3.2.3). They give explicit formulas for the areas of rectangles and parallelograms. Their complete proofs unfortunately contain elements that are beyond the scope of this text. Specifically, one runs into the difficulty inherent in proving propositions regarding line segments with irrational (non-fractional) lengths. These difficulties were first encountered by the Greeks in the sixth century BC and eventually surmounted by Eudoxus two hundred years later. Euclid's book did incorporate Eudoxus's treatment of irrational numbers, but it would be impractical to expound this theory here. Instead, a mere supporting argument for the fact that the area of a rectangle is given by the product of the lengths of its sides is offered. It is customary in today's high school geometry textbooks to circumvent these difficulties by stating this formula as yet another axiom, the *Rectangle Axiom*. In the author's opinion this is a misguided solution to a pedagogical problem since it opens up the possibility of stating many other interesting and non-trivial geometrical facts as axioms, even when elementary and convincing, albeit logically incomplete, arguments are available.

PROPOSITION 3.2.1. *If a rectangle has dimensions a units and b units then it has area ab unit^2.*

GIVEN: `$ABCD$ with sides a, b.

TO PROVE: Area of `$ABCD = ab$ unit^2.

SUPPORTING ARGUMENT: If a and b are positive integers then a rectangle of dimensions a and b can be divided into ab unit squares by means of straight lines that are parallel to its sides (Fig. 3.11). Consequently, by the Additivity Property, the given rectangle has area ab unit^2.

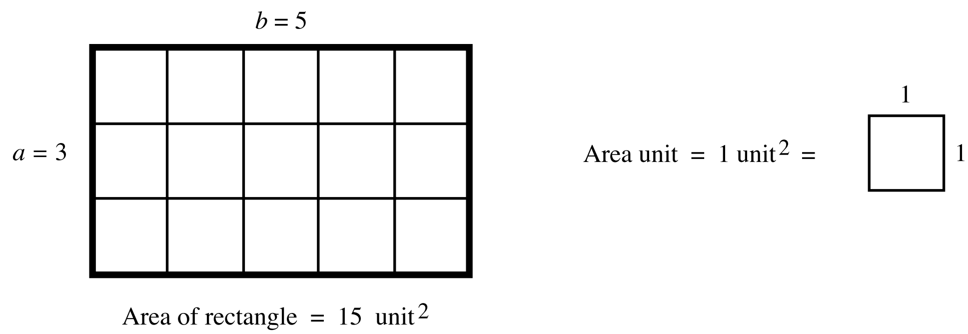


Figure 3.11

Similarly, if a rectangle has dimensions $a = 1/m$ and $b = 1/n$ for some positive integers m and n, then the unit square can be divided into mn copies of the given rectangle all of which, by the Invariance Property, have the same area (Fig. 3.12). Hence the given rectangle has area

$$\frac{1 \text{ unit}^2}{mn} = \frac{1}{m}\frac{1}{n} \text{ unit}^2 = ab \text{ unit}^2$$

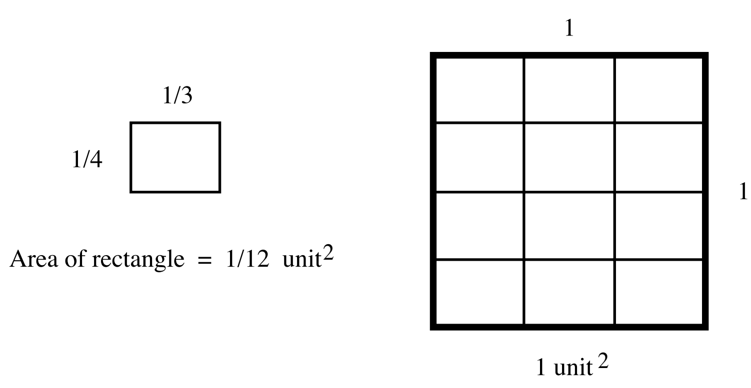


Figure 3.12

Next, if a rectangle has dimensions $a = m/n$ and $b = p/q$ where m, n, p, q are all positive integers, then it can be decomposed into mp rectangles each of which has dimensions $\frac{1}{n}$ and $\frac{1}{q}$ (Fig. 3.13) . Since each of these latter rectangles is now known to have area $\frac{1}{nq}$ unit^2, it follows from the Additivity Property that the given rectangle has area

$$mp(\frac{1}{nq}) \text{ unit}^2 \quad = \quad \frac{m}{n}\frac{p}{q} \text{ unit}^2 \quad = \quad ab \text{ unit}^2 .$$

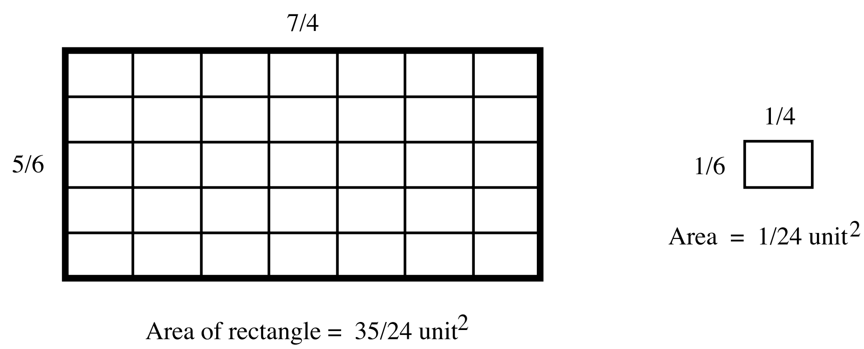


Figure 3.13

This verifies the proposition for all rectangles with fractional dimensions. As was mentioned above, the extension of this formula to rectangles with arbitrary real dimensions lies beyond the scope of this text. Q.E.D.

An *altitude* of a parallelogram is any line segment cut off by two opposite sides from a straight line that is perpendicular to both of them. It follows from Proposition 3.1.2 that all the altitudes joining the same pair of opposite sides of a parallelogram have equal length.

PROPOSITION 3.2.2. *The area of a parallelogram with base b units and altitude h units is bh unit^2.*

GIVEN: ▱ $ABCD$ with base b and altitude h (Fig. 3.14).

TO PROVE: Area of ▱ $ABCD$ = bh unit^2.

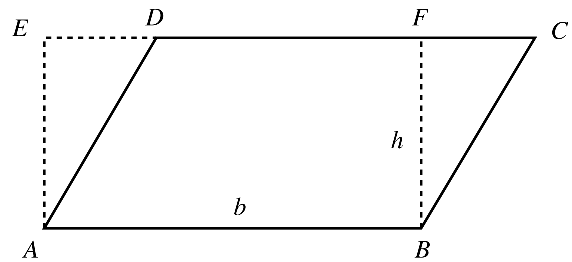


Figure 3.14

PROOF: In the given parallelogram draw AE ' CD and BF 'CD *with* E, F on CD. Thus, $ABFE$ is a rectangle with area bh unit^2. Since $\Delta ADE \cong \Delta BCF$ it follows that they have the same area, and hence, by the Additivity Property,

$$▱ ABCD \quad = \quad ▭ ABFE \quad = \quad bh \text{ unit}^2.$$

Q.E.D.

Euclid's version of Propositions 3.2.1 and 3.2.2 is now stated together with his proof as well as another proof that is more consistent with modern pedagogy.

PROPOSITION 3.2.3(I.35). *Parallelograms which are on the same base and in the same parallels are equal to one another.*

GIVEN: ▱ $ABCD$ and ▱ $EBCF$ such that A, D, E, F are collinear (Fig. 3.15).

