CHAPTER 2

The Neutral Geometry of the Triangle

1. Introduction

It is common knowledge that most of the methods and tools of elementary mathematics were developed independently thousands of years ago by many cultures, of which the most influential were those of Babylonia, Egypt, China, and India. These skills usually included a collection of rules for the computation of the areas of figures and plots, as well as the volumes of solids and containers, from which geometry is descended (in Greek *geo*-"earth" and *metron*-"a measure"). Such rules were often poorly justified, if at all, and in some cases provided only an approximation to the exact value in question. Moreover these rules were never stated in their generality. Instead, they were exemplified by means of specific computations. Thus, nowhere in the surviving writings of the Egyptian scribes is there found a formula for the calculation of the area of a circle of a given radius. Instead, the Rhind Mathematical Papyrus, which dates to 1650 B.C. or earlier, contains the following computation:

> Example of a round field of a diameter 9 khet. What is its area? Take away 1/9 of the diameter, 1; the remainder is 8. Multiply 8 times 8; it makes 64. Therefore it contains 64 setat of land.

It is, of course, reasonable to second-guess this papyrus's scribe and to claim that the general formula that underlies his worked out example is

$$A = (d - \frac{1}{9}d)^2 = (\frac{8d}{9})^2$$

but the fact of the matter is that no such explicit formulas have ever been found amongst any of the surviving Egyptian manuscripts. They probably did not exist and it is for this reason that this kind of geometry has been called *subconscious geometry*.

The naive geometry of these ancient cultures was transformed by the Greeks of the sixth and fifth centuries B.C. into a deductive science that is completely modern in spirit. Their contribution was twofold. They invented the abstract statements that are today called *theorems* and showed how these theorems could be *demonstrated* in a purely logical manner. The first steps in this direction seem to have been taken by Thales (*c*. 640 - *c*. 546 B.C.) of Miletus, on the west coast of Asia Minor. While nothing is known with certainty about him, he is credited with having either stated or proved the following results:

1. *A circle is bisected by any diameter.*
2. *The base angles of an isosceles triangle are equal to each other.*
3. *The vertical angles formed by two intersecting straight lines are equal.*
4. *Two triangles are congruent if they have two angles and one side in each respectively equal.*
5. *An angle inscribed in a semicircle is a right angle.*

These results are, of course, quite elementary and Thales's accomplishment lies in his recognition of the value of abstract statements. He is also reputed to have traveled widely and legend has it that he amazed the Egyptian sages by computing the heights of the pyramids (probably by means of similar triangles).

Pythagoras (*c*. 585 - *c*. 580 B.C.) is believed to have studied with Thales and then moved to southern Italy where he founded his own school. In this school he taught a mixture of philosophy, science, and religion that also included a fair amount of mathematics. One of the tenets of the Pythagorean faith was the belief that the positive integers were the ultimate components of the universe. For example, if one taut string is double the length of another, plucking them will result in two notes that are exactly one octave apart. If the lengths of the strings have ratio 3 to 2, then the shorter one will produce a note that is recognized by musicians as being a fifth above the note of the longer string. Similarly, the visual world could be reduced to numbers by endowing the latter with shapes. Thus, 1, 3, 6, 10, ... were called *triangular numbers* because the corresponding number of dots could be organized into triangles (Fig. 2.1). The *square*

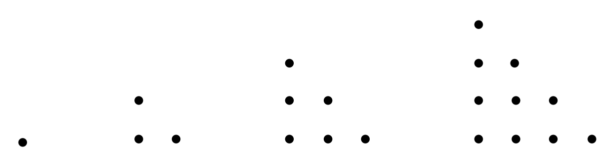

Figure 2.1

numbers were, of course, 1, 4, 9, 16, ... (Fig. 2.2). These visual representations of the

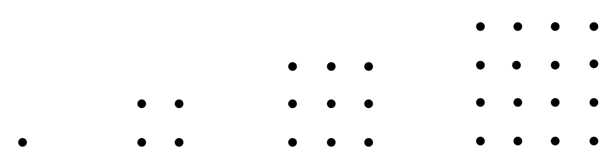

Figure 2.2

integers are mathematically fruitful. Figure 2.3 makes it clear that the sum of two

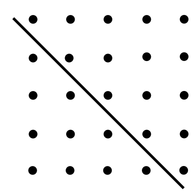

Figure 2.3

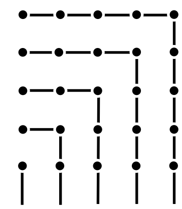

Figure 2.4

consecutive triangular numbers is a square number, and Figure 2.4 makes a convincing case that the sum of any sequence of consecutive odd integers that begins with 1 is in fact also a square number. Given their other discoveries, it is hard to believe that the Pythagoreans failed to notice these interesting relationships. They are supposed to have discovered all the theorems regarding triangles, polygons, and circles that are taught as part of today's high school curriculum, including the fact that the sum of the angles of a triangle is 180^{o} and, of course, the Theorem of Pythagoras. In addition, they also had some knowledge of the tilings and the regular solids that are discussed in Chapters 7 and 8 of this book.

The Pythagoreans' preoccupation with integers eventually led them to the realization that the length of the diagonal of a square of unit side is not expressible as the ratio of two integers. In other words, $\sqrt{2}$ is *irrational*. This discovery must have been disconcerting because it conflicted with the aforementioned doctrine that the universe can be expressed in terms of integers. Reportedly, its discoverer, Hippasus of Metapontum, was drowned for his bad tidings. Eventually, the Greeks, if not the Pythagoreans themselves, resigned themselves to the existence of irrational numbers and went on to develop a theory of incommensurable numbers that made it possible to incorporate them into the general framework of mathematics.

The irrationality of $\sqrt{2}$ is, of course, an interesting and important mathematical fact. Equally significant, however, are its implications regarding the high level of rigor the Greeks of the fifth century attained. All known proofs of this irrationality are subtle;

no visually immediate proof of the type displayed in Figures 2.3 and 2.4 exists. It is therefore safe to assume that by this time geometry had matured no only in the abstraction of its concepts but also in the rigor of its arguments. Anyone who can produce a convincing proof of such a surprising and counterintuitive fact as the irrationality of $\sqrt{2}$ must have logical standards at least equal to those of today's graduate mathematics students.

The sophists, a group of itinerant professional teachers, the earliest of whom studied with Pythagoras, did much to further the cause of geometry by incorporating it into their curriculum (which also included grammar, rhetoric, dialectics, eloquence, morals, astronomy, and philosophy). Since they promoted themselves as practical trainers in the art of persuasion as preparation for political and legal careers, they must have considered the abstraction and logical argumentation of geometry as a valuable pedagogical tool. As Plato says in *The Republic:*

> ... with respect to finer reception of all studies, we surely know there is a general and complete difference between the man who has been devoted to geometry and the one who has not.

None of several geometry books that preceded Euclid's have survived. The first of those was written by Hippocrates of Chios (circa 460-380 B.C.). The most important and innovative was, without doubt, that of Eudoxus (408 - *c*. 355 B.C.) who, amongst other accomplishments, invented the *method of exhaustion*, a calculus-like discipline.

Euclid of Alexandria wrote his famous book *The Elements* circa 300 BC. This book codified some of the state of the art of geometry at the time, although much was omitted too. It actually consists of thirteen books whose contents are

Books I-IV: The Geometry of Triangles and Circles

Books V-VI: Theory of Geometric Proportions

Books VII-IX: Theory of Numbers

Book X: Theory of Irrational Surds

Books XI-XIII: Solid Geometry.

For two thousands years *The Elements* remained the standard text for geometry throughout the world. It was translated into many languages and over a thousand printed editions have been published since 1482. After the Bible, it is Western Civilization's most influential book. The first alternative exposition of geometry was offered by the French mathematician Adrien-Marie Legendre (1752 - 1833) in 1794. The discovery of non-Euclidean geometry in the nineteenth century forced mathematicians to reexamine the foundations of geometry from a much more critical point of view. A variety of gaps and flaws were found and this work culminated in David Hilbert's (1862 - 1943) *The Foundations of Geometry* which appeared in 1899 and which set Euclidean geometry on completely rigorous grounds (see Appendix E). Unfortunately this development, as well as those promulgated subsequently by other mathematicians, suffers from the pedagogical defect that the proofs of many "obvious" facts are quite difficult. Current high school geometry texts resolve this difficulty by listing many such facts as axioms rather than theorems. This, of course, is problematic because it opens up the possibility of simply declaring all geometrical facts to be axioms. There has been a considerable amount of experimentation with innovative ways of teaching geometry during the last fifty years but, unfortunately, no consensus has been reached. In the process, much of the concern with rigor that characterized the old fashioned teaching style has been sacrificed in the name of reform and replaced with other, more accessible, visually appealing, or contemporary, topics.

EXERCISES 2.1A

1. Briefly describe the lives and accomplishments of the following predecessors of Euclid:

a)	Thales	b)	Pythagoras	c)	Democritus
d)	Anaxagoras	e)	Archytas	f)	Eudoxus
g)	Hippocrates of Chios				

2. Briefly describe the lives and accomplishments of the following successors of Euclid:

a)	Archimedes	b)	Apollonius	c)	Ptolemy
d)	Heron	e)	Menelaus	f)	Pappus
g)	Proclus				

2. Preliminaries

Quite appropriately, Euclid began his development with a list of definitions.

DEFINITIONS

1. *A* point *is that which has no part.*

By modern standards, at least, many of Euclid's definitions are deficient and it has been argued that they were added by a later commentator. This particular definition is not proper because it is impossible to define a term by listing only the qualities it does not have. Instead it should be viewed as an attempt on its author's part to informally convey the notion of a point - the ultimate indecomposable particle. Definitions 2 below should be understood in the same way. Points will be denoted by upper case letters *A, B, C, ...*

2. *A* line *is breadthless length.*

Euclid's *line* is today's *curve*.

3. *The extremities of a line are points*.

No new terms are defined here. Euclid is simply clarifying the relation between the points and the lines defined above.

4. *A* straight line *is a line which lies evenly with the points on itself.*

This sentence is obscure. In the author's opinion, the *points* of this definition are the two extremities mentioned in the previous one. In other words, of all the lines having the same extremities, the straight line is that one which lies exactly in between those extremities.

It is customary to conclude from Definitions 3 and 4 that Euclid implicitly assumed a straight line to have finite length so that it necessarily has endpoints. The phrasing of Propositions 2.3.10 and 2.3.15 (amongst others) below indicates that this view may not be entirely correct. When necessary, Euclid had no qualms about referring to the infinite line that extends indefinitely in both directions. Many of today's texts, although by no means all, use $\overline{AB}$ to denote the finite straight line segment joining A and B, AB for its length, and $\overleftrightarrow{AB}$ for the infinite straight line that contains both A and B. Like Euclid, this text will gloss over the distinction between the first two and use AB for both. When necessary, either the double arrow notation or a lower case letter will be used for the infinite line. Similarly, $\overrightarrow{AB}$ denotes the *ray* or *halfline* which consists of that half of $\overleftrightarrow{AB}$ that begins at A, extends indefinitely, and contains B.

5. *A* surface *is that which has length and breadth only*.

6. *The extremities of a surface are lines*.

It is implicit in this definition that every surface is necessarily of finite extent.

7. *A* plane surface *is a surface which lies evenly with the straight lines on itself*.

This should be interpreted in a manner similar to that of Definition 4. A finite plane surface is characterized by the fact that it lies exactly in between its border lines.

8. *A* plane angle *is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line*.

9. *And when the lines containing the angle are straight, the angle is called* rectilineal.

Notwithstanding the fact that Euclid seems to be displaying here an interest in curvilinear angles, he subsequently referred to such angles only once, in Proposition III.16, to make a point that is of little interest. It has been posited that the geometers of the time were dallying with such angles without coming to any serious conclusions, and that Euclid felt it necessary to acknowledge their efforts.

Since it is unclear what is meant by the *inclination* of two lines, an *angle* is defined here as either of the two portions of the plane bounded by a pair of rays that emanate from the same point. By extension, any pair of line segments with a common endpoint also determines two angles. The lines forming the angle are its *sides* and the intersection of the sides is the angle's *vertex*. An angle that has vertex A and sides AB and AC will be denoted by $\angle BAC$ (or $\angle CAB$). Definition 8 excludes the possibility that AB and AC determine the same infinite straight line. I.e., Euclid excluded the 0° and 180° angles from his definition, possibly because zero was not then recognized as a number. This has both advantages and disadvantages and this text will not follow his lead in this instance. Thus, if A, B, C are consecutive points on a straight line, then $\angle ABC$ is a *straight angle* and $\angle BAC$ and $\angle BCA$ are *zero angles*.

