CHAPTER 1

Other Geometries: A Computational Introduction

In order to provide a better perspective on Euclidean geometry, three alternative geometries are described. These are the geometry of the surface of the sphere, hyperbolic geometry, and taxicab geometry.

1. Spherical Geometry

Due to its relationship with geography and astronomy, spherical geometry was studied extensively by the Greeks as early as 300 BC. Menelaus (circa 100) wrote the book *Spherica* on spherical trigonometry which was greatly extended by Ptolemy (100-178) in his *Almagest*. Many later mathematicians, including Leonhard Euler (1707-1783) and Carl Friedrich Gauss (1777-1855) made substantial contributions to this topic. Here it is proposed only to compare and contrast this geometry with that of the plane. Because the time to develop spherical geometry in the same manner as will be done with Euclidean geometry is not available, this discussion is necessarily informal and frequent appeals will be made to the readers' visual intuition.

Strictly speaking there are no straight lines on the surface of a sphere. Instead it is both customary and useful to focus on curves that share the "shortest distance" property with the Euclidean straight lines. The following thought experiment will prove instructive for this purpose. Imagine that two pins have been stuck in a smooth sphere in points that are not diametrically opposite and that a (frictionless) rubber band is held by the pins in a stretched state. Rotate this sphere until one of the two pins is directly above the other right in front of your mind's eye. It is then hard to avoid the conclusion that the rubber band will be stretched out along the sphere in the plane formed by the two pins and the eye - the plane of the book's page in Figure 1.1. The inherent symmetry of the sphere dictates that this plane should cut the sphere into two identical hemispheres, in other words, that this plane should pass through the center of the sphere. It is also clear

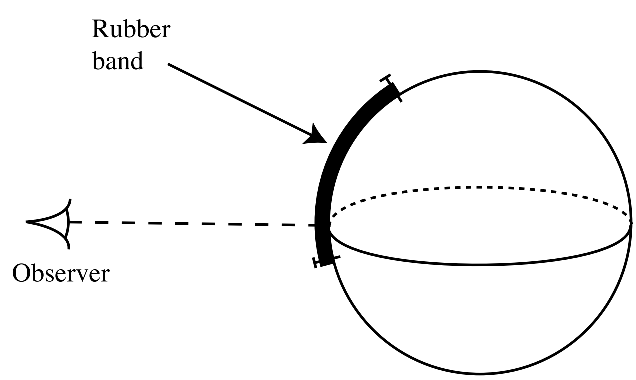

Figure 1.1 A geodesic on the sphere.

that the tension of the stretched rubber band forces it to describe the shortest curve on the surface of the sphere that connects the two pins. The following may therefore be concluded.

PROPOSITION 1.1.1 (Spherical geodesics). *If A and B are two points on a sphere that are not diametrically opposite, then the shortest curve joining A and B on* *the sphere is an arc of the circle that constitutes the intersection of the sphere with a plane that contains the sphere's center.*

Such circles are called *great circles* and these arcs are called *great arcs* or *geodesic segments*. They are the spherical analogs of the Euclidean line segments.

Diametrically opposite points on the sphere present a dilemma. A stretched rubber band joining them will again lie along a great circle, but this circle is no longer uniquely determined since these points can clearly be joined by an infinite number of great semicircles. For example, assuming for the sake of argument that the earth is an exact sphere, each meridian is a great semicircle that joins the north and south poles. Hence, the aforementioned analogy between the geodesic segments on the sphere and Euclidean line segments is not perfect. It is necessary to either exclude such meridians from the class of geodesic segments or else to accept that some points can be joined by many such segments. The first alternative is the one chosen in this text. Thus, by definition, the endpoints of geodesic segments on the sphere are never diametrically opposite.

Next, the spherical analog of the angle is defined. Any two great semicircles that join two diametrically opposite points A and B but are not contained in the same great circle divide the sphere into two portions each of which is called a *lune*, or a *spherical*

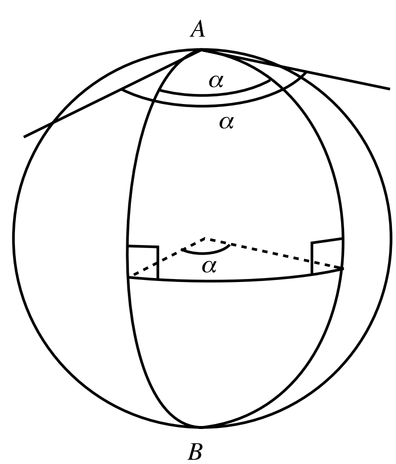

Figure 1.2 The lune α.

angle (Fig. 1.2). The measure of the spherical angle is defined to be the measure of the angle between their tangent lines at A (or at B). Alternately, this equals the measure of the angle formed by the radii from the center of the sphere to the midpoints of the bounding great semicircles. For example, each meridian forms a 90° angle with the equator at their point of intersection.