TO PROVE: ▱ $ABCD$ = ▱ $EBCF$.

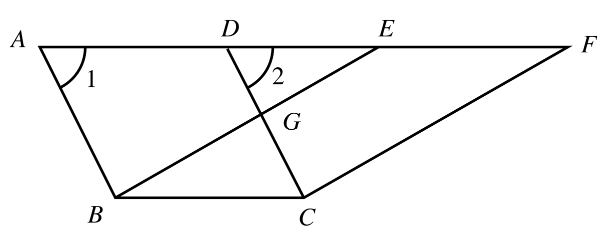


Figure 3.15

PROOF (Euclid): Since $AD = BC = EF$ [PN 34], it follows from CN 2 that $AE = DF$. Then $\Delta\, BAE \cong \Delta\, CDF$ by SAS because

	AE	$=$	DF	[See above]
	$\angle 1$	$=$	$\angle 2$	[Corresp. angles, $AB \parallel DC$, PN 3.1.1]
	AB	$=$	DC	[Parallelogram $ABCD$, PN 3.1.8]
$\therefore$	$\Delta\, EAB$	$=$	$\Delta\, FDC$	
$\therefore$	$ABGD$	$=$	$EGCF$	[Subtract $\Delta\, DGE$, CN 3]
$\therefore$	▱$ABCD$	$=$	▱$EBCF$	[Add $\Delta\, GBC$, CN 2]

Q.E.D.

Euclid's proof of Proposition 3.2.3 is incomplete (albeit easily fixed) because it depends on the relative position of the points A, D, E, F on their common line (Exercise 12).

PROOF (modern):

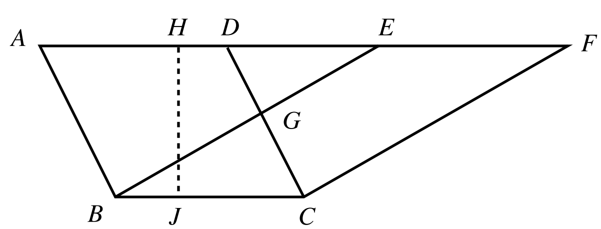


Figure 3.16

Draw HJ perpendicular to AD and BC (Fig. 3.16). It then follows from PN 3.2.2 that $\sim ABCD = BC \cdot HJ = \sim EBCF$.

Q.E.D.

PROPOSITION 3.2.4(I.36). *Parallelograms which are on equal bases and in the same parallels are equal to one another*.

See Exercise 1.

The area of the triangle will be given the same dual treatment as that of the rectangle. First the modern formula is offered.

PROPOSITION 3.2.5. *The area of a triangle with base* b *units and altitude* h *units is* $bh/2$ *unit*^{2}.

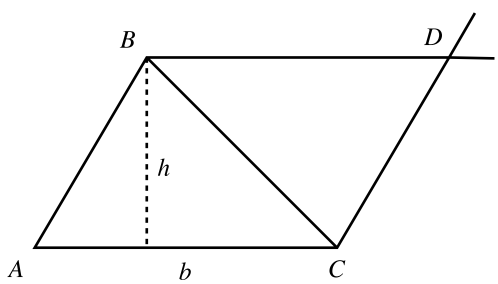


Figure 3.17

PROOF: Through the vertices B and C of ΔABC draw straight lines parallel to AC and AB respectively, and let their intersection be D (Fig. 3.17). It is clear that $ACDB$ is a parallelogram and hence, by Proposition 3.1.8,

$$\Delta ABC \quad = \quad \frac{1}{2} \square ACDB \quad = \quad \frac{bh}{2} \ \text{unit}^2.$$

Q.E.D.

Next comes Euclid's version.

PROPOSITION 3.2.6(I.37). *Triangles which are on the same base and in the same parallels are equal to one another.*

GIVEN: $\Delta ABC, \Delta DBC,\ \ AD \parallel BC$ (Fig. 3.18).

TO PROVE: $\Delta ABC = \Delta DBC$.

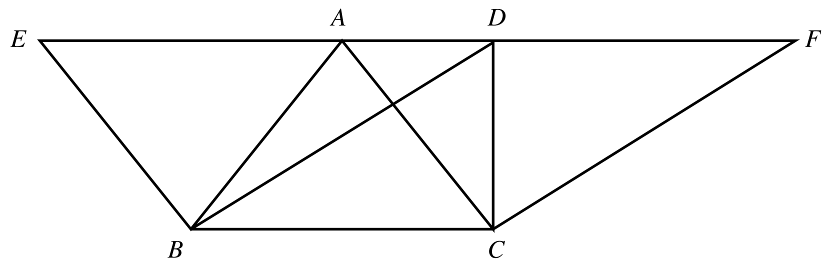


Figure 3.18

PROOF: Let E be the intersection of $\overleftrightarrow{AD}$ with the straight line through B parallel to AC and let F be the intersection of $\overleftrightarrow{AD}$ with the straight line through C parallel to BD [PN 3.1.4]. Then

$$
\begin{array}{llll}
 & ▱\, AEBC & = ▱\, DBCF & \text{[PN 3.2.2]} \\
 & \Delta ABC & = \tfrac{1}{2}\, ▱\, AEBC & \text{[PN 3.1.8]} \\
 & \Delta DBC & = \tfrac{1}{2}\, ▱\, DBCF & \text{[PN 3.1.8]} \\
\therefore & \Delta ABC & = \Delta DBC &
\end{array}
$$

Q.E.D.

PROPOSITION 3.2.7(I.38). *Triangles which are on equal bases and in the same parallels are equal to one another.*

See Exercise 2.

PROPOSITION 3.2.8(I.39). *Equal triangles which are on the same base and on the same side are also in the same parallels*

GIVEN: $\Delta ABC = \Delta DBC$, A and D are on the same side of BC.

TO PROVE: $AD \parallel BC$.

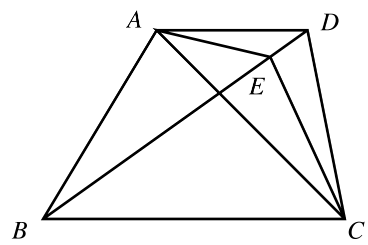


Figure **3.19**

PROOF: By contradiction. Suppose AD and BC are not parallel and let E be the intersection of BD with the straight line through A parallel to BC. Then

$$\begin{array}{llll} & \Delta ABC = \Delta EBC & & \text{[PN 3.2.6]} \\ & \Delta EBC < \Delta DBC & & \text{[CN 5]} \\ \therefore & \Delta ABC < \Delta DBC & & \end{array}$$

This, however, contradicts the give equality of the two triangles. Hence $AD \parallel BC$.

Q.E.D.

PROPOSITION 3.2.9(I.40). *Equal triangles which are on equal bases and on the same side are also in the same parallels.*

According to Heath, Proposition 3.2.9 is an interpolation into *The Elements* by a later geometer (Exercise 2).

PROPOSITION 3.2.10(I.41). *If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle.*

See Exercise 2.

There is no analog of Proposition 3.2.5 for the area of a general quadrilateral. In practice, any such quadrilateral can be divided into triangles by means of a diagonal and then the area of each of the parts can be evaluated by means of Proposition 3.2.5. A similar procedure can be used to dissect any polygon, regardless of the number of its sides, into triangles.

Neither Euclid's nor the modern approach to areas are applicable to spherical geometry. Both of these approaches rely heavily on the notion of parallelism, and the sphere has no parallel geodesics. Thus, another approach is required in order to develop a theory of spherical areas. As spherical polygons can also be dissected into spherical triangles, it suffices to provide a formula for the latter.