10. *When a straight line set up on another straight line makes the adjacent angles equal to one another, each of the equal angles is* right, *and the straight line standing on the other is called a* perpendicular *to that on which it stands*.

The existence of right angles is demonstrated in Proposition 2.3.11. This is also Euclid's first mention of the notion of equality, and it is not clear whether he is referring to equality of measure or congruence. This issue is discussed in detail in the paragraphs dealing with the Common Notions below.

11. *An* obtuse angle *is an angle greater than a right angle*.

12. *An* acute angle *is an angle less than a right angle*.

13. *A* boundary *is that which is an extremity of anything*.

14. *A* figure *is that which is contained by any boundary or boundaries*.

15. *A* circle *is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another*.

16. *And that point is called the* center *of the circle.*

17. *A* diameter *of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.*

18. *A* semicircle *is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.*

19. Rectilineal figures *are those which are contained by straight lines,* trilateral *figures* [or triangles] *being those contained by three,* quadrilateral *those contained by four, and* multilateral *those contained by more than four straight lines.*

These rectilineal figures are today's *polygons* and the straight line segments containing them are their *sides*. The endpoints of the sides of a polygons are its *vertices*. It is implicit in Euclid's definition that every vertex lies on only two sides and every two sides intersect only in a vertex. A straight line joining two vertices that are not the endpoints of a side is a *diagonal*. A polygon that contains all of its diagonals in its interior is said to be *convex*. The triangle with vertices A, B, C is denoted by ΔABC.

20. *Of trilateral figures, an* equilateral triangle *is that which has three sides equal, and* isosceles triangle *that which has two of its sides alone equal, and a* scalene triangle *that which has its three sides unequal.*

According to this definition equilateral triangles are *not* isosceles triangles. This unimportant distinction is contrary to modern usage, which sees an equilateral triangle as a special kind of isosceles triangle, and will be ignored here. It is customary to refer to a side of an isosceles triangle as its *base* provided the other two sides are equal to each other.

21. *Further, of trilateral figures, a* right-angled triangle [or right triangle] *is that which has a right angle, an* obtuse-angled triangle *that which has an obtuse angle, and an* acute-angled triangle *that which has its three angles acute*.

22. *Of quadrilateral figures, a* square *is that which is both equilateral and right-angled; an* oblong [rectangle] *that which is right-angled but not equilateral; a* rhombus *that which is equilateral but not right-angled; and a* rhomboid [parallelogram] *that which has its opposite sides and angles equal to one another but is neither equilateral not right-angled. And let quadrilaterals other than these be called* trapezia.

Once again, Euclid's definitions do not completely agree with modern usage. To us a parallelogram is a quadrilateral whose opposite sides are parallel (see Definition 23 below), a rectangle is a parallelogram whose angles are all right angles, a rhombus is a parallelogram all of whose sides are equal, and a square is a figure that is both a rectangle and a rhombus. The parallelogram $ABCD$ is denoted by ▱$ABCD$, the rectangle $ABCD$ is denoted by ▭$ABCD$, and the square $ABCD$ is denoted by □$ABCD$.

23. Parallel *straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*

If the straight lines AB and CD are parallel, this is denoted by $AB \parallel CD$.

By the time Euclid came to write *The Elements* the Greeks had amassed thousands of theorems. As is the case today, the proofs of many of these relied on other theorems. In order to eliminate the possibility of cyclical reasoning, it was necessary to create a list of the fundamental theorems in which each proof cited only definitions, previously proved theorems, or both. It must have come as a surprise to the Greek geometers to discover that such a list was impossible. The starting point of such a list could not consist of definitions alone - some theorems must be accepted without justifications. These unproven theorems were called *axioms* or *postulates*. Just which theorems should serve as postulates is a question that must be resolved on subjective grounds. It often happens that one mathematician's postulate is another's proven theorem. It is very likely that Euclid's choice of postulates was based to a large extent on the various textbooks to which he had access. Later generations modified his choices in many ways, and two well known systems appear in Appendices C, D, and E.

One informal principle that guides mathematicians in their selection of postulates is that of austerity. A postulate should be a simple, easily parsed, statement. Nevertheless, as will be seen below, there are important exceptions to this rule.

Euclid began with ten axioms. The first five are called postulates and the other *common notions* (the meaning of this term will be clarified soon). These were followed by 462 theorems that are called *propositions* (there is no difference between propositions and theorems). The truth of the matter is that the view of *The Elements* as a well grounded and logically consistent ordering of theorems is to be understood as an ideal only because Euclid's organization of geometry is flawed. There are several instances where undefined terms and unstated postulates appear in his arguments. Some of these are minor errors, but the correction of others would require considerable revision of the material. Nevertheless, because of its vision and because of its logical strength, Euclid's opus is justly regarded as one of the supreme achievements of Greek Civilization in particular and of the human mind in general. Euclid's choice of postulates are now listed.

POSTULATES

Let the following be postulated:

1. *To draw a straight line from any point to any point.*

Taken at face value, this postulate merely states that every pair of distinct points can be joined by a straight line. However, in view of some of the arguments given in the proof of Proposition 2.3.4 and elsewhere, it is necessary to conclude that Euclid understood this statement to include the additional assumption that any two points can be joined by at most one straight line.

2. *To produce a finite straight line continuously in a straight line.*

One of the implications of this postulate is that both the plane and space extend infinitely far in all directions. The reason that it is stated in terms of extendibility rather than extent is that the Greek mathematicians were aware of the logical complications inherent in the concept of infinity and therefore avoided its explicit mention whenever possible.

3. *To describe a circle with any center and distance.*

In this postulate Euclid is not asserting the existence of a circle. This existence actually follows from Postulate 2 which implies that given any point A, straight line m through A, and proposed radius r, there exists a point P on m such that $AP = r$. The set of all such points P is the proposed circle with center A and radius r (Fig. 2.5).

Rather, Euclid is stating here that he intends to use such circles as building blocks in the logical tower he is setting out to construct. Alternatively, one can think of Euclidean geometry as a solitaire game whose legitimate moves are these postulates.

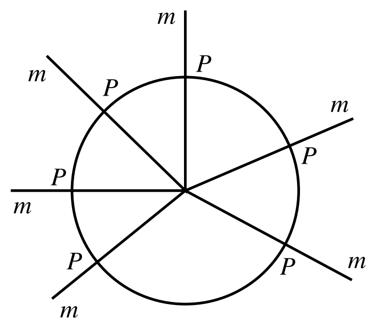

Figure 2.5 Regarding the existence of circles.

The statement of Proposition 2.3.2 below makes it clear that this postulate is to be interpreted in a very narrow sense. Namely, given a point A and a line segment AB there exists a circle with center A and radius AB. The postulate does <u>not</u> say that given a point A and a line segment BC there is a circle with center A and radius equal to BC. This latter statement, which is considerably stronger than Postulate 3, is in fact the content of Proposition 2.3.2 below. The distinction between Postulate 3 and Proposition 2.3.2 is rephrased by modern mathematicians by saying that their Greek predecessors used <u>collapsible</u> compasses whose legs lost their angle whenever the compass was lifted off the paper. This statement is, of course, a metaphor and should not be taken literally. The circle with center A and radius AB or r is denoted in this text by $(A; AB)$ or $(A; r)$.

4. *That all right angles are equal to one another.*

Euclid chose to use the right angle as his theoretical unit for measuring all rectilineal angles (degrees were used in practice) and this is only possible if it is known that all right angles are equal. It was therefore necessary for him to have either a theorem or a postulate that asserts this equality. It might be noted that this postulate is qualitatively different from the previous three. Those proclaimed the legitimacy of certain constructions whereas this one asserts the equality of some figures.

5. *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angle [in sum], the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.*

Like Postulate 4, this one differs from the first three in that it makes a statement that relates the several parts of a figure. Moreover, its statement is much more complicated than any of the preceding statements. For these reasons many of Euclid's successors believed that this postulate could be proved on the basis of the others and was therefore superfluous. However, their repeated attempts, over two millennia, to substantiate this feeling were invariably unsuccessful. Finally, in the nineteenth century, it was demonstrated beyond all doubt that all such efforts must of necessity fail. It is impossible to deduce Postulate 5 from the other postulates. This issue will be discussed in greater detail in Section 2.4 below.

It was mentioned above that Euclid's axiomatization was incomplete in the sense that his proofs make occasional use of unstated and unvalidated assumptions. This is the case in Propositions 2.3.1, 2.3.4, 2.3.8 and others. Two such assumptions are stated as additional postulates.

S (Separation).*The infinitely extended straight line, the triangle and the circle separate the plane into two portions such that any line joining a point of one portion to a point of the other intersects the separating figure. In the case of the line the two portions are called the line's* sides. *In the case of a triangle or a circle the portions are called the interior and exterior.*

A(Application). *Given two triangles ABC and $A'B'C'$, it is possible to apply ΔABC to $\Delta A'B'C'$ so that the vertex A falls on the vertex A', the side AB falls on the side $A'B'$, and the vertex C falls in the same side of $A'B'$ as C'.*

The Common Notions listed below are also postulates, and in some editions they are indeed grouped together with the previous five postulates. However, these Common Notions do share a common thread that sets them apart. They are all concerned with *equality* and *inequality*. While Euclid did not define these notions explicitly, his sense of these terms is clarified by the way he used them. For example, Proposition 3.2.3 states that *parallelograms which are on the same base and in [between] the same parallels are equal to one another*. The conclusion that Euclid was referring to *equality of size* rather than congruence is therefore inescapable. In other words, when Euclid said that two parallelograms were equal he meant that they had the same areas, when he said that two straight lines were equal he meant that they had the same lengths, and when he said that two angles were equal he meant something of the same nature. In order for this interpretation to work it is necessary to stipulate that Euclid had an underlying, albeit unstated, assumption that all geometric objects have an aspect of *numerical size* or *magnitude*. In lieu of defining these terms, their properties were set forth in the Common Notions, so named because they describe the properties that are shared by (or common to) all mensurations, regardless of whether they relate to length, area, volume, weight, or angular size. This explanation is supported by Euclid's failure to provide any other definition of the notions of area and volume notwithstanding his many propositions about these very concepts. When viewed in this light the Common Notions strikingly resemble the standard modern axiomatic definition of a *Haar Measure*. Section 3.2 contains a more detailed discussion of area.

The symbol "=" is used to denote equality in magnitude, regardless of the specific aspect that is being measured. Thus,

$$\Delta ABC \quad = \quad \Delta DEF$$

means that the two said triangles have equal areas. Similarly,

$$AB \quad = \quad CD$$

means that the straight line segments joining A to B and C to D have equal lengths. Of course, in this case, they are also congruent, but that is accidental. The equality in length of line segments and the equality of angular measures of angles happens to imply their congruences. On the other hand, the equality of the areas of regions and volumes of solids does not entail the stronger relation of congruence.

COMMON NOTIONS

1. *Things which are equal to the same thing are also equal to one another.*

2. *If equals be added to equals, the wholes are equal.*

3. *If equals be subtracted from equals, the remainders are equal.*

From the modern point of view the third Common Notion is redundant, since it seems to be already subsumed by the previous one. The Greeks, however, did not recognize the existence of negative numbers, and so it was necessary for Euclid to include both Common Notions 2 and 3 in his list.

4. *Things which coincide with one another are equal to one another.*

In view of the proof of Proposition 2.3.4 (see below) this should be understood as saying that things which can be *made to coincide* with one another have equal sizes. This can be interpreted as Euclid's first mention of congruence in the sense of a rigid transformation. The contradictory attitudes Euclid displayed in the proofs of Propositions 2.3.2 and 2.3.4 make it clear that he had ambivalent feelings about the use of such transformations. The first of these propositions could have been proved by simply moving a given line segment to a given location. Instead, Euclid produced an elaborate and ingenious proof that, quite properly, made no use of such movements. The proof of Proposition 2.3.4, on the other hand, begins with an *application* of one triangle to another. In other words, one triangle is lifted and placed on top of the other. Since none of the definitions or postulates provide for such applications, this is a clear-cut relinquishment of standards on the part of Euclid. The fact that Euclid used this device sparingly implies that he was in all likelihood aware of its impropriety. The lack of a framework for the treatment of transformations and congruence constitutes one of the more serious flaws of *The Elements*.

5. *The whole is greater than the part.*

This Common Notion turns out to be very useful in a variety of proofs by contradiction, that of Proposition 2.3.6 being the first instance. In addition, it also insures that, whatever aspect it is that is being measured, some geometrical figure has a non-zero size. This is logically necessary since otherwise it could be possible to trivialize the notion of size by assigning to every figure the measure of zero. Note that this trivial zero measure does have the properties stipulated in the first four notions and only fails to satisfy the fifth one.