In the Euclidean plane the relationships between lengths of straight line segments and measures of angles are given by well known trigonometric identities. Some fundamental theorems of spherical trigonometry are now stated without proof.

Any three points A, B, C on the sphere no two of which are diametrically opposite, constitute the *vertices* of a spherical triangle denoted by ΔABC. The three *sides* of this triangle are the geodesic segments that join each pair. The sides opposite the vertices A, B, C (and their lengths) are denoted a, b, c respectively. The *interior angle* α at the vertex A is the lune between AB and AC. The interior angles β and γ at B and C are defined in a similar manner.

PROPOSITION 1.1.2 (Spherical trigonometry). *On a sphere of radius $R = 1$, let ΔABC be a spherical triangle with sides a, b, c and interior angles α, β, γ. Then,*

i) $$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

i') $$\cos a = \cos b \cos c + \cos \alpha \sin b \sin c$$

ii) $$\cos a = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma}$$

ii') $$\cos \alpha = \cos a \sin \beta \sin \gamma - \cos \beta \cos \gamma$$

iii) $$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}.$$

These are known as the first spherical law of cosines, the second spherical law of cosines, and the spherical law of sines. It should be noted that i and i' are really the same equation as are ii and ii', although, as will be demonstrated by the examples below, their uses are different. The *solution* of a triangle consists of the lengths of its sides and the measures of its interior angles.

EXAMPLE 1.1.3. *Solve the spherical triangle with sides* $a = 1, b = 2,$ *and* $c = \pi/2$.

It follows from the first spherical law of cosines that

$$\cos\alpha = \frac{\cos 1 - \cos 2 \cos \pi/2}{\sin 2 \sin \pi/2} = \frac{\cos 1}{\sin 2}$$

so that

$$\alpha = \cos^{-1}\left(\frac{\cos 1}{\sin 2}\right) \approx 53.54^{\circ}.$$

The angles β and γ are similarly shown to have measures 119.64° and 72.91°.

EXAMPLE 1.1.4. *On a sphere of radius 4000 miles, solve the triangle in which an interior angle of* 50^{o} *lies between sides of lengths 7000 miles and 9000 miles respectively.*

Since the radius is the unit it follows that we may set

$$b = \frac{7000}{4000} = 1.75 \qquad c = \frac{9000}{4000} = 2.25\ .$$

Hence, from the first law of cosines,

$$a \quad = \quad \cos^{-1}(\cos a \sin b \sin c \ + \ \cos b \cos c) \quad \approx \quad .9221 \quad \approx \quad 3688 \text{ miles}$$

Now that all three sides of the triangle are known, the method of the previous example yields

$$\beta \quad = \quad \cos^{-1}\left(\frac{\cos b \ - \ \cos a \cos c}{\sin a \sin c}\right) \quad \approx \quad 71.05^{\circ}$$

$$\gamma \quad = \quad \cos^{-1}\left(\frac{\cos c \ - \ \cos a \cos b}{\sin a \sin b}\right) \quad \approx \quad 131.58^{\circ}\ .$$

Note that in both of the above examples the sum of the angles of the spherical triangle exceeds 180°. That is in fact true for all spherical triangles.

PROPOSITION 1.1.5. *The sum of the angles of every spherical triangle lies strictly between* 180^{o} *and* 540^{o}.