It is clear that any two lunes of the same angle α on the same sphere can be made congruent by a series of rotations of that sphere. Consequently, every two such lunes have the same area. This, in turn, implies that the area of a lune is proportional to its angle. Since the lune of angle 2π radians has area $4\pi R^2$ (the sphere's total surface area) the following lemma is obtained.

LEMMA 3.2.11. *On a sphere of radius R the area of a lune of angle α radians is $2\alpha R^2$ unit^2.*

The following theorem was first discovered by the Flemish mathematician Albert Girard (1595-1632). The proof presented here is due to Euler.

PROPOSITION 3.2.12. *On a sphere of radius R the area of the spherical* triangle *ABC with angles of radian measures α, β, γ, is $(\alpha + \beta + \gamma - \pi)R^2$ unit^2.*

GIVEN: Spherical ΔABC with interior angles α, β, γ (measured in radians).

TO PROVE: $\Delta ABC = (\alpha + \beta + \gamma - \pi)R^2$ unit^2.

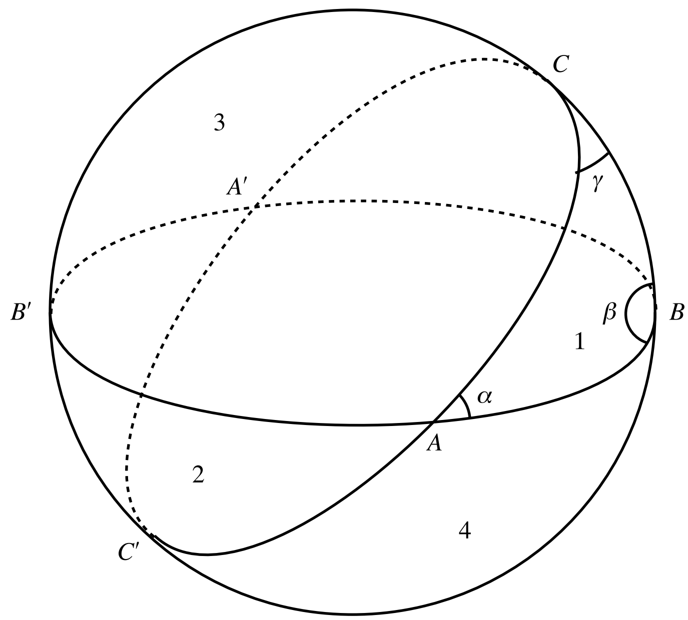


Figure 3.20

PROOF: Let A', B', C' be the respective antipodes of A, B, C (Fig. 3.20). Draw the great circles that contain the geodesic segments AB, BC, and CA. The hemisphere in front of the great circle $BCB'C'$ is thereby divided into four spherical triangles ABC, $AB'C'$, $AB'C$, ABC', whose areas are denoted, respectively, by T_1, T_2, T_3, T_4.

From the construction it follows that the spherical $\Delta\, A'BC$ is congruent to the Δ $AB'C'$ of area T_2. Hence,

$$T_1 + T_2 = \text{lune } \alpha$$

Similarly,

$$T_1 + T_3 = \text{lune } \beta$$

and

$$T_1 + T_4 = \text{lune } \gamma.$$

Consequently,

$$2T_1 = \text{lune } \alpha + \text{lune } \beta + \text{lune } \gamma - (T_1 + T_2 + T_3 + T_4)$$

$$= (2\alpha + 2\beta + 2\gamma - 2\pi)R^2 \; unit^2$$

and the statement of the theorem now follows immediately.

Q.E.D.

EXERCISES 3.2A

1. Prove Proposition 3.2.4.
2. Use Proposition 3.2.5 to prove
   a) Proposition 3.2.6; b) Proposition 3.2.7; c) Proposition 3.2.8;
   d) Proposition 3.2.9; e) Proposition 3.2.10.
3. One of the triangle's sides is divided into n equal segments and the division points are joined to the opposite vertex. Prove that the triangle is divided into n equal parts.
4. Prove that the area of the trapezoid equals the product of half the sum of its parallel sides with the distance between them.
5. Prove that the diagonals of a parallelogram divide it into four equal triangles.
6. Prove that the line segment joining the midpoints of two sides of a triangle cuts off a triangle that is equal to one fourth of the original triangle.
7. Prove that the parallelogram formed by the midpoints of the sides of a quadrilateral equals one half of that quadrilateral.
8. Prove that the triangle’s medians divide it into six equal triangles
9. The diagonals of a quadrilateral divide it into four equal triangles. Prove that the quadrilateral is a parallelogram.

10. Prove that if the point P lies in the interior of ▱$ABCD$ then the parallelogram equals twice the sum of ΔABP and ΔCDP.
11. Each of the sides AB, BC, CA of an equilateral triangle is extended by their common length to points $D, E, F,$ respectively, all in the same sense. Prove that $\Delta DEF = 7\Delta ABC$.
12. Complete Euclid's proof of Proposition 3.2.3.

Both the taxicab and maxi areas of a figure are defined to equal its Euclidean area.

13. Comment on Propositions 3.2.1, 3.2.2, and 3.2.5 in the context of taxicab geometry.
14. Comment on Proposition 3.2.3, 3.2.4, and 3.2.6-10 in the context of taxicab geometry.
15. Comment on Propositions 3.2.1, 3.2.2, and 3.2.5 in the context of maxi geometry.
16. Comment on Proposition 3.2.3, 3.2.4, and 3.2.6-10 in the context of maxi geometry.

Euclid's Propositions I.42-45 are of limited interest. They are included here only in order to facilitate the later discussion of the Golden Ratio (Proposition 3.4.1).

PROPOSITION 3.2.13(I.42). *To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.*

See Exercise 1.

The above proposition is an example of a *conversion* which consists of the construction of a polygon Π', of some prespecified nature, that is equal to a given polygon Π.

PROPOSITION 3.2.14(I.43). *In any parallelogram the complements of the parallelograms about the diagonal are equal to one another.*

GIVEN: ▱$ABCD$, K is a point on the diagonal AC, ▱$BGKE$, ▱$KFDH$ (Fig. 3.21).

TO PROVE: ▱$BGKE$ = ▱$KFDH$.

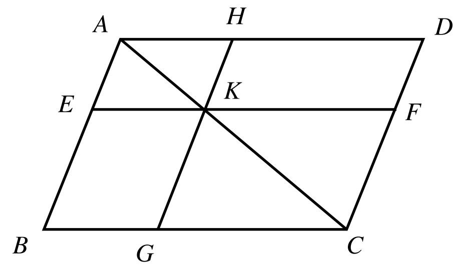


Figure 3.21

PROOF: See Exercise 2.

PROPOSITION 3.2.15(I.44). *To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle.*

See Exercise 3.

PROPOSITION 3.2.16(I.45). *To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.*

See Exercise 4.

EXERCISES 3.2B

1. Prove Proposition 3.2.13.
2. Prove Proposition 3.2.14.
3. Prove Proposition 3.2.15.
4. Prove Proposition 3.2.16.
5. Convert a given parallelogram into a rectangle with the same base.
6. Convert a given parallelogram into a rhombus with the same base.
7. Convert a given parallelogram into another parallelogram with the same base and a given angle.
8. Convert a given parallelogram into a triangle with the same base and a given angle.
9. Convert a given triangle into a right triangle with the same base.
10. Convert a given triangle into an isosceles triangle with the same base.
11. Convert a given triangle into another triangle with the same base and a given angle.