Euclid's postulates are now reexamined in the context of the surface of the sphere. Since great semicircles are not geodesics, diametrically opposite points on the sphere cannot be joined by a geodesic and so, strictly speaking, Euclid's Postulate 1 fails to hold on the sphere. However, the next best thing is true: *every two points on the sphere that are not diametrically opposite can be joined by a (unique) geodesic segment*. The uniqueness is guaranteed by the fact that the plane containing the geodesic segment must also contain the center of the sphere.

Euclid's Postulate 2 fails to hold on the sphere in a rather dramatic fashion. *No geodesic on the sphere of radius R can be extended to a length equal to or greater than* $2\pi R$. In fact, it could be argued that no spherical geodesic can be extended to a length of πR or more, since if that length were reached the geodesic would have to contain a pair of diametrically opposite points.

Euclid's Postulate 3 does hold on the sphere. Given any point C and any geodesic segment CD, there clearly is a circle of spherical radius CD on the sphere (see Fig. 2.6).

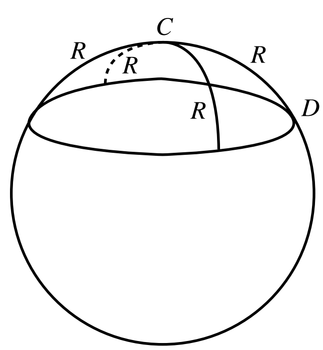

Figure 2.6 A spherical circle of spherical radius R.

Euclid's Postulate 4 also holds on the sphere. A right spherical angle is formed by any two geodesic arcs whose defining planes are at right angles to each other. For example, every meridian cuts the equator in a spherical right angle. It is also true that given any two spherical right angles, there is a rotation of the sphere that transforms one into the other.

Postulate 5 holds trivially, regardless of the angles between the lines, since any two geodesic segments, if sufficiently extended, will intersect (in a pair of diametrically opposite points). The reason for this is that both segments extend to great circles each of which divides the sphere into two equal halves. If these circles did not intersect these equalities could not hold.

With the proper reinterpretation Postulate S can be made to hold on the sphere. Infinitely extended straight lines have to be replaced with great circles, and the exterior sides are no longer infinite. With these modifications the figures in question all separate the surface of the sphere into two parts that can be seen as sides.

Postulate A also holds on the sphere. The requisite applications are rotations of the sphere about an axis that passes through its center and reflections in planes that contain the center.

EXERCISES 2.2A

1. Let A and B be two distinct points on the surface of a sphere. Describe a rigid motion of the sphere that moves A to B's location.
2. Let AB and CD be two geodesic segments on the surface of the sphere. Describe a rigid motion that moves AB to the position of CD. (Note: It is permissible to describe a rigid motion as a composition of several rigid motions.)
3. Let $\Delta\ ABC$ and $\Delta\ DEF$ be equal spherical angles. Describe a rigid motion of the sphere that moves the first angle onto the position of the second so that BA falls along ED and BC falls along EF.

Euclid's postulates are now reexamined in the context of hyperbolic geometry. Euclid's Postulate 1 holds here too. To verify this assertion two cases must to be considered. If the two points in question have coordinates (a, b_1) and (a, b_2) then clearly the vertical Euclidean line segment joining them is the only geodesic that joins them. If, on the other hand, they have coordinates (a_1, b_1) and (a_2, b_2) with $a_1 \neq a_2$, then there is one and only one semicircle that contains them both and has its center on the x-axis. This semicircle's center is the intersection of the x-axis with the perpendicular bisector to the line segment joining those two points.

Euclid's Postulate 2 holds for hyperbolic geometry because the x-axis is infinitely far for all of its inhabitants (see Section 1.2).

As was noted in the first discussion of Postulate 3 , the existence of the circle $(A; AB)$ follows from Postulate 2 which is already known to hold in the hyperbolic plane. Consequently all such circles exist in hyperbolic geometry as well. It follows that these circles are legitimate building blocks in this context too.

Since two geodesics form a hyperbolic right angle if and only if their tangents form a Euclidean right angle, the equality of hyperbolic right angles follows from the equality of Euclidean right angles. Thus, Postulate 4 holds in the hyperbolic plane.

Rather surprisingly, Postulate *A* does hold for the hyperbolic plane. This is difficult to explain at this point because not enough tools have been developed to describe the rigid motions of the hyperbolic plane. However, this statement is borne out by the subsequent Exercises 7.3.9-11.

Postulate *S* also holds in the hyperbolic plane too. This is intuitively plausible since, in fact, every closed curve C that does not intersect itself has a well defined inside and outside, and every curve that joins the inside to the outside must of necessity cross C at some point.

Since Euclid's Postulates 1, 2, 3, 4, *A, S* hold for the hyperbolic plane it follows that all the propositions that can be proved on the basis of these postulates alone also hold for the hyperbolic plane. The set of propositions that follow from these common postulates are collectively known as *neutral geometry*, or *absolute geometry*. This includes Propositions 1-28 of Book I of Euclid as well as a variety of others that are listed in the neutral sections of this text. Postulate 5, on the other hand, does not hold in the hyperbolic plane. In Figure 2.7 the geodesics p and q make interior angles α and β

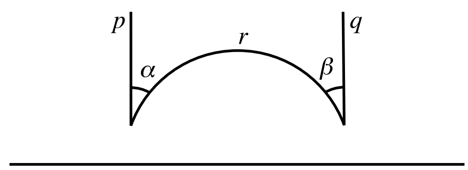

Figure 2.7 A hyperbolic counterexample to Euclid's Postulate 5.

with the geodesic r, where the sum of α and β is quite clearly less than 180°, and yet the geodesics p and q, no matter how far extended, do not meet. Postulate 5 is therefore not a part of neutral geometry.

EXERCISES 2.2B

1. Comment on the following postulates in the context of taxicab geometry:
   a) 1; b) 2; c) 3; d) 4; e) 5;
   f) S; g) A.
2. Comment on the following postulates in the context of maxi geometry:
   a) 1; b) 2; c) 3; d) 4; e) 5;
   f) S; g) A.
3. Draw a taxicab circle of radius 1.
4. Draw a maxi circle of radius 1.

3. Propositions 1 - 28

Having laid a foundation in the form of definitions, postulates, and common notions, Euclid proceeded to list several hundred logical conclusions each of which is called a *proposition*. There are two kinds of propositions. One kind asserts a *relationship* between some geometrical objects. Such, for example is Proposition 2.3.5 which states that the angles at the base of an isosceles triangle are equal to each other. The second kind of proposition asserts the *constructibility* of a certain geometrical object. Thus, Proposition 2.3.9 states that angles can be bisected, or, in other words, that angle bisectors can be constructed. Since Euclid's postulates only mention the existence of points, straight lines, and circles, it has become customary to say that the only tools that are allowed in Euclidean constructions are rulers and compasses. This too is to be understood only as a metaphor. Surely the idea of using a physical ruler to draw a straight line is just as objectionable as that of using the top of a can to draw a circle. It would be more accurate to say that Euclid constrained himself to a discussion of logical constructs definable by means of points, straight lines, and circles alone.

Ideally, the justification of both kinds of propositions must rely only on definitions, postulates, common notions, and previously justified propositions. In fact, both Euclid and the author allow for some deviation from this strict standard.

Euclid's conventions also allowed him to only use and/or make relational propositions about geometrical figures whose constructibility had been demonstrated in a previous constructive proposition. Thus, Proposition 2.3.20, whose proof requires the midpoint of the side of a triangle must be preceded by Proposition 2.3.10 which asserts the constructibility of the midpoint of a straight line segment.

The format that Euclid used for stating and validating his propositions is still in use today. The verbal description of the proposition is followed by a symbolic description of the context (GIVEN) and the relational or constructive asseveration (TO PROVE or TO CONSTRUCT). In the case of a relational proposition this is followed by a proof (PROOF) that validates the proposition. In the justifications of the steps of a proof the terms *definition, common notion, postulate,* and *proposition* will be respectively abbreviated as DFN, CN, PT, and PN.

Euclid invariably signed off on his proofs with the phrase "what it was required to prove" which is rendered as "quod erat demonstrandum" in Latin. The initials of this phrase, Q.E.D. served as the traditional end of proof symbol for hundreds of years until it became fashionable in the last thirty or so years to replace it with some variant of a small rectangle. In this text the traditional "Q.E.D." denotes the end of a proof. In the case of a constructive proposition, the above mentioned asseveration is followed by a description of the construction (CONSTRUCTION). This, in turn, should be followed by a proof that the constructed figure does indeed possess the required properties. However, the proof of the validity of a construction is often so straightforward that it can, and should be, omitted.

As part of his exposition Euclid incorporated many propositions that constitute geometrical analogs of algebraic rules. Because of the lack of algebraic symbols in his times, this resulted in many obvious propositions with tedious proofs. In order to avoid these inconveniences basic algebraic manipulations will be permitted in this text’s proofs.

For the most part, the statements of the propositions and proofs in this text are essentially the same as those that appear in Sir Thomas L. Heath's century old translation of *The Elements*. For the sake of clarity the proofs are presented with modern notation and some of the statements of the propositions are paraphrased. Some of Euclid's more cumbersome or erroneous proofs are replaced by improved ones. All such instances are explicitly noted.

PROPOSITIONS

PROPOSITION 2.3.1(I.1). *On a given finite straight line to construct an equilateral triangle*.

GIVEN: Line segment AB (Fig 2.8)

TO CONSTRUCT: ΔABC such that $AB = BC = CA$.

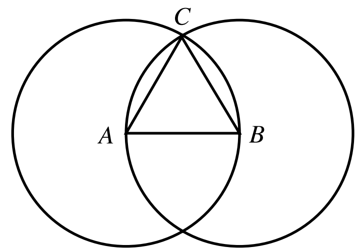

Figure 2.8

CONSTRUCTION: Draw the circles $(A; AB)$ and $(B; AB)$ [PT 3]. Let C be one of their intersections [PT S]. Then ΔABC is the required triangle.

PROOF:	$AB = AC$	[DFN 15]
	$BA = BC$	[DFN 15]
$\therefore$	$AC = BC$	[CN 1]

Q.E.D.

This proposition demonstrates both some of Euclid's strengths and some of his weaknesses. On the positive side he was a careful thinker and expositor who was unwilling to accept as obvious the existence of a triangle which most people take for granted. Unfortunately, he was not careful enough. Specifically, he implicitly accepted that the two auxiliary circles drawn in this proof necessarily intersected (there is no version of Postulate S in *The Elements)*. Now, as physical objects, these figures must clearly intersect, but as abstract entities, whose properties must be reducible to Euclid's definitions, postulates, and common notions, this claim calls for verification. This is not a minor point. The fact is that Euclid failed to provide a framework within which the interiors and exteriors of configurations can be discussed and this is one of the major defects of his logical edifice. The need for such a framework is underscored by the paradoxical nature of Exercise 3.3C.10. In this text, of course, it is Postulate S that provides the rationale for the existence of the intersection point C in the above proof.

There was no compelling reason for Euclid to choose the construction of equilateral triangles as his first proposition. Other texts employ different starting points. It is noteworthy that the last few propositions of the last of the thirteen books that comprise Euclid's *The Elements* deal with the construction of the five regular solids (see Section 8.1) that are the three dimensional analogs of the equilateral triangle in particular and of the regular polygons in general. In fact, the faces of three of these five solids are themselves equilateral triangles. Thus, Euclid may have chosen his starting point and ending point on esthetic grounds. They gave his work an artistic form.

It is convenient to denote the intersection of two lines p and q by $p \cap q$. In the case where p or q is a circle, this will be used to denote only one of the intersection points.

PROPOSITION 2.3.2(I.2). *To place at a given point (as an extremity) a straight line equal to a given straight line.*

GIVEN: Point A, line segment BC (Fig 2.9).

TO CONSTRUCT: A point D such that $AD = BC$.

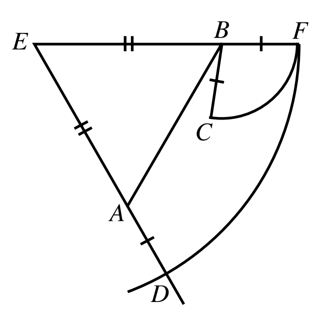

Figure 2.9

CONSTRUCTION: Let ΔABE be equilateral [PN 2.3.1], let $F = EB \cap (B; BC)$

[PT 3], and let $D = EA \cap (E; EF)$ [PT 3]. Then D is the required point.