A spherical triangle the sum of whose angles is close to 180° is formed by the equator together with two close meridians. Thus, the sum of the angles of the spherical ΔABC of Figure 1.3 is $90^{\circ} + 90^{\circ} + \alpha \ = \ 180^{\circ} + \alpha.$ A spherical triangle $A'B'C'$ with

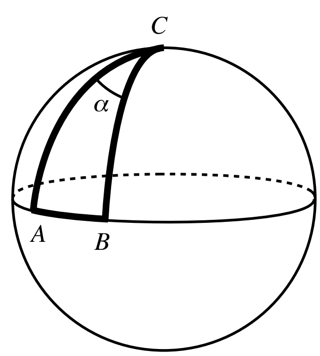

Figure 1.3 A thin spherical triangle.

angle sum near 540^o is described in Figure 1.4. where A, B, C are points that are equally spaced along a great circle. As A', B', C' approach A, B, C respectively, the angles they form are flattened out and come arbitrarily close to 180^o each. For example, since the spherical distance between any two of the points A, B, C is $2\pi/3$ it might be assumed that the spherical distance between any two of the points A', B', C' is $a = 2\pi/3 - 0.00001$, in which case each of the angles of $\Delta A'B'C'$ is

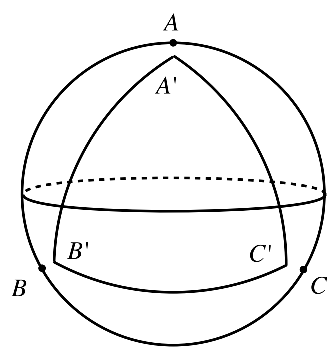

Figure 1.4 A nearly maximal spherical triangle.

$$\cos^{-1}\left(\frac{\cos a - \cos a \cos a}{\sin a \sin a}\right) \approx 179.52^o$$

and their sum is 538.56^o.

Since, by definition, each of the interior angles of the spherical triangle is less than 180^o, it follows that the sum of these angles can never equal 540^o. Similarly, as will be shown momentarily, the sum of these angle cannot equal the lower bound of 180^o either.

The area of the spherical triangle is also of interest. An elegant proof of this formula is offered in Section 3.2.

PROPOSITION 1.1.6. *If a triangle on a sphere of radius R has angles with radian measures* α, β, γ *then it has area* $(\alpha + \beta + \gamma - \pi)R^2$.

For example, the spherical triangle formed by the equator, the Greenwich meridian and the 90^o East meridian has all of its angles equal to $\pi/2$ and hence its area is

$$\left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} - \pi\right)R^2 = \frac{\pi R^2}{2}.$$

This answer is consistent with the fact that the said triangle constitutes one fourth of a hemisphere. Since the surface area of the sphere is $4\pi R^2$, this triangle has area

$$\frac{1}{4}\,\frac{4\pi R^2}{2} = \frac{\pi R^2}{2}$$

which agrees with the previous calculation.

The quantity $\alpha + \beta + \gamma - \pi$ is called the *excess* of the spherical $\Delta\, ABC$. The above theorem in effect states that the area of a spherical triangle is proportional to its excess. This assertion is supported by the triangle below which has excess $\pi/2 + \pi/2 + \alpha - \pi = \alpha$ and whose area is clearly proportional to α as long as A and B vary along the equator and C remains at the north pole (see Fig. 1.5).

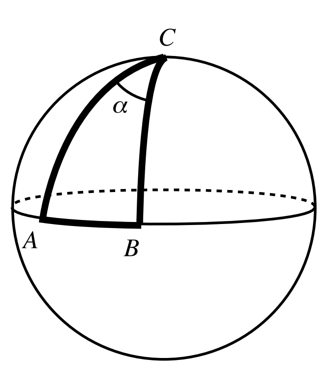

Figure 1.5 A spherical triangle.

This area formula can be used to close a gap in the above discussion. Since every triangle has positive area, it follows that the sum of the angles of a spherical triangle never equals π, or 180°, although, as was seen above, it can come arbitrarily close to this lower bound.