12. Bisect the area of a parallelogram by means of a straight line that is parallel to a given straight line.
13. Given ΔABC, construct a point O in its interior such that the triangles AOB, BOC, COA all have equal areas.

3. The Theorem of Pythagoras

The Theorem of Pythagoras was discovered independently by several cultures and has been given more different proofs than any other theorem. It is considered by many mathematicians to be the most important of all theorems, and has the dubious distinction of being misquoted in the classic movie *The Wizard of Oz* and of being the subject of popular jokes. It will be presented following an easy lemma.

PROPOSITION 3.3.1(I.46). *On a given straight line to describe a square.*

GIVEN: Line segment AB (Fig. 3.22).

TO CONSTRUCT: $\square ABCD$.

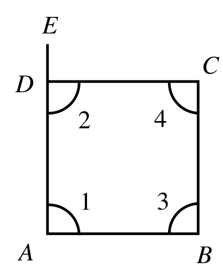


Figure 3.22

CONSTRUCTION: Draw $EA \perp AB$ [PN 2.3.11] and let D on AE be such that $AD = AB$. Let C be the intersection of the straight lines through B and D that are parallel to AD and AB respectively [PN 3.1.4]. Then quadrilateral $ABCD$ is the required square.

PROOF: By construction, $ABCD$ is a parallelogram. Since $AB = AD$ it follows that $AB = AD = DC = CB$. It remains to show that all of the angles of $ABCD$ are right angles. However,

	$\angle 1 + \angle 2 = 2$ right angles	[PN 3.1.1]
$\therefore$	$\angle 2 =$ right angle	[$\angle$ 1 is a right angle]
$\therefore$	$\angle 3 = \angle 4 =$ right angle	[PN 3.1.8]

Q.E.D.

PROPOSITION 3.3.2(I.47, The Theorem of Pythagoras)**.** *In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.*

GIVEN: ΔABC, $\angle BAC$ = right angle, $\square ABFG$, $\square ACKH$, $\square BCED$ (Fig. 3.23).

TO PROVE: $\square BCED = \square ABFG + \square ACKH$.

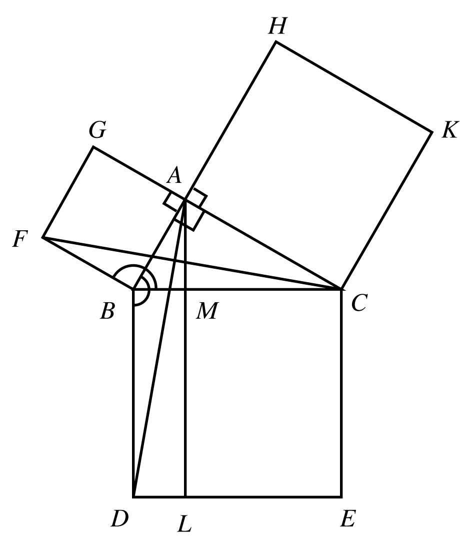


Figure 3.23

PROOF: Let L, M be the respective intersections of the straight lines DE and BC with the straight line through A parallel to BD and CE [PN 3.1.4]. Note that

the points G, A, C are collinear [$\angle GAB + \angle BAC =$ 2 right angles, PN 2.3.18]

the points B, A, H are collinear [$\angle HAC + \angle BAC =$ 2 right angles, PN 2.3.18]

$\Delta ABD \cong \Delta FBC$ by SAS because

$$BD = BC \qquad \text{[Sides of the same square]}$$

$$\angle ABD = \angle FBC \qquad \text{[Both equal } \angle ABC + \text{right angle]}$$

$$AB = FB \qquad \text{[Sides of the same square]}$$

$$\therefore \quad \Delta ABD = \Delta FBC$$

$$\therefore \quad ▭BDLM = \square ABFG \qquad \text{[Doubles of equal triangles, PN 3.2.10]}$$

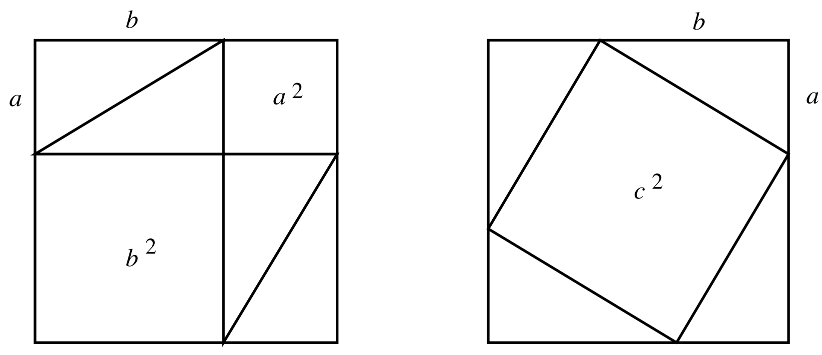
A similar argument yields the equation $▭CELM = \square ACKH$ and hence

$$\square BCED = ▭BDLM + ▭CELM = \square ABFG + \square ACKH$$

Q.E.D.

Two other proofs of this theorem are now sketched out.

If a and b are the legs and c is the hypotenuse of a right triangle then the square of side $a + b$ can be dissected in the two ways depicted in Figure 3.24. The dissection of I calls for no explication. That of II requires a proof that the interior quadrilateral labeled as c^2 is indeed a square (see Exercise 8). However, once these dissections are granted, it is clear from Figure 3.24 that $a^2 + b^2 = c^2$. This proof is attributed to the Chinese mathematician Chou-pei Suan-ching who lived circa 250 B.C.



I II

Figure 3.24

The next proof is due to the Indian mathematician Bhaskara (1114-1185). The square of side c can be dissected in the manner depicted in Figure 3.25. It then follows that

$$c^2 = 4\frac{ab}{2} + (a-b)^2 = 2ab + a^2 - 2ab + b^2 = a^2 + b^2 .$$

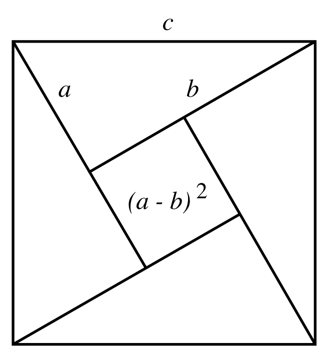


Figure 3.25

Yet another proof of the Theorem of Pythagoras in indicated in Exercise 17. This one is due to president James Garfield (1831-1881).

The Theorem of Pythagoras has a converse.

PROPOSITION 3.3.3(I.48). *If in a triangle the square on one of the sides be equal to the squares on the remaining two sides of the triangle, the angle contained by the remaining two sides is right.*

See Exercise 9.

Since $3^2 + 4^2 = 5^2$ it follows that any triangle whose sides have lengths 3 units, 4 units, and 5 units is necessarily a right triangle. So is the triangle whose sides have lengths 5, 12, 13. Triples of integers a, b, c such that

$$a^2 + b^2 = c^2$$

are known as *Pythagorean triples*, but the interest in such triples precedes Pythagoras by over a thousand year. The Babylonian tablet PLIMPTON 322, dated between 1900 and 1600 BC contains fifteen Pythagorean triples the largest of which consists of 12709, 13500, and 18541. Although it is highly unlikely that the Babylonians found these numbers by trial and error, it is not known what method they used to generate these triples. Not surprisingly, the earliest method for generating Pythagorean triples appears in Euclid's *The Elements*. Lemma 1 to Proposition 29 of Book X states that if $m > n$ are any positive integers, then

$$(2mn)^2 + (m^2 - n^2)^2 = (m^2 + n^2)^2$$

(Exercise 13) so that $2mn,\ m^2 - n^2,\ m^2 + n^2$ form a Pythagorean triple. For example, $m = 5$ and $n = 4$ yield the triple

$$(2\cdot5\cdot4)^2 + (5^2 - 4^2)^2 = (5^2 + 4^2)^2$$

or

$$40^2 + 9^2 = 41^2.$$

Pierre Fermat (1601? - 1665) took it for granted that Euclid's method can be used to generate all the Pythagorean triples and this fact was proven by Euler a hundred years later. Specifically, Euler proved that if a, b, c are numbers whose only common divisor is 1 and which constitute a Pythagorean triple, then there exists a pair of relatively prime integers m, n such that

$$\{a, b, c\} = \{2mn,\ m^2 - n^2,\ m^2 + n^2\}.$$

All other Pythagorean triple of course proportional to these.