PROOF:	ED	$=$	EF	[Radii of the same circle]
	EA	$=$	EB	[Sides of an equilateral triangle]
$\therefore$	AD	$=$	BF	[CN 3]
but	BF	$=$	BC	[Radii of the same circle]
$\therefore$	AD	$=$	BC	[CN 1]

Q.E.D.

Informally speaking, this proposition asserts that any construction that can be carried out with a non-collapsible compass, i.e., a compass that does not lose its angle when lifted off the plane, can also be accomplished with a collapsible one, albeit by a process that requires several more steps. More succinctly, *the constructive power of the collapsible compass equals that of the rigid compass*.

This proposition and its relation to Postulate 3 clarify one of the principles that underlie Euclid's strategy in choosing his postulates. Postulates should have as little content as possible. After all, Euclid could have chosen Proposition 2.3.2 as Postulate 3 with no resulting loss of any subsequent propositions or complications in any subsequent proofs. That he chose not to do this indicates that he enjoyed flexing his mental muscles just for the joy of using them.

PROPOSITION 2.3.3(I.3). *Given two unequal straight lines, to cut off from the greater a straight line segment equal to the less*.

GIVEN: Line segments $AB > CD$ (Fig 2.10).

TO CONSTRUCT: A point E on AB such that $AE = CD$.

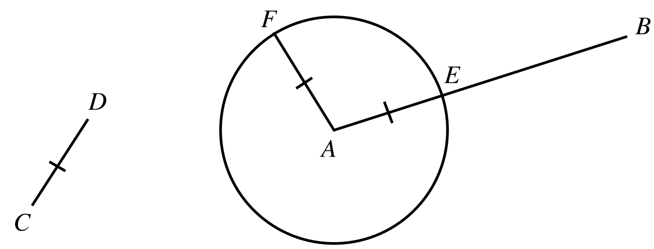

Figure 2.10

CONSTRUCTION: Let F be a point such that $AF = CD$ [PN 2.3.2] and let $E = AB \cap (A; AF)$ with AB. Then E is the required point.

PROOF: By construction, $AE = AF = CD$.

Q.E.D.

EXERCISES 2.3A

1. Construct a line segment whose length is double that of a given line segment.
2. At a given point P, construct a line segment whose length is double that of a given line segment AB.
3. Construct a line segment whose length is the sum of the lengths of two given line segments.
4. At a given point P construct a line segment whose length equals the sum of the lengths of two given line segments AB and CD.
5. At a given point P construct a line segment whose length equals the difference of the lengths of two given unequal line segments AB and CD.
6. At a given point P, construct a line segment whose length is triple that of a given line segment AB.
7. Comment on Proposition 2.3.1 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.
8. Comment on Proposition 2.3.2 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.
9. Comment on Proposition 2.3.3 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

10(C). Perform the construction of Proposition 2.3.1 using a computer application.

11(C). Perform the construction of Proposition 2.3.2 using a computer application.

12(C). Perform the construction of Proposition 2.3.3 using a computer application.

The next proposition is the well known SAS (side-angle-side) congruence theorem.

PROPOSITION 2.3.4(I.4). *If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which equal sides subtend.*

GIVEN: $\Delta ABC, \Delta DEF,\ AB = DE,\ AC = DF,\ \angle BAC = \angle EDF$ (Fig. 2.11).

TO PROVE: $BC = EF,\ \Delta ABC = \Delta DEF,\ \angle ABC = \angle DEF,\ \angle ACB = \angle DFE$.

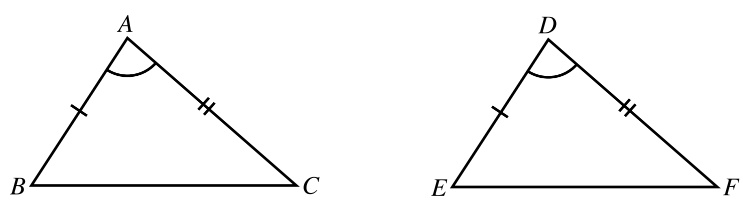

Figure 2.11

PROOF: Apply ΔABC to ΔDEF so that A falls on D and AB falls along DE [PT A]. Then

	B falls on E	[$AB = DE$, given]
	AC falls along DF	[$\angle BAC = \angle EDF$, given]
	C falls on F	[$AC = DF$, given]
	BC falls on EF	[PT 1]
$\therefore$	$BC = EF,\ \Delta ABC = \Delta DEF,$	
	$\angle ABC = \angle DEF,\ \angle ACB = \angle DFE$	[CN 4]

Q.E.D.

This is the first of several instances wherein Euclid uses the undefined notion of application. Postulate A can be used to justify this step.

The triangles ABC and DEF of Proposition 2.3.4 are such that

$$AB = DE, \quad BC = EF, \quad AC = DF,$$
$$\angle BCA = \angle EFD, \angle CAB = \angle FDE, \angle ABC = \angle DEF$$

i.e., their respective sides and their respective angles are equal. Such triangles are today called *congruent* since it is intuitively clear that given any two physical congruent triangles, one of them can be picked up and placed on top of the other so that the corresponding sides and angles will coincide. It is important to note, though, that this term does not appear in *The Elements*. The congruence of these triangles is denoted today as $\Delta ABC \cong \Delta DEF$. It is both customary and helpful to list the vertices of these triangles in an order that is consistent with the equality of their parts. In other words, if $\Delta PQR \cong \Delta XYZ$ then it is implicitly understood that $\angle QRP = \angle YZX$, $PR = XZ$, and so on.

EXERCISES 2.3B

1. Use the notion of an application to create a simpler "construction" for Proposition 2.
2. Use the notion of an application to provide a "proof" of Postulate 4 .
3. Use the notion of an application to prove the ASA congruence theorem [PN 2.3.29], i.e. that if two triangles have two angles of one equal respectively to two angles of the other, and if the sides joining these angles are also equal, then the triangles are congruent.
4. Comment on Proposition 2.3.4 in the context of the following geometries:

   a) spherical; b) hyperbolic.

5. Let $A = (0, 0)$, $B = (1, 1)$, $C = (-1, 1)$, $A' = (2, 0)$, $B' = (4, 0)$, $C' = (2, -2)$. Show that ΔABC and $\Delta A'B'C'$ satisfy the hypothesis of Proposition 2.3.4 in taxicab geometry but are not congruent in it.
6. Let $\Delta\ ABC$ be given. Show that there exists a $\Delta\ DEF$ such that $\Delta\ ABC \ncong \Delta\ DEF$ in the Euclidean sense but not in the taxicab sense.
7. Comment on Proposition 2.3.4 in the context of maxi geometry.

PROPOSITION 2.3.5(I.5). *In isosceles triangles the angles at the base are equal to one another; and, if the equal straight lines be produced further, the angles under the base will be equal to one another.*

GIVEN: $\Delta\ ABC$, $AB = AC$ (Fig 2.12).

TO PROVE: $\angle 1 = \angle 2$, $\angle 3 = \angle 4$.

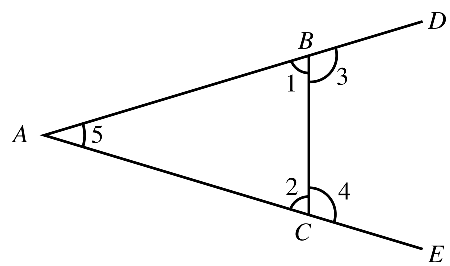

Figure 2.12

PROOF: $\Delta\ ABC \cong \Delta\ ACB$ by SAS because

	$AB = AC$	[Given]
	$AC = AB$	[Given]
	$\angle 5 = \angle 5$	
$\therefore$	$\angle 1 = \angle 2$	
but	$\angle ABD = \angle ACE$	[Both are straight angles]
$\therefore$	$\angle 3 = \angle 4$	[CN 3]

Q.E.D.

Euclid's proof of Proposition 2.3.5 is surprisingly long and intricate. This earned it the nickname *Pons Asinorum* or *Ass's Bridge* in the days when every schoolboy studied Euclid. The 19th century logician and philosopher Charles S. Pierce commented that it "... made so many boys conclude they have no capacity for geometry because this proof, the first one of any difficulty in Euclid, leaves the proposition to their minds less evident than they found it." This is the attitude that is evident in the following modern day limerick:

In Greek mathematical forum
Your Euclid was present to bore 'em.
He spent all his time
Drawing circles sublime
And crossing the Pons Asinorum.

The proof given here is due to Pappus (ca 300), also an Alexandrine, who wrote a now lost commentary on Euclid's *Elements*.

EXERCISES 2.3C

1. Prove that the three angles of an equilateral triangle are all equal to each other. What adjustments would your proof require if you accepted Euclid's definition of an isosceles triangle?
2. In ΔABC, K and L are points on the equal sides AB and AC respectively, such that $AK = AL$. Prove that $CK = BL$.
3. In ΔABC, $AB = AC$ and D and E are points on the side BC such that $BD = CE$. Prove that $AD = AE$.
4. Suppose $AC = AD$ and $BC = BD$ where C and D are points in the opposite sides of the straight line AB. Prove that $\Delta ABC \cong \Delta ABD$. (Hint: Draw CD and consider three cases.)
5. Comment on Proposition 2.3.5 in the context of the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

PROPOSITION 2.3.6(I.6)**.** *If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.*

GIVEN: $\Delta\, ABC$, $\angle 1 = \angle 2$ (Fig 2.13).

TO PROVE: $AC = AB$.

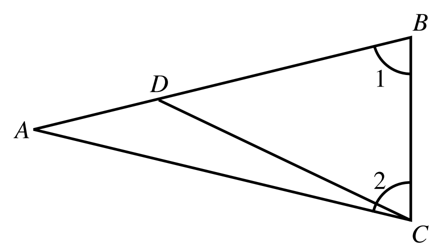

Figure 2.13

PROOF: By contradiction. Suppose AB and AC are unequal and assume without loss of generality that $AB > AC$. Let D be that point in the interior of side AB such that $BD = AC$. Then $\Delta\, BCD \cong \Delta\, CBA$ by SAS because

$$DB = AC \qquad \text{[Construction]}$$

$$BC = CB$$

$$\angle 1 = \angle 2 \qquad \text{[Given]}$$

$$\therefore \quad \Delta\, BCD = \Delta\, CBA$$

This, however, contradicts Common Notion 5 since $\Delta\, DCB$ is clearly only a part of $\Delta\, ACB$. Hence, the original supposition was false and so $AB = AC$.

Q.E.D.

The *converse* of a statement of the form *if* p *then* q is the statement *if* q *then* p. Thus, the converse of the statement *if* $a = b$ *then* $a^2 = b^2$ is the statement *if* $a^2 = b^2$ *then* $a = b$. Similarly, each of Propositions 2.3.5 and 2.3.6 is the other's converse. As these examples attest, there is no apparent necessary relation between the logical validity of a statement and that of its converse. In the first instance the original statement *if* $a = b$ *then* $a^2 = b^2$ is valid whereas its converse *if* $a^2 = b^2$ *then* $a = b$ is demonstrably false. Similarly, the false statement *if* $a^2 = b^2$ *then* $a = b$ has the valid converse *if* $a = b$ *then* $a^2 = b^2$. Finally, the converse Propositions 2.3.5 and 2.3.6 are both valid.

EXERCISES 2.3D

1. Prove that if all three angles of a triangle are equal to each other then the triangle is equilateral.
2. Comment on Proposition 2.3.6 in the context of the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.
3. Give an example of a proposition and its converse that are both false.

PROPOSITION 2.3.7(I.7). *Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities) and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely, each to that which has the same extremity.*

GIVEN: Line segment AB, points C, D on the same side of AB,

$CA = DA$, $CB = DB$ (Fig 2.14).

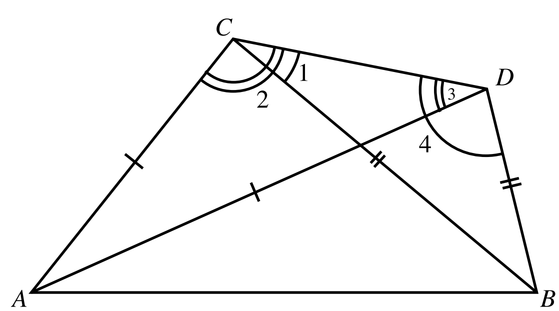

Figure 2.14

TO PROVE: The points C and D coincide.