EXERCISES 1.1

1. Let ABC be a spherical triangle with a right angle at C. Use the formulas of spherical trigonometry to prove that:
   a) $\sin a = \sin\alpha \sin c$ b) $\tan a = \tan\alpha \sin b$
   c) $\tan a = \cos\beta \tan c$ d) $\cos c = \cos a \cos b$
   e) $\cos\alpha = \sin\beta \cos a$ f) $\sin b = \sin\beta \sin c$
   g) $\tan b = \tan\beta \sin a$ h) $\tan b = \cos\alpha \tan c$
   i) $\cos c = \cot\alpha \cot\beta$ j) $\cos\beta = \sin\alpha \cos b$.
2. On a sphere of radius R, solve the spherical triangle with angles
   a) $60^{\circ}, 70^{\circ}, 80^{\circ}$ b) $70^{\circ}, 70^{\circ}, 70^{\circ}$
   c) $120^{\circ}, 130^{\circ}, 140^{\circ}$ d) θ, θ, θ, where $60^{\circ} < \theta < 120^{\circ}$.
3. On a sphere of radius R, solve the spherical triangle with sides
   a) R, R, R b) $R, 1.5R, 2R$
   c) $\pi R/2, \pi R/2, \pi R/2$ d) d, d, d, where $0 < d < 2\pi R$
   e) $.2R, .3R, .4R$ f) $.02R, .03R, .04R$
   g) $2R, 3R, 4R$
4. On a sphere of radius R, solve the spherical triangle with
   a) $a = .5R,\ \beta = 60^{\circ},\ \gamma = 80^{\circ}$ b) $b = R, \alpha = 40^{\circ}, \gamma = 100^{\circ}$
   c) $a = 2R, \beta = \gamma = 10^{\circ}$ d) $a = 2R, \beta = \gamma = 170^{\circ}$
5. On a sphere of radius R, solve the spherical triangle with
   a) $a = 2R, b = R, \gamma = 100^{\circ}$ b) $b = .5R, c = 1.2R, \alpha = 100^{\circ}$
   c) $a = 2R, b = R, \gamma = 120^{\circ}$ d) $b = .5R, c = 1.2R, \alpha = 120^{\circ}$
6. On a sphere of radius 75 cm, solve the spherical triangle with
   a) $a = 100$ cm, $b = 125$ cm, $c = 140$ cm b) $\alpha = 100^{\circ}, \beta = 125^{\circ}, \gamma = 140^{\circ}$
   c) $\alpha = 100^{\circ}$, $b = 125$ cm, $c = 125$ cm d) $a = 100$ cm, $\beta = 125^{\circ}, \gamma = 125^{\circ}$
7. Evaluate the limits of the angles of the spherical triangles below both as $x \to 0$ and as $x \to \pi$
   a) $a = b = c = x$ b) $a = b = x,\ c = 2x$
   c) $a = x, b = c = 2x$
8. Which of the following congruence theorems hold for spherical triangles? Justify your answer.

a) SSS b) SAS c) ASA d) SAA e) AAA.

9*. Prove that the three angles α, β, γ are the three interior angles of a spherical triangle if and only if they satisfy all of the following conditions:

$$\alpha + \beta + \gamma > \pi, \quad \alpha + \pi > \beta + \gamma, \quad \beta + \pi > \alpha + \gamma, \quad \gamma + \pi > \alpha + \beta.$$

2. Hyperbolic Geometry

Imagine a two dimensional universe, with a superimposed Cartesian coordinate system, in which the x-axis is infinitely cold. Imagine further, that as the objects of this universe approach the x-axis, the drop in temperature causes them to contract (see Fig. 1.6). Thus,

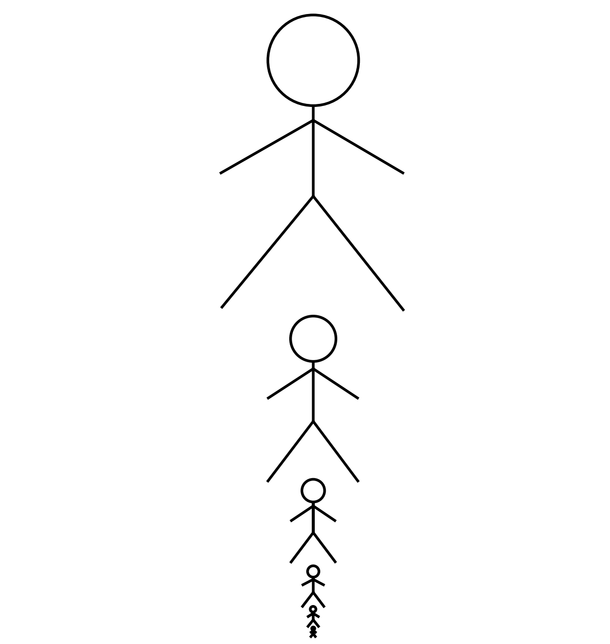

Figure 1.6 The shrinkage that defines the hyperbolic plane.