EXERCISES 3.3A

1. Which of the following triples of numbers are the lengths of the sides of a right triangle:
   a) 7, 10, 15 b) 5, 12, 13
   c) 203750, 364056, 417194 d) 57302, 491714, 650463
2. Show that an equilateral triangle of side a has area $\frac{a^2\sqrt{3}}{4}$.
3. An isosceles right triangle has a hypotenuse of length c. Compute its other sides and its area.
4. A right triangle has an angle of 30° and a hypotenuse of length 1. Compute its other sides and its area.
5. Compute the area of a rhombus whose sides equal 13 and one of whose diagonals has length 10.
6. Compute the area of a parallelogram whose sides have lengths 11 and 8 and one of whose angles is 45°.
7. The diagonals and one side of a parallelogram have lengths 30, 16, 17, respectively. Prove that it is a rhombus and compute its area.
8. Show that the interior quadrilateral in Dissection II of Figure 3.24 is indeed a square of area c^2.
9. Prove Proposition 3.3.3.
10. Find the error in the following "proof" of the "proposition" that every triangle is isosceles:
    GIVEN: ΔABC

TO PROVE: $AB = AC$

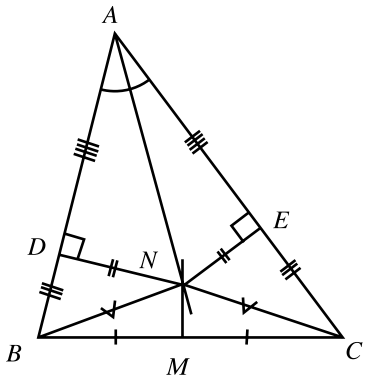


Figure 3.26

PROOF: Let N be the intersection of the bisector of $\angle ABC$ and the perpendicular bisector of side BC, M the midpoint of BC, and ND ' AB, NE ' AC. Then

	ND	=	NE	[PN 2.3.33]
$\therefore$	AD	=	AE	[Pythagoras]
Also	BN	=	CN	[PN 2.3.14]
$\therefore$	BD	=	CE	[Pythagoras]
$\therefore$	AB	=	AC	[CN 2]

Q.E.D.

11. Given a square of side a, construct a square of double its area.
12. Given a square of side a and a positive integer n, construct a square whose area equals n times that of the given square.
13. Construct a square whose area equals the sum of three given squares.
14. Use algebra to prove that Euclid's method does indeed generate Pythagorean triples.
15. Assume that a line segment of length 1 inch is given. Prove that line segments of the following lengths can be constructed.

    a) $\sqrt{2}$ inch b) $\sqrt{3}$ inch c) $\sqrt{5}$ inch

    d) $\sqrt{n}$ inch, where n is any positive integer e) $1/\sqrt{2}$ inch .
16. Find two non-congruent isosceles triangles whose sides have integer lengths and whose perimeters and areas are equal.
17. Prove the Theorem of Pythagoras by applying Exercise 3.2A.4 to Figure 3.27.

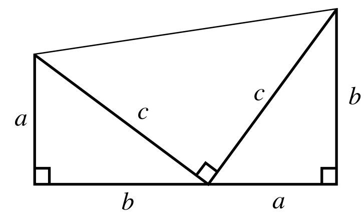


Figure 3.27

18(C). Perform the construction of Proposition 3.3.1 using a computer application.

19(C). Use a computer application to verify the Theorem of Pythagoras.

Both spherical and hyperbolic geometry have their own versions of the Theorem of Pythagoras. Their proofs follow directly from the appropriate trigonometries (Exercises 1, 3).

PROPOSITION 3.3.4(The spherical Theorem of Pythagoras). *If the spherical* ΔABC *has a right angle at* C, *then*

$$\cos c = \cos a \cos b.$$

PROPOSITION 3.3.5(The hyperbolic Theorem of Pythagoras). *If the hyperbolic* ΔABC *has a right angle at* C, *then*

$$\cosh c = \cosh a \cosh b.$$

EXERCISES 3.3D

1. Derive the spherical Theorem of Pythagoras from Proposition 1.1.2.

2. Find the length of the hypotenuses of the three spherical isosceles right triangles whose legs have lengths 1, .1, .01 respectively. Compare the answers to the lengths of the hypotenuses of the three Euclidean isosceles right triangles both of whose legs have lengths 1, .1, .01 respectively.
3. Derive the hyperbolic Theorem of Pythagoras from Propositions 1.2.2.
4. Find the length of the hypotenuses of the three hyperbolic right triangles whose legs have lengths 1, .1, .01 respectively. Compare the answers to the lengths of the hypotenuses of the three Euclidean triangles both of whose legs have lengths 1, .1, .01 respectively.
5. Is there a taxicab version of the Theorem of Pythagoras?
6. Is there a maxi version of the Theorem of Pythagoras?

4. Consequences of the Theorem of Pythagoras (optional)

Book II of Euclid's *Elements* contains a variety of consequences of the Theorem of Pythagoras of which only a sample are presented here. The first of these is tantamount to a construction of the *Golden Ratio*. This proposition will be used later in the construction of the regular pentagon.

PROPOSITION 3.4.1(II.6). *To cut a given line segment so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.*

GIVEN: Line segment AB (Fig. 3.28).

TO CONSTRUCT: A point C on AB such that $AB \cdot BC = AC^2$.

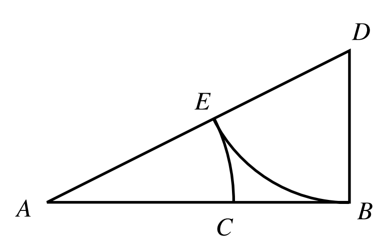


Figure 3.28

CONSTRUCTION: At B construct BD ' AB and $BD = \frac{1}{2}AB$. Join AD, let E be the point on AD such that $DE = DB$ and let C be the point on AB such that $AC = AE$. Then C is the required point.

PROOF: Set $AB = 2a$. Then $DE = BD = a$ and

$$AC = AE = AD - DE = \sqrt{(2a)^2 + a^2} - a = (\sqrt{5} - 1)a$$

so that

$$AC^2 = (\sqrt{5} - 1)^2 a^2 = (5 - 2\sqrt{5} + 1)a^2 = (6 - 2\sqrt{5})a^2$$

and

$$AB{\cdot}BC = AB(AB - AC) = AB(AB - AE) = AB[AB - (AD - DE)] =$$
$$2a[2a - (\sqrt{5}a - a)] = 2(3 - \sqrt{5})a^2 = AC^2$$

Q.E.D.