PROOF: By contradiction. Suppose C and D are distinct and draw the line segment CD. Since ΔACD is isosceles, it follows from Proposition 2.3.5 that $\angle 2 = \angle 3$. Consequently,

$$\angle 1 < \angle 2 = \angle 3 < \angle 4 .$$

This, however, contradicts the fact that $\angle 1 = \angle 4$ [because ΔBDC is also isosceles].

Hence the points C and D cannot be distinct.

Q.E.D.

EXERCISES 2.3E

1. The proof of Proposition 2.3.7 is incomplete because it depends on the position of D relative to ΔABC. Complete this proof by considering three additional cases: one where D is inside ΔABC, one where it is in the interior of one of the sides, and one where it coincides with one of the vertices.
2. Comment on Proposition 2.3.7 in the context of the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

PROPOSITION 2.3.8(I.8). *If two triangles have their respective sides equal then they are congruent.*

GIVEN: ΔABC, ΔDEF, $AB = DE$, $BC = EF$, $AC = DF$ (Fig 2.15).

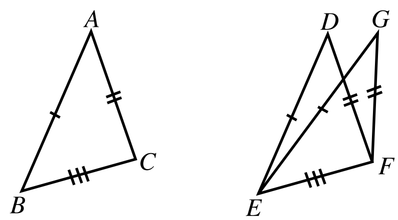

Figure 2.15

TO PROVE: $\Delta\, ABC \cong \Delta\, DEF$.

PROOF: Apply $\Delta\, ABC$ to $\Delta\, DEF$ so that B, C fall on E, F respectively and so that the vertex A falls in that side of EF that contains the point D [PT A]. Then

$$GE = AB = DE \quad \text{and} \quad GF = AC = DF$$

and hence, by Proposition 2.3.7, it follows that the points G and D coincide. Consequently, $\angle\, ABC = \angle\, GEF = \angle\, DEF$ and so, $\Delta\, ABC \cong \Delta\, DEF$ [SAS].

Q.E.D.

Proposition 2.3.8 is called the SSS congruence theorem.

EXERCISES 2.3F

1. Use Exercise 2.3C.4 to provide an alternate proof of Proposition 2.3.8 that does not rely on Proposition 2.3.7.
2. Comment on Proposition 2.3.8 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

PROPOSITION 2.3.9(I.9). *To bisect a given rectilineal angle.*

GIVEN: $\angle\, BAC$ (Fig 2.16).

TO CONSTRUCT: A line AF such that $\angle\, BAF = \angle\, FAC$.

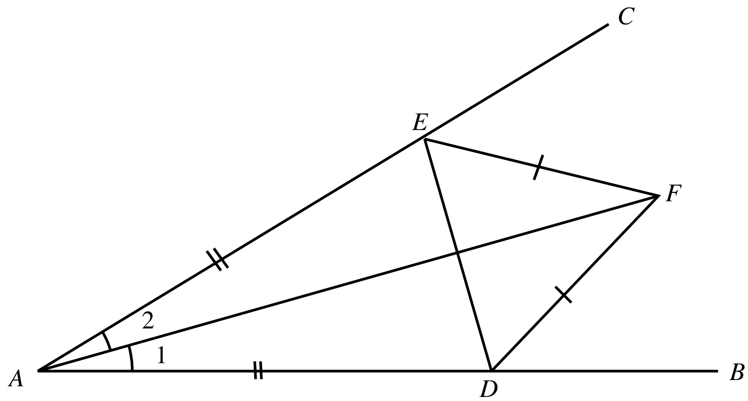

Figure 2.16

CONSTRUCTION: Let D be any point on AB and let E be a point on AC such that $AE = AD$ [PN 2.3.3]. Let DEF be an equilateral triangle [PN 2.3.1]. Then $\angle 1 = \angle 2$.

PROOF: $\Delta DAF \cong \Delta EAF$ by SSS because

$AD = AE$		[Construction]
$DF = EF$		[Construction]
$AF = AF$		

$\therefore \quad \angle 1 = \angle 2.$

Q.E.D.

The line AF of this proposition is called the *angle bisector* of $\angle BAC$. In a ΔABC it is common to refer to the portion of the angle bisector of $\angle BAC$ that lies between the vertex A and the side BC also as one of the triangle's angle bisectors. The method used by Euclid to construct angle bisectors is different from the one commonly taught in high schools wherein the equilateral ΔDEF is replaced by two arcs of equal radii that are centered at D and E respectively and intersect at F. This latter construction is properly speaking not Euclidean since it assumes the compass to be rigid. Of course, this method could be justified by Proposition 2.3.2, but Euclid chose to work with Proposition 2.3.1 instead.

Proposition 2.3.9 begs the question of how to divide an arbitrary angle into any number of equal parts. There is of course no difficulty in dividing an arbitrary angle into 4, 8, or 2^n equal parts, where n is any positive integer. On the other hand, dividing an angle into three equal parts, otherwise known as the *angle trisection* problem turns out to be impossible. This problem was first formulated by the Greeks and continued to draw the interest of both professional and amateur mathematicians for over 2000 years. In 1837 this impossibility of this construction was finally demonstrated by Pierre Laurent Wantzel (1814-1848). Unfortunately, the proof is not easy and lies beyond the grasp of many amateurs who still continue to search for a ruler and compass angle trisection. Sometimes they "succeed" and produce a construction that either misuses rulers and/or compasses or else simply produces an approximation. An example of such a popular misconstruction appears in Exercise 3.1C.32

EXERCISES 2.3G

1. Find an alternate method of bisecting a given angle using Euclid's "collapsible" compass.
2. Prove that the angle bisectors of an isosceles triangle divide the equal sides into respectively equal line segments.
3. Prove that the bisector of the angle opposite to the base of an isosceles triangle also bisects the base and is perpendicular to it.
4. Comment on Proposition 2.3.9 in the context of the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.
5. Divide a given angle into four equal parts.
6. Given positive integers m and n and an angle of measure α, prove that it is possible to construct an angle of measure $\frac{m\alpha}{2^n}$.

7(C). Perform the construction of Proposition 2.3.9 using a computer application.

PROPOSITION 2.3.10(I.10). *To bisect a given finite straight line.*

GIVEN: Line segment AB (Fig 2.17).

TO CONSTRUCT: A point D on AB such that $AD = DB$.

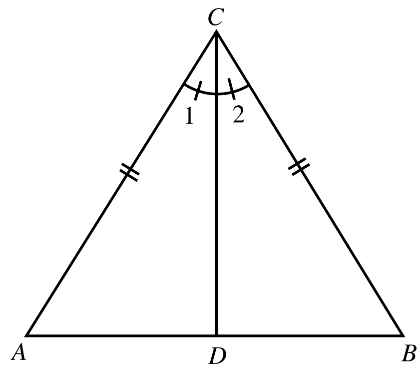

Figure 2.17

CONSTRUCTION: Construct an equilateral ΔABC [PN 2.3.1] and let D be the intersection of the bisector of $\angle ACB$ with AB. Then $AD = DB$.

PROOF: $\Delta DAC \cong \Delta DBC$ by SAS because

$AC = BC$ [Construction]

$\angle 1 = \angle 2$ [Construction]

$CD = CD$

$\therefore \quad AD = DB$

Q.E.D.

That point M of the line segment AB such that $AM = MB$ is called the *midpoint* of AB. The line segment joining a vertex of a triangle to the midpoint of the opposite side is called a *median*. The median joining vertex A of ΔABC to the midpoint of the side BC is denoted by either m_a or m_{BC}.

EXERCISES 2.3H

1. Prove that the triangle formed by joining the midpoints of the three sides of an isosceles triangle is also isosceles.
2. Prove that the triangle formed by joining the midpoints of the three sides of an equilateral triangle is also equilateral.
3. Prove that the medians to the equal sides of an isosceles triangle are equal to each other.
4. Prove that the medians to the equal sides of an isosceles triangle divide each other into respectively equal segments.
5. Prove that the median to the base of an isosceles triangle is perpendicular to the base and bisects the opposite angle.
6. Divide a given line segment into four equal parts.
7. Comment on Proposition 2.3.10 in the context of the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

8(C). Perform the construction of Proposition 2.3.10 using a computer application.

PROPOSITION 2.3.11(I.11). *To draw a straight line at right angles to a given straight line.*

GIVEN: Point C on straight line AB (Fig 2.18).

TO CONSTRUCT: Straight line $CF \perp AB$.

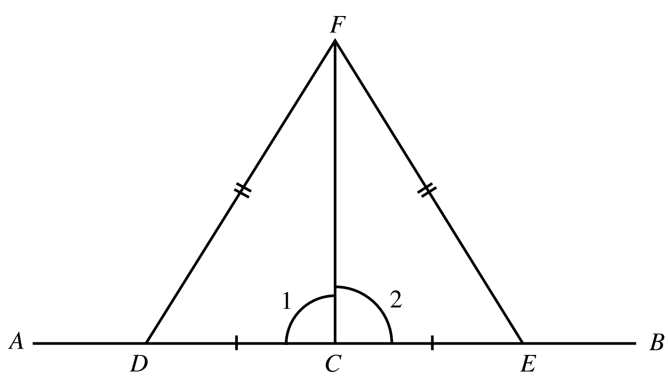

Figure 2.18

CONSTRUCTION: Let D be any point on AC and let E be point on BC such that $CD = CE$ [PN 2.3.3]. Construct equilateral ΔDEF. Then $CF \perp AB$.

PROOF: $\Delta DCF \cong \Delta ECF$ by SSS because

$DF = EF$ [Construction]

$DC = EC$ [Construction]

$FC = FC$

$\therefore \quad \angle 1 = \angle 2$

$\therefore \quad CF \perp AB$ [DFN 10]

Q.E.D.

Given a line segment AB, the straight line through its midpoint that is also perpendicular to AB is called its *perpendicular bisector*. The next two propositions about perpendicular bisectors do not appear in Euclid's *Elements* but are nevertheless quite useful.

PROPOSITION 2.3.12. *Every point on the perpendicular bisector of a line segment is equidistant from the segment's endpoints.*
See Exercise 1.

PROPOSITION 2.3.13. *Every point that is equidistant from the endpoints of a line segment is on its perpendicular bisector.*
See Exercise 2.

The geometrical word for the notion of *set* or *collection* is *locus*. For example, given a point C and a real number r, the locus of all the points at distance r from C is the circle $(C; r)$. This term is frequently used to describe collections of points that have geometrically interesting properties. Thus, it is customary to combine Propositions 2.3.12-13 into the following one.

PROPOSITION 2.3.14. *The locus of all the points that are equidistant from two distinct points is the perpendicular bisector to the line segment determined by these points.*

As this proposition does not say anything that isn't already contained in either Proposition 2.3.12 or Proposition 2.3.13, no proof is required.

PROPOSITION 2.3.15(I.12)**.** *To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.*

GIVEN: Straight line $\overleftrightarrow{AB}$; point C not on $\overleftrightarrow{AB}$ (Fig 2.19).

TO CONSTRUCT: A straight line CH such that $CH \perp \overleftrightarrow{AB}$.

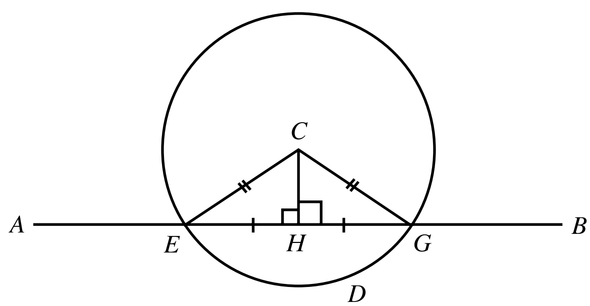

Figure 2.19

CONSTRUCTION: Let D be any point on the side of AB that does not contain C. Let the circle ($C; CD$) [PT 3] intersect the straight line AB in the points E and G [PT S]. Let H be the midpoint of EG [PN 2.3.10]. Then $CH \perp AB$.

PROOF: Draw CG and CE. Then $\Delta\, CGH \cong \Delta\, CEH$ by SSS because

CG	$=$	CE	[Radii of the same circle]
GH	$=$	EH	[Construction]
CH	$=$	CH	

$$\therefore \quad \angle GHC = \angle EHC$$

$\therefore \quad CH \perp AB$ [DFN 10]

Q.E.D.

The straight line segment that joins a vertex of a triangle to a point on the opposite side and is perpendicular to that side is called an *altitude* of the triangle. The altitudes that contain the vertices A, B, C are denoted by h_a, h_b, h_c respectively, or else h_{BC}, h_{AC}, h_{AB}.

EXAMPLE 2.3.16. *Construct* ΔABC *given the data* β, a, m_c. By this is meant that a ΔABC is to be constructed in which the magnitudes of $\angle ABC$, the side BC, and the median from C to AB are prespecified. Thus, following the standard format for construction problems:

GIVEN: Angle β, line segments a, m_c (Fig 2.20).