the inhabitants of this fictitious land will find that it takes them less time to walk along a horizontal line from $A(0, 1)$ to $B(1, 1)$ (Fig. 1.7) than it takes to walk along a horizontal line from $C(0, .5)$ to $D(1, .5)$. Since their rulers contract just as much as they do, this observation will not seem at all paradoxical to them. If it is assumed that the contraction is such that the outside observer sees the length of any object as being proportional to its distance from the x-axis, then the inhabitants will find that walking from $C(0, 0.5)$ to $D(1, .5)$ takes twice as long as walking from $A(0, 1)$ to $B(1, 1)$ and one fifth of the time

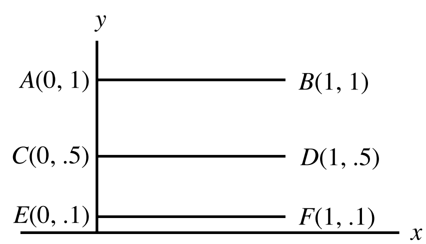

Figure 1.7 Paths of unequal hyperbolic lengths.

of walking from $E(0, .1)$ to $F(1, .1)$. To differentiate between the Euclidean length of such a segment and its length as experienced by these fictitious beings, it is customary to refer to the latter as the *hyperbolic length* of the segment. Accordingly, the hyperbolic lengths of the segments AB, CD, and EF of Figure 1.7 are 1, 2, and 10 respectively. In general, the hyperbolic length of a horizontal line segment at distance y from the x-axis is given by the formula

$$\text{hyperbolic length} = \frac{\text{Euclidean length}}{y} \qquad (1).$$

Other curves also have a hyperbolic length and a method for computing this is given in Exercise 16 below.

Not surprisingly, perhaps, the Euclidean straight line segment joining two points does not constitute the curve of shortest hyperbolic length between them. When setting out from $A(0, 1)$ to $B(1, 1)$ the inhabitants of this strange land may find that if they bear a little to the north their journey will be somewhat shorter because, unbeknownst to them,

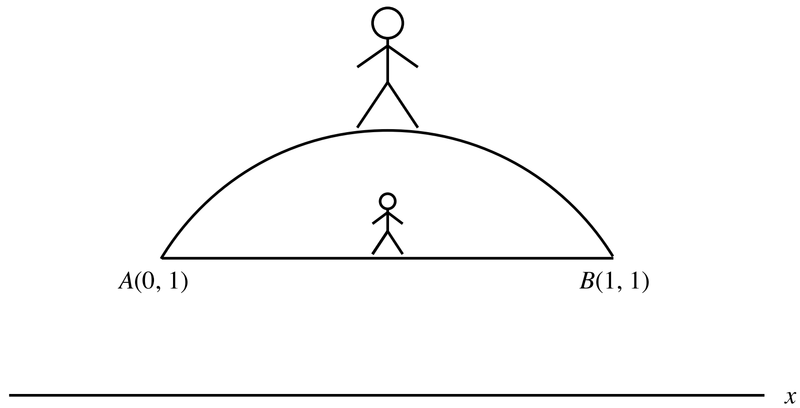

Figure 1.8 Which path has shorter hyperbolic length?

their legs are longer on this route (Fig. 1.8). However, if they stray too far north the length of the detour will offset any advantages gained by the elongation of their stride and they will find the length of the tour to be excessive. They are therefore faced with a trade-off problem. Some deviation to the north will shorten the duration of the trip from A to B, but too much will extend it. Which path, then, is it that makes the trip as short as possible?

The answer to this question is surprisingly easy to describe, though not to justify. The path of shortest hyperbolic length that connects $A(0, 1)$ to $B(1, 1)$ is the arc of the

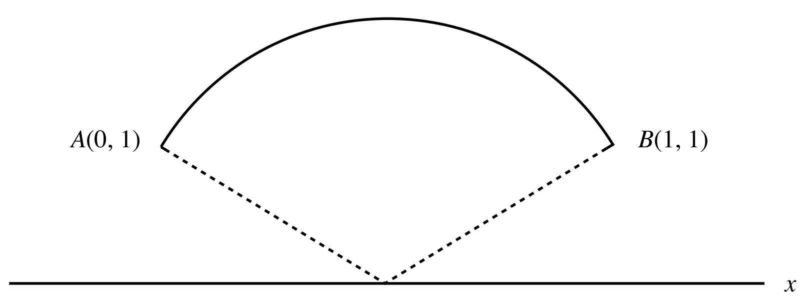

Figure 1.9 A hyperbolic geodesic.

circle that is centered at $(.5, 0)$ and contains A and B (Fig. 1.9). Its hyperbolic length turns out to be $0.962...$ in contrast with the hyperbolic length 1 of the horizontal segment AB. More drastically, the arc of the semicircle centered at $(50, 0)$ and which joins the points $A(0, 1)$ and $X(100, 1)$ has hyperbolic length 9.21, a mere 9% of the hyperbolic length of the segment AX.