In the context of the above proposition, the common value τ of the ratios

$$\frac{BC}{AC} = \frac{AC}{AB} = \frac{AD - DE}{AB} = \frac{(\sqrt{5} - 1)a}{2a} = \frac{\sqrt{5} - 1}{2} = 0.618...$$

is called the *Golden Ratio*. While of demonstrated mathematical interest, this quantity has also been the subject of much nonsensical speculation. Typical of this latter variety is an article that reports that the average ratio of the height of a man's navel off the ground to his height equals the Golden ratio.

The following two propositions constitute Euclid's analog of the modern day Law of Cosines. Their proofs are relegated to the exercises.

PROPOSITION 3.4.2(II.12). *In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse* *angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.*

GIVEN: ΔABC, $\angle BAC$ > right angle, $CD' AB$ (Fig. 3.29).

TO PROVE: $BC^2 = AB^2 + AC^2 + 2AB \cdot AD$.

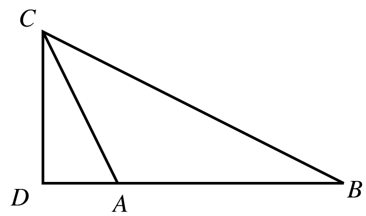


Figure 3.29

PROOF: See Exercise 2.

PROPOSITION 3.4.3(II.13). *In acute-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.*

GIVEN: ΔABC, $\angle BAC$ < right angle, $CD' AB$ (Fig. 3.30).

TO PROVE: $BC^2 = AB^2 + AC^2 - 2AB \cdot AD$.

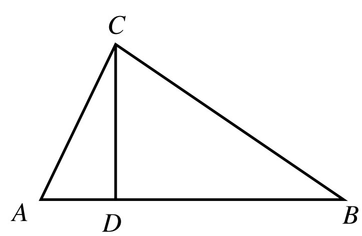


Figure 3.30

PROOF: See Exercise 3.

EXERCISES 3.4A

1. Explain the relation of Proposition 3.4.2 to the Law of Cosines of the trigonometry of the Euclidean plane.
2. Prove Proposition 3.4.2.
3. Explain the relation of Proposition 3.4.3 to the Law of Cosines of the trigonometry of the Euclidean plane.
4. Prove Proposition 3.4.3.

Assuming a unit length and a segment of length r units, the next proposition deals with the construction of a line segment of length $\sqrt{r}$ units. The statement that appears here is weaker than that of Euclid's, but it is sufficient for this text's purposes and obviates the need for the omitted Proposition I.45. Euclid's version appears in Exercise 2.

PROPOSITION 3.4.4(II.14). *To construct a square equal to a given rectangle.*

GIVEN: ▭ *BCDE* (Fig. 3.31).

TO CONSTRUCT: Line segment EH such that EH^2 = ` $BCDE$.

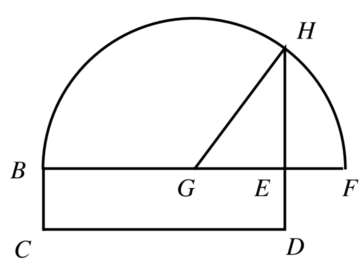


Figure 3.31

CONSTRUCTION: If $BE = ED$ then BE is the required line segment. Otherwise, it may be assumed without loss of generality that $BE > ED$. Extend BE to F so that EF

$= DE$ and let G be the midpoint of BF. Let H be an intersection of the straight line through E perpendicular to BF with the circle $(G; BG)$. Then EH is the required line segment.

PROOF: By the Theorem of Pythagoras,

$$EH^2 = GH^2 - GE^2 = (GH + GE)(GH - GE) = (BG + GE)(GF - GE)$$

$$= BE \cdot EF = BE \cdot ED = \square BCDE .$$

Q.E.D.

An alternative proof of the equation $EH^2 = BE \cdot EF$ appears in Exercise 3.5C.16.

EXERCISES 3.4B

1. Assume that a line segment of length 1 inch is given. Construct a line segment whose length is:

   a) $\sqrt{6}$ inch b) $\sqrt{11}$ inch c) $\sqrt[4]{2}$ inch

   d) $\sqrt[4]{6}$ inch e) $\sqrt[4]{11}$ inch

2. Prove Euclid's Proposition II.14: To construct a square equal to a given polygon.

5. Proportion and Similarity

Euclid's definition of proportion is too intricate for the context and purpose of this text. Instead, a shortcut provided by the real number system is used. Recall Euclid's tacit assumption that all geometrical figures have an aspect of size or magnitude. Thus, lines, regions, solids, and angles, have lengths, areas, volumes, and angular measure respectively. If two figures of the same types have sizes a and b (relative to some unit), it is said that their *ratio* is the real number $\frac{a}{b}$. The numbers $a, b, c, \ldots$ are said to be *proportional* to the numbers $a', b', c', \ldots$ provided that

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \ldots .$$

The next five propositions set up some algebraic preliminaries. The first of these is, of course, none other than the Distributive Law and so requires no proof. The rest are basic observations regarding proportions.

PROPOSITION 3.5.1(V.1). *If* $m, a, b, c,$ *are any numbers, then*

$$ma + mb + mc + \ldots = m(a + b + c + \ldots)$$

PROPOSITION 3.5.2(V.12). *If* $a, b, c, \ldots$ *are proportional to* $a', b', c', \ldots$ *then*

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \ldots = \frac{a + b + c + \ldots}{a' + b' + c' + \ldots} .$$

GIVEN: $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \ldots .$

TO PROVE: $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \ldots = \frac{a + b + c + \ldots}{a' + b' + c' + \ldots} .$

PROOF: Let $k = \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \ldots$. Then

$$a = ka', \qquad b = kb', \qquad c = kc', \qquad \ldots$$

$$\therefore \quad a + b + c + \ldots = ka' + kb' + kc' + \ldots = k(a + b + c + \ldots)$$

$$\therefore \quad \frac{a + b + c + \ldots}{a' + b' + c' + \ldots} = k = \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \ldots .$$

Q.E.D.

PROPOSITION 3.5.3(V.16). $\frac{a}{a'} = \frac{b}{b'}$ *if and only if* $\frac{a}{b} = \frac{a'}{b'}$.

See Exercise 8.

PROPOSITION 3.5.4(V.17). *If* $\frac{a}{a'} = \frac{b}{b'}$ *then* $\frac{a - a'}{a'} = \frac{b - b'}{b}$.

See Exercise 9.

PROPOSITION 3.5.5(V.18). *If* $\frac{a}{a'} = \frac{b}{b'}$ *then* $\frac{a + a'}{a'} = \frac{b + b'}{b}$.

See Exercise 10.

EXERCISES 3.5A

In Exercises 1-7 prove the stated equalities on the basis of the assumption that a, b, c, d are proportional to a', b', c', d'.

1. $\frac{a}{a'} = \frac{4a - 3b + 2c + 7d}{4a' - 3b' + 2c' + 7d'}$
2. $\frac{a^2}{a'^2} = \frac{4a^2 - 3b^2 + 2c^2 + 7d^2}{4a'^2 - 3b'^2 + 2c'^2 + 7d'^2}$
3. $\frac{a}{a'} = \frac{\sqrt{a^2 + 2b^2 - 3c^2 + 5d^2}}{\sqrt{a'^2 + 2b'^2 - 3c'^2 + 5d'^2}}$
4. $\frac{a^3}{a'^3} = \frac{a^3 - b^3 + c^3 - d^3}{a'^3 - b'^3 + c'^3 - d'^3}$
5. $\frac{a + 3b}{a - 3b} = \frac{c + 3d}{c - 3d}$
6. $\frac{a - 3c}{a + 3c} = \frac{b - 3d}{b + 3d}$
7. $\frac{a^2 + 3d^2}{a^2 - 3d^2} = \frac{c^2 + 3b^2}{c^2 - 3b^2}$
8. Prove Proposition 3.5.3.
9. Prove Proposition 3.5.4.
10. Prove Proposition 3.5.6.