TO CONSTRUCT: ΔABC such that $\angle ABC = \beta$, $BC = a$, $m_{BC} = m_a$.

CONSTRUCTION: Label the vertex of β with B, and its two sides with BP and BQ.

Set $C = BP \cap (B; a)$ [PN 2.3.3]

Set $D = BQ \cap (C; m_c)$ [PN 2.3.3]

Set $A = BQ \cap (D; DB)$ [PT 2.3.3]

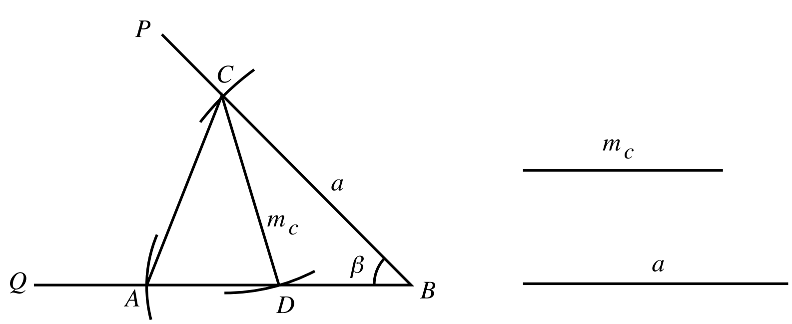

Figure 2.20

PROOF: Self-evident.

Q.E.D.

COMMENTS: If m_c is too short, there will be no solution. If m_c is long enough, there may be two solutions, depending on whether the given angle β is acute, right, or obtuse.

EXERCISES 2.3I

1. Prove Proposition 2.3.12.
2. Prove Proposition 2.3.13.
3. Prove that if one of the altitudes of a triangle is also a median, then the triangle is isosceles.
4. Construct an isosceles triangle in which a and h_a are given (b and c being the equal sides).
5. The point D lies either inside or outside $\angle BAC$. Construct a straight line that contains D and cuts off equal segments on the sides of $\angle BAC$.
6. Construct ΔABC given the data b, c, h_a.
7. Construct ΔABC given the data a, b, h_b.
8. Construct ΔABC given the data a, m_a, h_b.
9. Construct ΔABC given the data a, m_b, h_b.
10. Construct ΔABC given the data c, h_b, m_a.
11. Construct ΔABC given the data a, m_a, β.
12. Construct ΔABC given the data $a, b + c, \beta$.
13. Construct ΔABC given the data $a, b - c, \gamma$.
14. Construct ΔABC given the data $a, b - c, \beta$.
15. Comment on Proposition 2.3.11 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
16. Comment on Proposition 2.3.12 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
17. Comment on Proposition 2.3.13 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
18. Comment on Proposition 2.3.14 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
19. Comment on Proposition 2.3.15 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

20(C). Perform the construction of Proposition 2.3.11 using a computer application.

If A is a point not on the straight line $\overleftrightarrow{CD}$ and B is a point on it between C and D, then $\angle ABC$ and $\angle ABD$ are said to be *supplementary* (Fig 2.21).

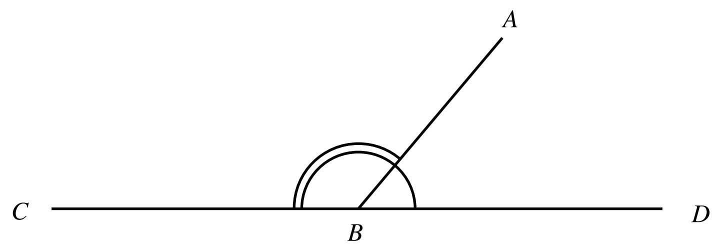

Figure 2.21

PROPOSITION 2.3.17(I.13)**.** *The sum of two supplementary angles is equal to two right angles.*

GIVEN: Straight lines AB and CD with B on CD.

TO PROVE: $\angle CBA + \angle ABD = 2$ right angles.

This proposition will not be proved in this text as the sum of supplementary angles quite obviously equals a straight angle. Euclid, however, is somewhat hobbled by his artificial exclusion of the straight angle from the realm of angles (see DFN 8). Consequently he finds it necessary to replace this obvious equality by the statement that the sum of supplementary angles equals that of two right angles. Moreover, he gives a proof that is, in the author's opinion, unnecessary. The same attitude is adopted regarding the next proposition which is the converse of the previous one.

PROPOSITION 2.3.18(I.14)**.** *If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.*

GIVEN: $\angle ABC$ and $\angle ABD$ with C and D lying in distinct sides of AB.

$\angle ABC + \angle ABD = 2$ right angles.

TO PROVE: BC and BD form a single straight line.

EXERCISES 2.3J

1. Prove that the bisectors of $\angle ABC$ and $\angle ABD$ of Proposition 2.3.17 are perpendicular to each other.
2. Comment on Proposition 2.3.17 in the context the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.
3. Comment on Proposition 2.3.18 in the context the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

PROPOSITION 2.3.19(I.15). *If two straight lines cut one another, they make the vertical angles equal to one another*.

GIVEN: Straight lines AB and CD intersecting at E (Fig 2.22).

TO PROVE: $\angle 1 = \angle 2$.

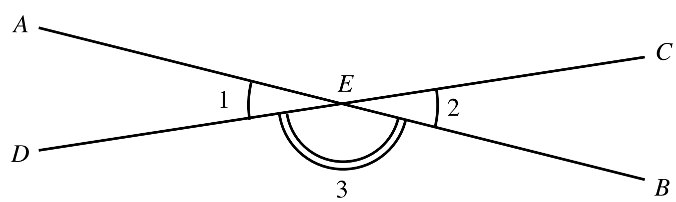

Figure 2.22

PROOF: $\angle 1 + \angle 3 = 2$ right angles [PN 2.3.17]

$\angle 2 + \angle 3 = 2$ right angles [PN 2.3.17]

$\therefore \quad \angle 1 = \angle 2$ [By subtraction of $\angle 3$ & CN 3]

Q.E.D.

The angles whose equality is asserted by Proposition 2.3.19 are said to be *vertically opposite*.

EXCERCISES 2.3K

1. Prove that if an angle's bisector is extended into the vertically opposite angle then it bisects the latter too.
2. Prove that the bisectors of vertically opposite angles lie in the same straight line.
3. Prove that if one of the triangle's medians is also an angle bisector then the triangle is isosceles. (Hint: extend the median by its own length)
4. Prove that if a quadrilateral's diagonals bisect each other then the quadrilateral's opposite sides are equal to each other and so are its opposite angles.
5. Comment on Proposition 2.3.19 in the context of the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

PROPOSITION 2.3.20(I.16)**.** *In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.*

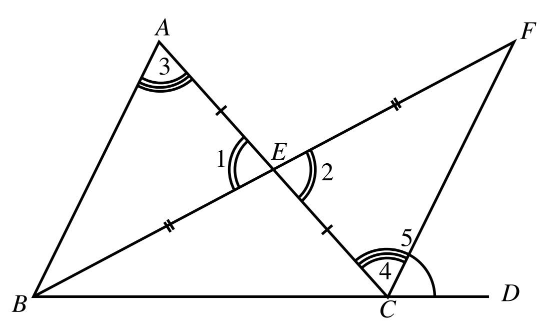

Figure 2.23

GIVEN: ΔABC, side BC produced to BD (Fig 2.23).

TO PROVE: $\angle 5 > \angle 3$.

PROOF: Let E be the midpoint of AC [PN 2.3.10] and extend BE to F so that $BE = EF$ [PT 2, PN 2.3.3]. Then $\Delta AEB \cong \Delta CEF$ by SAS because

	$AE = EC$	[Construction]
	$\angle 1 = \angle 2$	[PN 2.3.19]
	$BE = FE$	[Construction]
$\therefore$	$\angle 3 = \angle 4$	
But	$\angle 4 < \angle 5$	[CN 5]
$\therefore$	$\angle 3 < \angle 5$	

Q.E.D.

EXERCISES 2.3L

1. Let D be a point in the interior of ΔABC. Prove that $\angle ADC > \angle ABC$.
2. Points are said to be *collinear* if they lie on one straight line. Prove that three collinear points cannot all be equidistant from the same point.
3. Prove that through a given point P there is only one straight line perpendicular to a given straight line AB.
4. Comment on Proposition 2.3.20 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

PROPOSITION 2.3.21(I.17). *In any triangle two angles taken together in any manner are less than two right angles*.

GIVEN: ΔABC (Fig 2.24).

TO PROVE: $\angle 1 + \angle 2 < 2$ right angles.

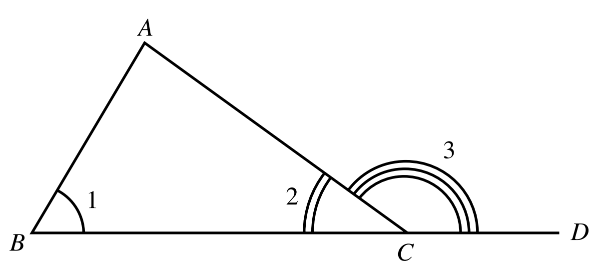

Figure 2.24

PROOF: Extend BC to D [PT 2]. Then

$\angle 3 + \angle 2 = 2$ right angles [PN 2.3.17]

$\angle 3 > \angle 1$ [PN 2.3.20]

$\therefore$ $\angle 1 + \angle 2 < 2$ right angles

Q.E.D.

EXERCISES 2.3M

1. One of the angles of ΔABC is obtuse. Prove that the other two are acute.
2. Prove that if one of a triangle's angles is right, then the other two are acute.
3. Prove that the angles at the base of an isosceles triangle are acute.
4. Prove that every triangle has at least one altitude that is interior to it.
5. One side of a triangle is extended in both directions. Prove that the sum of the two exterior angles so formed is greater than two right angles.
6. Comment on Proposition 2.3.21 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

PROPOSITION 2.3.22(I.18)**.** *In any triangle the greater side subtends the greater angle.*

GIVEN: ΔABC, $AC > AB$ (Fig 2.25).

TO PROVE: $\angle 1 > \angle 4$.

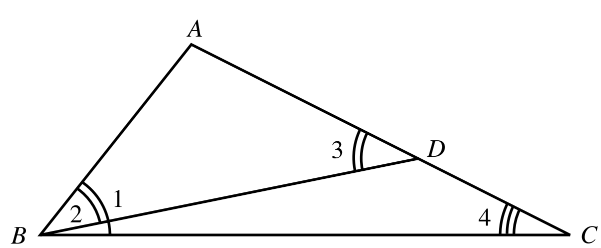

Figure 2.25

PROOF: Let D be that point inside AC such that $AD = AB$ [PN 2.3.3], and draw BD. Then

	$\angle 1$	$>$	$\angle 2$	[CN 5]
	$\angle 2$	$=$	$\angle 3$	[Δ ABD is isosceles, PN 2.3.5]
	$\angle 3$	$>$	$\angle 4$	[Exterior versus interior, PN 20]
$\therefore$	$\angle 1$	$>$	$\angle 4$	

Q.E.D.

PROPOSITION 2.3.23(I.19). *In any triangle the greater angle is subtended by the greater side.*

GIVEN: $\Delta\, ABC$, $\angle 1 > \angle 2$ (Fig 2.26).

TO PROVE: $AC > AB$.

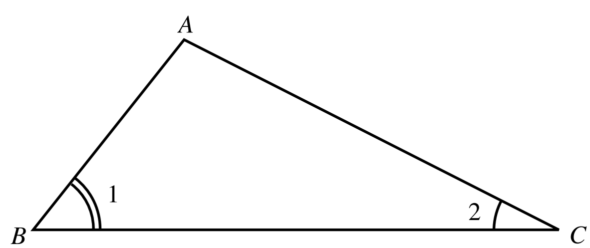

Figure 2.26

PROOF: By contradiction. Suppose that $AC \not> AB$. Then either $AC = AB$ or $AC < AB$. However,

if	$AC = AB$	then	$\angle 1 = \angle 2$	[PN 2.3.5]
if	$AC < AB$	then	$\angle 1 < \angle 2$	[PN 2.3.22]

Since both of the above conclusions contradict the given $\angle 1 > \angle 2$, it follows that $AC > AB$.

Q.E.D.