Given any two points, their *hyperbolic distance* is the minimum of the hyperbolic lengths of all the curves joining them. As was the case for spherical geometry, the *geodesic segments* of hyperbolic geometry are those curves that realize the hyperbolic distance between their endpoints. The *hyperbolic plane* consists of the portion of the Cartesian coordinate system that lies above the x-axis.

PROPOSITION 1.2.1 (Hyperbolic geodesics). *The geodesic segments of the hyperbolic plane are arcs of circles centered on the x-axis and Euclidean line segments that are perpendicular to the x-axis.*

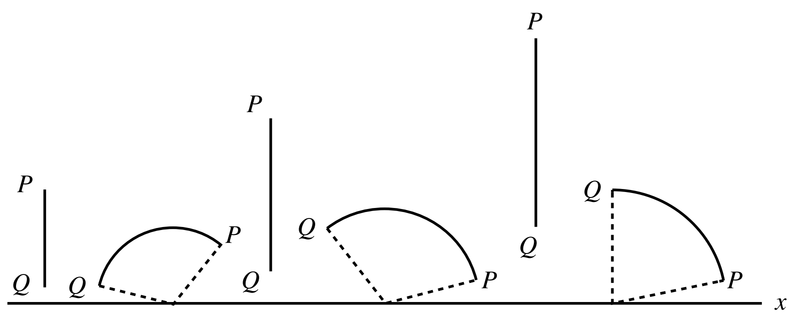

Figure 1.10 Six hyperbolic geodesics.

The geodesics of the first variety are called *bowed* geodesics, whereas the vertical ones are the *straight* geodesics (see Fig. 1.10). This distinction is only meaningful to the outside observer. The inhabitants of this geometry perceive no difference between these two kinds of geodesics.

It so happens that as the inhabitants of the hyperbolic plane approach the x-axis they shrink at such a rate as to make the x-axis unattainable. Technically speaking, the hyperbolic lengths of all of the geodesic segments in Figure 1.10 diverge to infinity as the endpoints Q approach the x-axis. This claim will be given a quantitative justification at the end of this section.

A *hyperbolic angle* is the portion of the hyperbolic plane between two geodesic rays (Fig. 1.11). The measure of the angle between two geodesics is, by definition, the

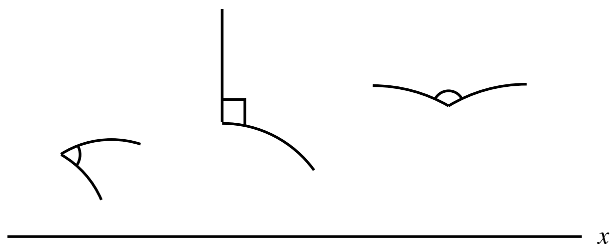

Figure 1.11 Three hyperbolic angles.

measure of the angle between the tangents to the geodesics at the vertex of the angle. Accordingly, two geodesics are said to form a hyperbolic right angle if and only if their tangents are perpendicular to each other as Euclidean straight lines (Fig. 1.12). Given any three points that do not lie on one hyperbolic geodesic, they constitute the *vertices* of a

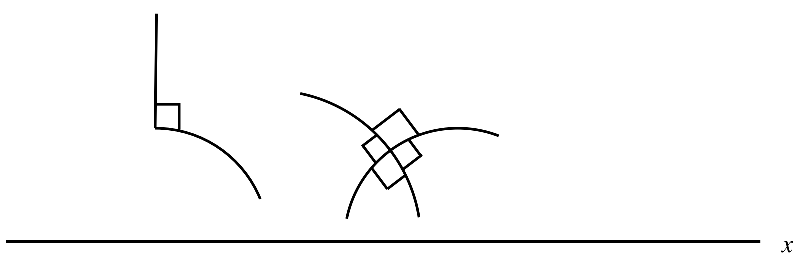

Figure 1.12 Hyperbolic right angles.

hyperbolic triangle formed by joining the vertices, two at a time, with hyperbolic geodesics (Fig. 1.13).