PROPOSITION 3.5.6(VI.2). *If a straight line meets two sides of a triangle, then it is parallel to the third side if and only if it cuts them into proportional segments*.

GIVEN: ΔABC, points D and E on $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ respectively (Fig. 3.32).

TO PROVE: 1. If $DE \parallel BC$ then $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

2. If $\dfrac{AD}{DB} = \dfrac{AE}{EC}$ then $DE \parallel BC$.

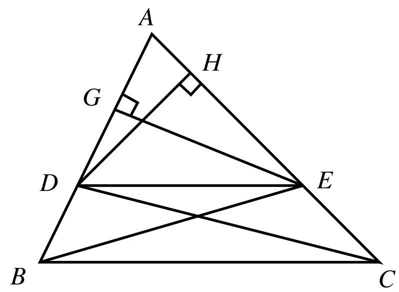


Figure 3.32

Proof of 1: Join CD and BE. Then, since $DE \parallel BC$

$\Delta BDE = \Delta CDE$ [PN 3.2.6]

$\Delta ADE = \Delta ADE$

$\therefore \quad \Delta ABE = \Delta ACD$ [CN 2]

$$\therefore \quad \frac{\Delta ABE}{\Delta ADE} = \frac{\Delta ACD}{\Delta ADE}$$

Let EG and DH be altitudes of ΔADE. Then it follows from the above that

$$\frac{AB \cdot GE}{AD \cdot GE} = \frac{AC \cdot DH}{AE \cdot DH} \qquad \text{[PN 3.2.5]}$$

$$\therefore \quad \frac{AB}{AD} = \frac{AC}{AE}$$

$$\therefore \quad \frac{DB}{AD} = \frac{EC}{AE} \qquad \text{[PN 3.5.4]}$$

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}$$

Proof of 2: Using the same construction as above, it is only necessary to reverse the order of the steps of the above argument (Exercise 1).

Q.E.D.

The proof of the above theorem depends superficially on the additional assumption that the point D lies between A and B. Exercises 15-17 rectify this minor flaw.

EXERCISES 3.5B

1. Complete the proof of Proposition 3.5.6.
2. Prove that the straight line that bisects one side of a triangle and is parallel to a second side also bisects the third side.
3. Use Proposition 3.5.6 (twice) to prove that the line segment joining the midpoints of two sides of the triangle is parallel to the third side and equals half its length.
4. The point K is on the side AB of ΔABC, points L, M are on side AC, so that $KL \parallel BM$ and $KM \parallel$ BC. Prove that $\frac{AL}{AM} = \frac{AM}{AC}$.
5. Point O is not on any of the sides of ΔABC or their extensions, and K, L, M are on OA, OB, OC respectively so that $KL \parallel AB$ and $LM \parallel BC$. Prove that $KM \parallel AC$.
6. Prove that if D is any point on the side BC of ΔABC, then AD bisects the interior angle at A if and only if $\frac{AB}{AC} = \frac{BD}{DC}$.
7. Prove that if E is any point on the extension of side BC of ΔABC, then AE bisects the exterior angle at A if and only if $\frac{AB}{AC} = \frac{BE}{EC}$.
8. Prove that if the straight lines $m_1, m_2, ..., m_n$ are all parallel to one side of a triangle and they cut off equal segments on a second side, then they also cut off equal segments on the third side.
9. Divide a given line segment into three equal parts.
10. Let n be a given positive integer. Divide a given line segment into n equal parts.
11. Let m and n be given positive integers. Divide a given line segment into two parts whose ratio is m/n.
12. Let a, b, c be three given line segments. Divide a into two parts whose ratio equals b/c.

13. Let a, b, c be three given line segments. Construct a line segment x such that $\frac{a}{b} = \frac{c}{x}$.

14. Let a, b be two given line segments. Construct a line segment x such that $\frac{a}{x} = \frac{x}{b}$.

15. Show that the proof of Proposition 3.5.6 still holds, with minor modifications, when A lies in between B and D.

16. Show that the proof of Proposition 3.5.6 still holds, with minor modifications, when B lies in between A and D.

17. Does the proof of Proposition 3.5.6 require any other corrections?

18. Comment on Proposition 3.5.6 in the context of the following geometries:

    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

19. Use the spherical trigonometry formulas to experiment with comparing the length of the line joining the midpoints of two sides of a spherical triangle with that of the third side (see Exercise 2 above). Form a conjecture regarding the relative sizes of these two geodesic segments.

20. Use the hyperbolic trigonometry formulas to experiment with comparing the length of the line joining the midpoints of two sides of a hyperbolic triangle with that of the third side (see Exercise 2 above). Form a conjecture regarding the relative sizes of these two geodesic segments..

21. Comment on Exercise 19 in the context of taxicab geometry.

22. Comment on Exercise 19 in the context of maxi geometry.

23. Construct ΔABC given the following data: a) a, m_b, m_c b) m_a, m_b, m_c.

24(C). Use a computer application to verify part 1 of Proposition 3.5.6.

Similar polygons are those whose corresponding angles are equal and whose corresponding sides are proportional. Congruent triangles are similar (Exercise 2) and the relation of similarity is transitive (Exercise 3). If ΔABC and ΔDEF are similar this is denoted by $\Delta ABC \sim \Delta DEF$ where it is implicit that A, B, C correspond to D, E, F respectively, as was the case for congruent triangles. The next proposition is known as the AAA similarity theorem.

PROPOSITION 3.5.7(VI.4). *Equiangular triangles are similar.*

GIVEN: $\Delta ABC, \Delta DEF$ $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$, $\angle BAC = \angle EDF$ (Fig. 3.33)

TO PROVE: $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$.

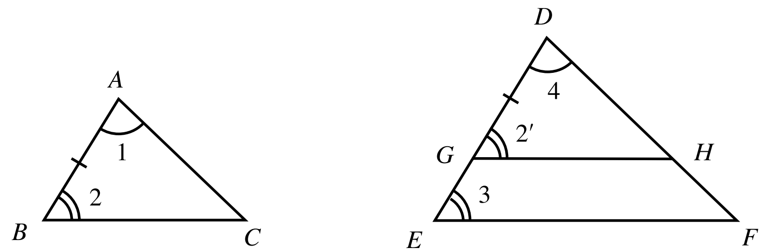


Figure 3.33

PROOF: If any side of $\Delta\, ABC$ equals the corresponding side of $\Delta\, DEF$ then the two triangles are congruent [ASA] . Hence it may be assumed without loss of generality that $AB < DE$. Let G be a point in the interior of DE such that $DG = AB$. Let H be the intersection of the straight line parallel to EF through G with DF. Then

$$\frac{GE}{DG} = \frac{HF}{DH} \qquad \text{[PN 3.5.6]}$$

$$\frac{DE}{DG} = \frac{DF}{DH} \qquad \text{[PN 3.5.5]}$$

However, $\Delta\, ABC \cong \Delta\, DEF$ by ASA because

	$\angle 2$	$=$	$\angle 2'$	[Both equal $\angle 3$]
	AB	$=$	DG	[Construction]
	$\angle 1$	$=$	$\angle 4$	[Given]
$\therefore$	AC	$=$	DH	

$$\therefore \quad \frac{AB}{DE} = \frac{DG}{DE} = \frac{DH}{DF} = \frac{AC}{DF}$$

A similar argument can be used to prove that $\dfrac{BC}{EF} = \dfrac{AC}{DF}$

Q.E.D.