EXERCISES 2.3N

1. If the bisector of $\angle BAC$ of ΔABC intersects the side BC in the point D, then $AB > BD$ and $AC > CD$.
2. If in ΔABC and $\Delta A'B'C'$ $AB = A'B'$, $BC = B'C'$, $AB < BC$, and $\angle BAC = \angle B'A'C'$, then the two triangles are congruent. (This is sometimes dubbed SSA, although it is clearly not as powerful as the other congruence theorems.)
3. Prove that if the leg and hypotenuse of one right triangle are equal to the leg and hypotenuse of another, respectively, then the two triangles are congruent.
4. In ΔABC $AB < AC$ and N is the intersection of the bisectors of the angles at B and C. Prove that $NB < NC$.
5. Prove that the line segment joining any point in the interior of the base of an isosceles triangle to the opposite vertex is shorter than the two other sides.
6. Prove that in a right triangle the *hypotenuse* (the side opposite the right angle) is greater than either of the *legs* (the other two sides).
7. In ΔABC $AB < BC$ and E is the midpoint of AC. Prove that $\angle CBE < \angle ABE$. (Hint: Extend BE by its own length.)
8. In ΔABC $AB < BC$ and the bisector of $\angle ABC$ intersects the side AC in the point D. Prove that $AD < CD$.
9. One of the angle bisectors of a triangle is also a median. Prove that the triangle is isosceles (Hint: Use Exercises 2.3K.4.)
10. Comment on Proposition 2.3.22 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

11 Comment on Proposition 2.3.23 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

The following useful proposition does not appear explicitly in *The Elements*, but it is implicit in Definition 5 of Book III.

PROPOSITION 2.3.24. *Of all the line segments joining a point to a straight line, the shortest is the one that is perpendicular to the given straight line.*

GIVEN: Straight line $\overleftrightarrow{AB}$, point C not on $\overleftrightarrow{AB}$, points $P \neq Q$ on $\overleftrightarrow{AB}$, $CP \perp \overleftrightarrow{AB}$

(Fig 2.27).

TO PROVE: $CP < CQ$.

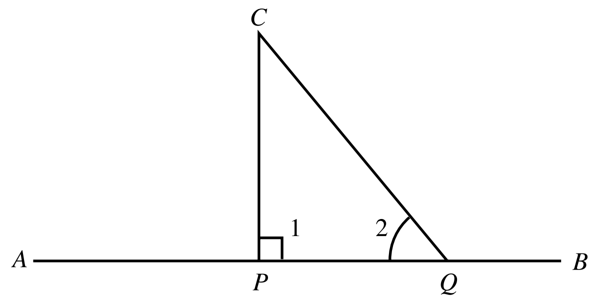

Figure 2.27

PROOF: $\angle 1 + \angle 2 < 2$ right angles [PN 2.3.21]

$\angle 1 =$ right angle [PC ' AB]

$\therefore$ $\angle 2 <$ right angle $= \angle 1$

$\therefore$ $CP < CQ$ [PN 2.3.23]

Q.E.D.

The *distance* of the line $\overleftrightarrow{AB}$ from a point C not on it is the shortest of the segments CP where P is any point on AB. It follows from Proposition 2.3.24 that this distance is realized by that point P such that CP 'AB.

PROPOSITION 2.3.25(I.20). *In any triangle two sides taken together in any manner are greater than the remaining one.*

GIVEN: ΔABC (Fig 2.28).

TO PROVE: $AB + AC > BC$.

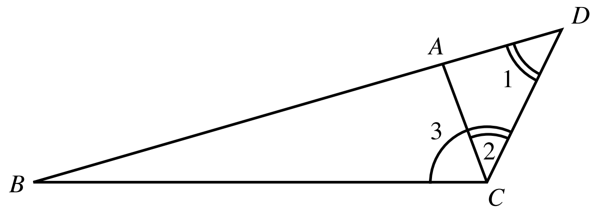

Figure 2.28

PROOF: Extend BA to a point D such that $AD = AC$ [PT 2, PN 2.3.3] and draw CD. Then,

	$\angle 3$	$>$	$\angle 2$	[CN 5]
	$\angle 2$	$=$	$\angle 1$	[Δ *ACD* is isosceles, PN 2.3.5]
$\therefore$	$\angle 3$	$>$	$\angle 1$	
$\therefore$	BD	$>$	BC	[PN 2.3.23 in Δ *BCD*]

$$\therefore \quad AB + AC = BA + AD = BD > BC$$

Q.E.D.

The *perimeter* of a triangle is the sum of the lengths of its sides.

EXERCISES 2.3O

1. Prove that the difference of the lengths of two sides of a triangle is less than the third side.
2. Prove that each of the triangle's sides is less than half of the triangle's perimeter.
3. Prove that half of the triangle's perimeter is greater than any line segment that joins a vertex to a point in the interior of the opposite side.
4. Prove that in a triangle one side's median is less than half the sum of the other two sides.
5. Prove that the sum of the lengths of the three line segments that join any point to the three vertices of a triangle is greater than half of that triangle's perimeter. (The point may be inside the triangle, outside it, or on one of its sides.)
6. Prove that the sum of the lengths of the four line segments that join a point to the vertices of a quadrilateral is greater than the sum of the lengths of the quadrilateral's diagonals, unless that point is the intersection of the diagonals.
7. Prove that the sum of the lengths of a quadrilateral's diagonals is less than its perimeter.

8. Prove that the sum of the lengths of the diagonals of a convex quadrilateral is greater than half its perimeter.
9. Prove that if AD is the bisector of the exterior $\angle CAE$ of ΔABC then $BD + DC > BA + AC$.
10*. Let A, B be two points on the same side of the straight line m, and let P be an arbitrary point on m. Prove that $AP + PB$ is least when m forms equal angles with AP and BP. (Hint: Use the previous exercise.)
11. Comment on Proposition 2.3.24 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
12. Comment on Proposition 2.3.25 in the context of the following geometries:
    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

PROPOSITION 2.3.26(I.21)**.** *If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.*

See Exercise 1.

PROPOSITION 2.3.27(I.22)**.** *Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one.*

GIVEN: Line segments a, b, c, $a + b > c$, $b + c > a$, $c + a > b$ (Fig 2.29).

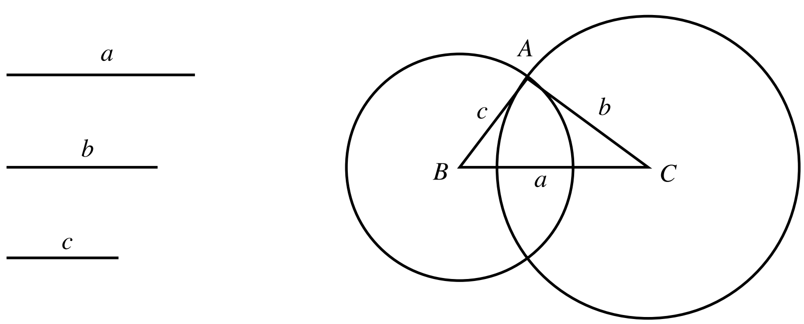

Figure 2.29

TO CONSTRUCT: ΔABC such that $BC = a,\ CA = b,\ AB = c$.

CONSTRUCTION: Because of the symmetry of the given inequalities, it may be assumed, without loss of generality that $a \geq b \geq c$. Let BC be any line segment of length a. If $A = (B; c) \cap (C; b)$ then ΔABC is the required triangle.

PROOF: It follows from the inequality $b + c > a = BC$ that the two drawn circles intersect in a point A not on BC. The choice of radii makes it clear that $AC = b$ and $AB = c$.

Q.E.D.

Euclid's proof of the above proposition is unsatisfactory because he does not prove that his circles intersect. This was fixed in this text by assuming that $a \geq b \geq c$.

PROPOSITION 2.3.28(I.23). *On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.*

GIVEN: Straight line AB and $\angle DCE$ (Fig 2.30).

TO CONSTRUCT: $\angle FAB = \angle DCE$.

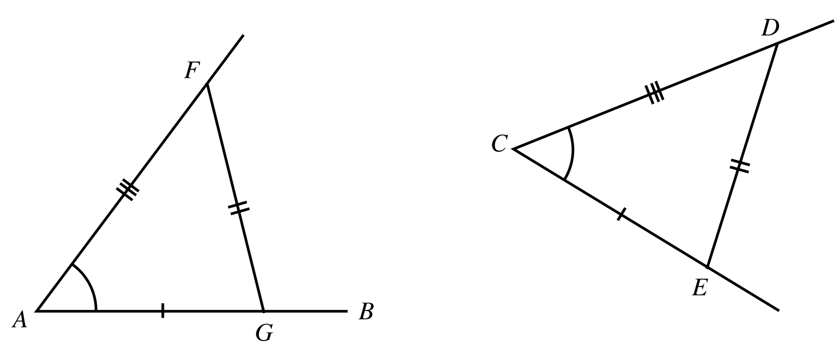

Figure 2.30

CONSTRUCTION: Draw DE. Construct ΔAFG so that G is on AB, $AG = CE$, $AF = CD$, and $FG = DE$ [PN 2.3.27]. Then $\angle FAG$ is the required angle.

PROOF: $\Delta AFG \cong \Delta CDE$ by SSS and so $\angle FAG = \angle DCE$.

Q.E.D.

It would have been much easier to accomplish this construction by applying the given angle to the given line. That Euclid chooses not to do so indicates that he was uncomfortable with this device.

EXERCISES 2.3P

1. Prove Proposition 2.3.26.
2. For which of the following triples of numbers does there exist a triangle whose sides have those lengths?

   a) $6, 7, 8$ b) $4, 6, 8$ c) $3, 5, 8$

   d) $10^3, 10^4, 10^5$
3. Find all numbers x such that there is a triangle the lengths of whose sides are $6, 8, x$.
4. Let a and b be two positive numbers. Find all numbers x such that there is a triangle the lengths of whose sides are a, b, x.
5. Construct an isosceles triangle each of whose equal sides is double the third side.
6. Construct an angle that equals the sum of two given angles.
7. Construct an angle that equals the difference of two given angles.
8. Construct ΔABC given the data a, B, C.
9. Construct ΔABC given the data A, B, h_c.
9. Construct ΔABC given the data A, B, h_a.
10. Construct ΔABC given the data b, h_a, A.
11. Comment on Proposition 2.3.26 in the context of the following geometries:

    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
12. Comment on Proposition 2.3.27 in the context of the following geometries:

    a) spherical; b) hyperbolic; c) taxicab; d) maxi.
13. Comment on Proposition 2.3.28 in the context of the following geometries:

    a) spherical; b) hyperbolic; c) taxicab; d) maxi.

14(C). Perform the construction of Proposition 2.3.28 using a computer application.

As Propositions I.24 and I.25 are not important for the development in this text, their statements and proofs have been relegated to Exercises 5 and 6 respectively.

Euclid's Proposition I.26 consists of two congruence theorems that are different enough to require separate proofs. These two are stated here separately. The following proposition is known as the ASA congruence theorem. Proposition 2.3.30, on the other hand, could be called the AAS congruence theorem. An alternative proof for ASA that makes use of Postulate A was mentioned in Exercise 2.3B.3.

PROPOSITION 2.3.29(I.26)**.** *If two triangles have the two angles equal to the two angles respectively, and the sides joining the angles are also equal, then they are congruent.*

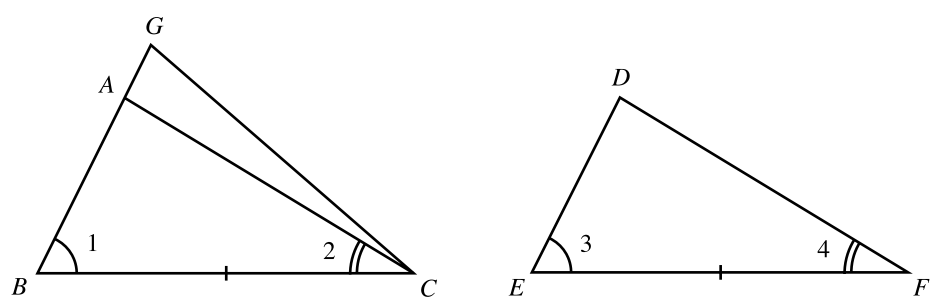

Figure 2.31

GIVEN: $\Delta\, ABC, \Delta\, DEF,\ BC = EF,\ \angle\, 1 = \angle\, 3,\ \angle\, 2 = \angle\, 4$ (Fig 2.31).

TO PROVE: $\Delta\, ABC \cong \Delta\, DEF$

PROOF: It is first proven, by contradiction, that $AB = DE$. Suppose not, then it may be assumed without loss of generality that $AB < DE$. Let G be a point in the extension of the segment AB such that $GB = DE$. Then,

$\Delta\, GBC \cong \Delta\, DEF$ by SAS because

$GB = DE$		[Construction]
$\angle\, 1 = \angle\, 3$		[Given]
$BC = EF$		[Given]

$\therefore \quad \angle GCB = \angle 4$

$\therefore \quad \angle GCB = \angle 2 \qquad [\angle 4 = \angle 2$, given]

This last conclusion, however, contradicts CN 5, and hence it may be concluded that $AB = DE$. That $\Delta ABC \cong \Delta DEF$ now follows by SAS.