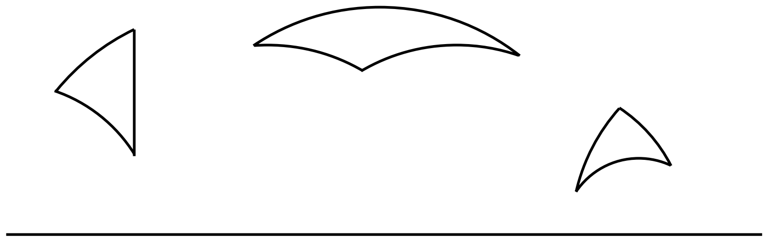

Figure 1.13 Three hyperbolic triangles.

The geometry of the hyperbolic plane has been studied extensively. Its trigonometric laws are surprisingly, not to say mysteriously, similar to those of spherical geometry.

PROPOSITION 1.2.2 (Hyperbolic trigonometry). *Let* ΔABC *be a hyperbolic triangle with sides* a, b, c *and interior angles* α, β, γ. *Then*

i) $$\cos\alpha = \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}$$

i') $$\cosh a = \cosh b \cosh c - \cos\alpha \sinh b \sinh c$$

ii) $$\cosh a = \frac{\cos\alpha + \cos\beta \cos\gamma}{\sin\beta \sin\gamma}$$

ii') $$\cos\alpha = \cosh a \sin\beta \sin\gamma - \cos\beta \cos\gamma$$

iii) $$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}$$

EXAMPLE 1.2.3. *Solve the hyperbolic triangle with sides* $a = 1, b = 2$, *and* $c = \pi/2$.

It follows from Formula i of hyperbolic trigonometry that

$$\cos\alpha = \frac{\cosh 2 \cosh \pi/2 - \cosh 1}{\sinh 2 \sinh \pi/2} \approx .9461$$

and so $\alpha \approx \cos^{-1}(.9461) \approx 18.89^{\circ}$. Similarly $\beta \approx 87.67^{\circ}$ and $\gamma \approx 39.34^{\circ}$.

EXAMPLE 1.2.4. *Solve the hyperbolic triangle with two sides of lengths 2, 3 respectively, if they are to include an angle of 30^{o}.*

Set $\alpha = 30^{\circ}$, $b = 2$, $c = 3$. It follows from Formula i') of hyperbolic trigonometry that

$$a = \cosh^{-1}(\cosh b \cosh c - \cos \alpha \sinh b \sinh c) = 2.545...$$

Now that all three sides of the triangle are known, the method of the previous example yields

$$\beta = \cos^{-1}\left(\frac{\cosh a \cosh c - \cosh b}{\sinh a \sinh c}\right) \approx 16.64^{\circ}$$

$$\gamma = \cos^{-1}\left(\frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}\right) \approx 52.28^{\circ}$$

As for the sum of the angles of a hyperbolic triangle, the situation is diametrically opposite to that on the sphere.

PROPOSITION 1.2.5. *The sum of the angles of every hyperbolic triangle is less than 180^{o}.*

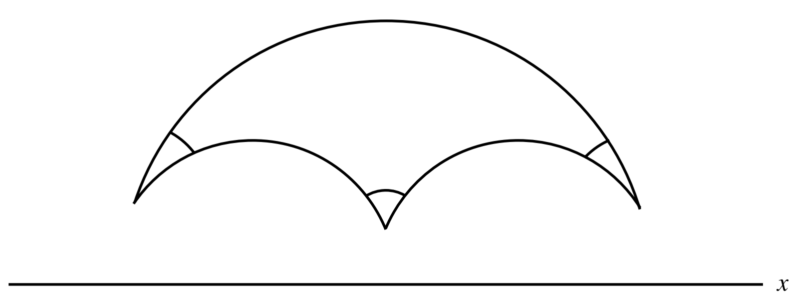

Figure 1.14 A hyperbolic triangle with three small angles.