The following similarity theorems are known as the SSS and SAS similarity theorems, not to be confused with the SSS and SSA congruence theorems, and their proofs are relegated to Exercises 5 and 6 respectively.

PROPOSITION 3.5.8(VI.5). *If two triangles have their sides proportional then the triangles are similar.*

PROPOSITION 3.5.9(VI.6). *If two triangles have one angle equal to one angle and the sides about those angles are proportional, then the triangles are similar.*

In view of the fact that the sum of the angles of every Euclidean triangle is 180°, it follows that the conclusion of Proposition 3.5.7 holds even when only <u>two</u> of the angles of one triangle are known to be equal to the corresponding angles of the other triangle.

EXERCISES 3.5C

1. Complete the proof of Proposition 3.5.7.
2. Prove that congruent triangles are similar.
3. Prove that if ΔABC is similar to $\Delta A'B'C'$ and $\Delta A'B'C'$ is similar to $\Delta A''B''C''$, then ΔABC is similar to $\Delta A''B''C''$.
4. Prove Proposition 3.5.8.
5. Prove Proposition 3.5.9.
6. Prove that in similar triangles corresponding altitudes are proportional to corresponding sides.
7. Prove that in similar triangles corresponding medians are proportional to corresponding sides.
8. Prove that in similar triangles the corresponding angle bisectors are proportional to the corresponding sides.
9. Prove that the areas of similar triangles are proportional to the squares of their corresponding sides.
10. Prove that the areas of similar polygons are proportional to the squares of their corresponding sides (VI.19).

11. Three parallel straight lines cut the straight lines m and n in the points $A, B, C,$ and $K, L, M.$ respectively. Prove that $\dfrac{AB}{BC} = \dfrac{KL}{LM}$.

12. In $\sim ABCD$ the straight line $BKLM$ cuts the diagonal AC in the point K and the (possibly extended) sides AD and CD in the points L and M respectively. Prove that $\dfrac{BK}{KL} = \dfrac{KM}{BK}$.

13. In $\sim ABCD$ the straight line BKN cuts the (possibly extended) sides CD and AD in the points K and N respectively. Prove that $\dfrac{AD}{DN} = \dfrac{CK}{KD}$

14. Prove that the intersection point of two of a triangle's altitudes divides them so that the product of each altitude's segments equals the product of the other's segments.

15. Prove Euclid's Proposition VI.8: If in a right triangle a perpendicular be drawn from the right angle to the opposite side, then the triangles so formed are similar to each other and to the given triangle.

16. Prove Euclid's Proposition VI.13: The square of the altitude to the hypotenuse of a right triangle equals the product of the segments it determines on the hypotenuse.

17. Prove Euclid's Proposition VI.31: If similar polygons are constructed on the sides of a right triangle, then the polygon on the hypotenuse equals the sum of the polygons on the other two sides.

18. Describe the locus of all the points whose distances from the sides of a given angle have a ratio equal to that of two given line segments.

19. Given an angle and a point A inside it, construct through A a straight line whose portion between the sides of the angle is divided by A into segments whose ratio equals that of two given line segments.

20. Given an angle and a point A inside it, find a point P on one side of the angle whose distance from A equals its distance from the other side of the angle. (How many solutions are there?)

21. Prove that the perimeters of similar triangles are proportional to their corresponding sides.

22. Prove that the perimeters of similar polygons are proportional to their corresponding sides.

23. A straight line through the intersection of the diagonals of a trapezoid is parallel to its parallel sides. Prove that the segment between the non-parallel sides is bisected by the intersection of the diagonals.

24. Prove that in a trapezoid the line joining the midpoint of one of the parallel sides to the intersection of the diagonals bisects both the parallel sides.

25. Prove that in a trapezoid which is not a parallelogram the straight line joining the intersection of the diagonals to the intersection of the non-parallel sides bisects both the parallel sides.

26. Comment on Proposition 3.5.7 in the context of the following geometries:

    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

CHAPTER REVIEW EXERCISES

1. Let P be a point in the interior of the equilateral ΔABC. Prove that the sum of the perpendicular segments from P to the sides of the triangle is constant.
2. The diagonals of a trapezoid cut each other into segments that are proportional to the parallel sides of the trapezoid.
3. Suppose the non-parallel sides of a trapezoid are equal. Prove the following.
   - i) The two angles adjacent to the same base are equal.
   - ii) The diagonals are equal.
   - iii) The diagonals intersect in a point that lies on the straight line joining the midpoints of the unequal sides.
   - iv) The diagonals divide each other into respectively equal segments.
   - v) The midpoints of the four sides form a rhombus.
4. Find a simple expression for the angle between two of a triangle's angle bisectors.
5. A straight line through the vertex of a triangle divides it into two triangles that are similar to each other and to the original triangle. Prove that the given triangle is a right triangle.
6. In $\sim ABCD$ a straight line parallel to AB intersects AD, AC, and BC in the points P, Q, R respectively. Prove that $\Delta APR = \Delta AQD$.
7. In ΔABC, $m_a = BC/2$. Prove that $\angle CAB$ is a right angle.
8. In ΔABC, $\angle BAC = 2\angle ABC$. Prove that $a^2 = b(b + c)$.
9. In $\sim ABCD$ M and N are the midpoints of the opposite sides AB and CD. Prove that the straight lines DM and BN divide the diagonal AC into three equal segments.
10. Suppose that $\angle ACB$ of ΔABC is obtuse and the perpendicular bisectors to AC and BC intersect AB in the points D and E respectively. Prove that $\angle DCE = 2(\angle ACB - 90^o)$.
11. From a point on the base of an isosceles triangle straight lines parallel to the triangle's other sides are drawn. Prove that the perimeter of the parallelogram thus formed is independent of the position of the point on the base.
12. Prove that the bisectors of the two angles formed by the opposite pairs of sides of a convex quadrilateral intersect in angle that equals half the sum of two opposite angles of the quadrilateral.
13. Prove that the median and the altitude to the hypotenuse of a right triangle form an angle that equals the difference of the triangle's other two angles.

14*. In ΔABC, $AB = AC$, E and D are on the sides AB and AC respectively, and $\angle ABD = 20^0$, $\angle CBD = 60^o$, $\angle BCE = 50^o$, and $\angle ACE = 30^o$. Find $\angle EDB$.

15. Are the following statements true or false? Justify your answers.
    - a) Playfair's postulate is valid in neutral geometry.

b) Playfair's postulate is valid in Euclidean geometry.

c) Playfair's postulate is valid in spherical geometry.

d) Playfair's postulate is valid in hyperbolic geometry.

e) Playfair's postulate is valid in taxicab geometry.

f) If in a quadrilateral one pair of opposite sides are equal, as are one pair of opposite angles, then the quadrilateral is a parallelogram.

g) There is a Euclidian right triangle with sides $287, 816, 865$.

h) There is a neutral right triangle with sides $287, 816, 865$.

i) There is a spherical right triangle with sides $287, 816, 865$.

j) There is a hyperbolic right triangle with sides $287, 816, 865$.

k) There is a taxicab right triangle with sides $287, 816, 865$.

l) If the corresponding sides of two triangles are proportional then so are their corresponding angles.

m) Equiangular triangles are similar.

n) The corresponding sides of equiangular quadrilaterals are proportional.

o) If the corresponding angles of two quadrilaterals on the surface of a sphere are equal, then so are their areas.

p) Equiangular quadrilaterals are similar.