Q.E.D.

EXERCISES 2.3Q

1. Prove that if one of the triangle's altitudes is also an angle bisector then the triangle is isosceles.
2. Prove that if a triangle has two equal angles then their bisectors are equal.
3. In quadrilateral $ABCD$ the diagonal AC bisects the interior angles at A and C. Prove that the interior angles at B and D are equal.
4. In ΔABC $AB = AC$ and D, E are points in the interiors of sides AB and AC respectively such that $AD = AE$. Prove that if CD and BE intersect in the point O then AO bisects $\angle BAC$.

5*. In ΔABC and ΔDEF $AB = DE$, $AC = DF$, and $\angle BAC > \angle EDF$. Prove that $BC > EF$. (This is Proposition I.24.)

6. In ΔABC and ΔDEF $AB = DE$, $AC = DF$, and $BC > EF$. Prove that $\angle BAC > \angle EDF$. (This is Proposition I.25.)
7. Comment on Proposition 2.3.29 in the context of the following geometries:

   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

PROPOSITION 2.3.30(I.26)**.** *If two triangles have two angles equal to two angles respectively, and the side opposite one of the angles equal to the side opposite the corresponding angle, then the two triangles are congruent.*

See Exercise 1.

The following two propositions about angle bisectors do not appear in *The Elements*. They are, however, standard fare in most elementary geometry texts.

PROPOSITION 2.3.31. *Every point on an angle's bisector is equidistant from that angle's sides*.

See Exercise 5.

PROPOSITION 2.3.32. *Every point inside an angle that is equidistant from its sides lies on its bisector*.

See Exercise 6.

The next proposition merely combines the previous two. It requires no new proof.

PROPOSITION 2.3.33. *The locus of all the points that are equidistant from two intersecting lines are the two straight lines that bisect the four angles formed by them*.

EXERCISES 2.3R

1. Prove Proposition 2.3.30. (Hint: Assume the triangle sides joining the given angles are not equal and obtain a contradiction using PN 2.3.20.)
2. Prove that the altitudes to the equal sides of an isosceles triangle are also equal.
3. Prove that the altitudes to the equal sides of an isosceles triangle divide each other into segments that are respectively equal.
4. Prove that the altitude to the base of an isosceles triangle bisects both the base and the angle opposite to it.
5. Prove Proposition 2.3.31.
6. Prove Proposition 2.3.32.
7. Comment on Proposition 2.3.30 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.
8. Comment on Proposition 2.3.31 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

9. Comment on Proposition 2.3.32 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.
10. Comment on Proposition 2.3.33 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

If a straight line p intersects two other straight lines m and n, then it is convenient to attach the following nomenclature to the listed angles and pairs of angles (Fig 2.32).

Alternate angles:	$\{\varepsilon, \gamma\}$, $\{\delta, \theta\}$, $\{\alpha, \psi\}$, $\{\beta, \eta\}$
Corresponding angles:	$\{\alpha, \varepsilon\}$, $\{\beta, \theta\}$, $\{\delta, \eta\}$, $\{\gamma, \psi\}$
Interior angles:	$\gamma, \delta, \varepsilon, \theta$
Exterior angles:	$\alpha, \beta, \eta, \psi$

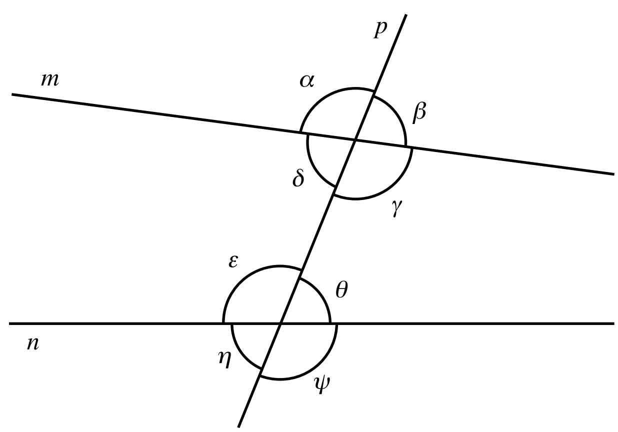

Figure 2.32

PROPOSITION 2.3.34(I.27). *If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another*.

GIVEN: Straight line EF intersects the straight lines AB and CD in the points E and F respectively, $\angle 1 = \angle 2$ (Fig 2.33).

TO PROVE: $AB \parallel CD$.

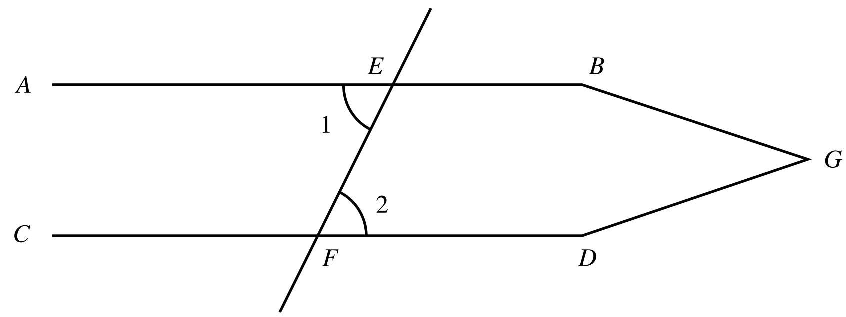

Figure 2.33

PROOF: By contradiction. Suppose AB and CD are not parallel. Then it may be assumed without loss of generality that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at some point G that lies on the same side of EF as B and D. Then, however, $\angle 2$ is interior to ΔEFG whereas $\angle 1$ is exterior to it. By Proposition 2.3.20, $\angle 2 < \angle 1$, contradicting the given equality. Hence $AB \parallel CD$.

Q.E.D.

PROPOSITION 2.3.35(I.28). *If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another.*

See Exercise 3.

EXERCISES 2.3S

1. If both pairs of the opposite sides of a quadrilateral are equal to each other then they are also parallel to each other.
2. If the diagonals of a quadrilateral bisect each other then the quadrilateral's opposite sides are parallel to each other.

3. Prove Proposition 2.3.35.
4. Comment on Proposition 2.3.34 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.
5. Comment on Proposition 2.3.35 in the context of the following geometries:
   a) spherical; b) hyperbolic; c) taxicab; d) maxi.

4. Postulate 5 Revisited

So far no use whatsoever has been made of Postulate 5. It is somewhat puzzling that Euclid should shy away from the use of this postulate for so long, especially as this deferral results in the need for an unnatural, though ingenious, proof for the apparently incomplete Proposition 2.3.20(I.16). The reason this proposition might be viewed as incomplete is that, as will be seen in Proposition 3.1.6(I.32), the exterior angle actually equals the sum of the two interior and opposite angles - a considerably more elegant and satisfying observation. Some scholars read into this ordering of the propositions a conscious reluctance on Euclid's part to make use of the fifth postulate. They believe that Euclid felt that this postulate was unnecessary and that, with enough effort, it could be omitted by proving that it is a consequence of the other postulates and common notions. By 450 A.D. the search for such a proof of the Postulate 5 became an acknowledged goal of geometry. Over the centuries, the existence of a proof of this postulate turned into a holy grail for mathematicians. Many geometers occupied themselves with this project and many purported proofs were produced. All of these proofs were subsequently demonstrated to be faulty in that they relied on yet other unstated assumptions that were in fact logically equivalent to Euclid's Postulate 5. By the early 1800s the mathematical community began acknowledging the possibility that such a proof might not exist. János Bolyai (1802-1862), Nikolay Ivanovich Lobachevsky (1792-1856), and Ernst Minding

(1806-1885) found strong evidence supporting this view and in 1868 Eugenio Beltrami (1835-1900) proved conclusively that

Postulate 5 cannot be proved on the basis of the other postulates and common notions alone.

Beltrami accomplished this feat by pointing out that with the exception of Postulate 5, all of Euclid's postulates and common notions were valid in the hyperbolic geometry that had been created by Joseph Liouville (1809-1882) two decades earlier in a completely different context and for completely different purposes, and which was described in Section 1.2. Postulate 5, on the other hand, does not hold in this geometry as is illustrated by Figure 2.7 where $\alpha + \beta < 180^{\circ}$ and yet the geodesics p and q do not intersect. This difference between Postulate 5 and the other postulates and common notions demonstrates that this postulate cannot be a logical consequence of them alone.

CHAPTER REVIEW EXERCISES

1. Let E be the midpoint of the median AD of the equilateral ΔABC. Prove that $AE < BE$.
2. The isosceles triangles CAD and CBD have the common base CD. Prove that AB bisects both $\angle CAD$ and $\angle CBD$.
3. Let P be a point in the interior of ΔABC in which $\angle BAC$ is obtuse. Prove that if $D = BP \cap AC$ and $E = CP \cap AB$ then $BD + CE > BE + ED + DC$.
4. Let P be a point in the interior of ΔABC and set $D = AP \cap BC$ and $E = BP \cap AC$. Prove that if $PA = PB$ and $PD = PE$, then ΔABC is isosceles.
5. Prove that the sum of the three altitudes of a triangle is less than the sum of its three sides.
6. Given ΔABC and point P in its interior, let $D = AP \cap BC$ and $E = BP \cap AC$. Prove that if $\angle PBC = \angle PAC$ and $\angle PBA = \angle PAB$, then ΔABC is isosceles.
7. Show that if a quadrilateral is convex then the sum of the diagonals is greater than the sum of each pair of opposite sides.

8. In quadrilateral $ABCD$, $AB = AD$ and $\angle ABC = \angle ADC$. Prove that AC is the perpendicular bisector of BD.
9. Let D be the midpoint of the base BC of the isosceles ΔABC, and let E be any point on AC. Prove that the difference between DB and DE is smaller than the difference between AB and AE.
10. Let C be the midpoint of AB and let D and E be points on the same side of AB such that $AD = BE$ and $CD = CE$. Prove that $AE = BD$.
11. Let BD, CE, AF be equal segments on the respective sides BC, CA, AB of the equilateral ΔABC. Prove that if the points $M = AD \cap BE, N = BE \cap CF, P = CF \cap AD$ are distinct then they form an equilateral triangle.
12. Given points A and B on the same side of the straight line m, determine on m a point P such that the difference $AM - MB$ is as large as possible.
13. It is planned to place a bridge across a straight river so that each of two given locations on the river's opposite sides are at equal distances from the nearest entry to the bridge. Where should the bridge be placed, assuming that it runs at right angles to the banks? (See Fig. 2.34)

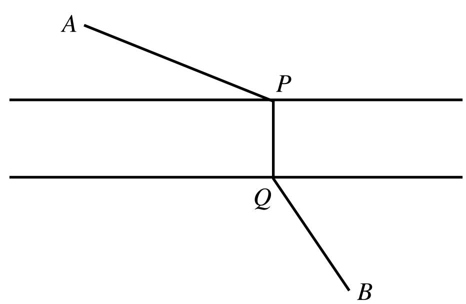

Figure 2.34

14**. Prove that if two of the angle bisectors of a triangle are equal then the triangle is isosceles. (Hint: Proceed by contradiction.)
15. Explain the difference between Euclid's axioms and common notions.
16. Explain the difference between collapsible and rigid compasses. What is the significance of this difference in Euclidean geometry?
17. Are the following statements true or false? Justify your answers.
    a) Euclid’s development of geometry is error-free.
    b) Euclid gave an axiomatic definition of area.
    c) Every proposition of Euclidean geometry is also valid in neutral geometry.
    d) Every proposition of neutral geometry is also valid in Euclidean geometry.
    e) Every proposition of spherical geometry is also valid in neutral geometry.
    f) Every proposition of neutral geometry is also valid in spherical geometry.

g) Every proposition of hyperbolic geometry is also valid in neutral geometry.

h) Every proposition of neutral geometry is also valid in hyperbolic geometry.

i) Every proposition of taxicab geometry is also valid in neutral geometry.

j) Every proposition of neutral geometry is also valid in taxicab geometry.

k) Parallel straight lines exist in neutral geometry.

l) Parallel straight lines exist in spherical geometry.

m) Parallel straight lines exist in Euclidean geometry.

n) Parallel straight lines exist in hyperbolic geometry.

o) Parallel straight lines exist in taxicab geometry.