This proposition is borne out by the above two examples. Figure 1.14 demonstrates that this sum can be quite small. In fact, in the hyperbolic triangle with sides $a = b = c = 10$ each angle equals

$$\alpha \quad = \quad \cos^{-1}\left(\frac{\cosh 10 \cosh 10 - \cosh 10}{\sinh 10 \sinh 10}\right) \quad \approx \quad .77^{\circ}.$$

EXAMPLE 1.2.6. *Solve the hyperbolic triangle with* $a = 2, \beta = \gamma = 60^{o}$. By formula ii' of hyperbolic trigonometry

$$\cos \alpha \quad = \quad \cosh 2 \sin 60^{\circ} \sin 60^{\circ} - \cos 60^{\circ} \cos 60^{\circ} \quad \approx \quad 2.57 \quad .$$

Since the cosine of an angle cannot exceed 1, such a hyperbolic triangle does not exist. Note that a Euclidean triangle with the same specifications does exist. Exercises 4, 5 contain some related information.

The area of the hyperbolic triangle is of course of interest too. Its formula is quite surprising.

PROPOSITION 1.2.7. *The area of the hyperbolic triangle whose angles have radian measures* α, β, γ *is* $\pi - \alpha - \beta - \gamma$.

This formula is given some support by Figure 1.15. Note that the sum of the angles of the larger hyperbolic ΔABC is less than the sum of the angles of the smaller hyperbolic $\Delta AB'C'$. The quantity $\pi - \alpha - \beta - \gamma$ is called, by analogy with its spherical counterpart, the *defect* of the hyperbolic triangle. Thus, the above theorem asserts that the area of a hyperbolic triangle is equal to its defect.

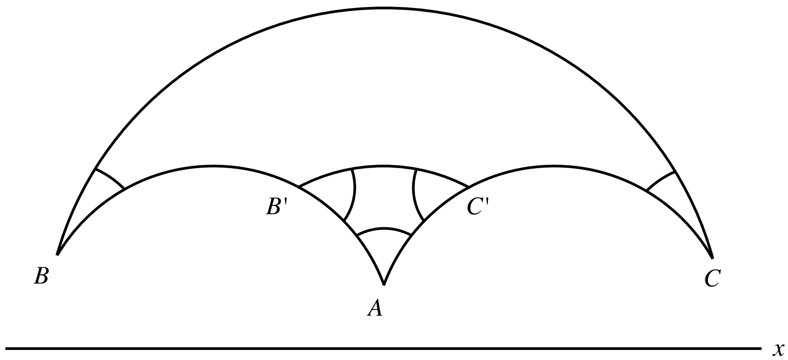

Figure 1.15 Two hyperbolic triangles with different areas and defects.

A somewhat peculiar aspect of the hyperbolic notion of length is its independence of the choice of unit of length. Regardless of what scale is chosen for the Cartesian coordinate system that is used to define the hyperbolic version of length, the hyperbolic distance between any two points remains the same. Note that the three parts of Figure 1.16 correspond to three different scales, and yet, according to Formula (1) above, in

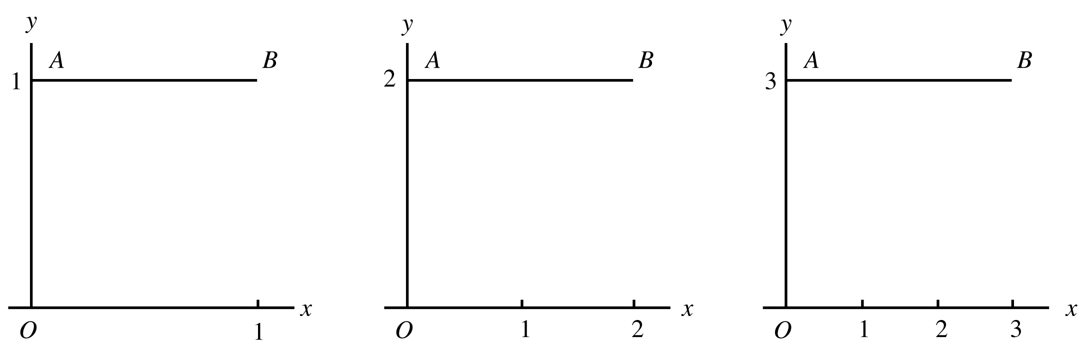

Figure 1.16 A curve with hyperbolic length 1.

each of the three cases the hyperbolic length of AB is

$$\frac{1}{1} = \frac{2}{2} = \frac{3}{3} = 1.$$

It follows from this independence of scale that in hyperbolic geometry it is not necessary to specify units of length.

It might be instructive to show that this independence holds in the vertical direction as well. The hyperbolic length of a vertical segment of the hyperbolic plane is easily computed with the aid of calculus. For, if dy denotes the Euclidean length of the

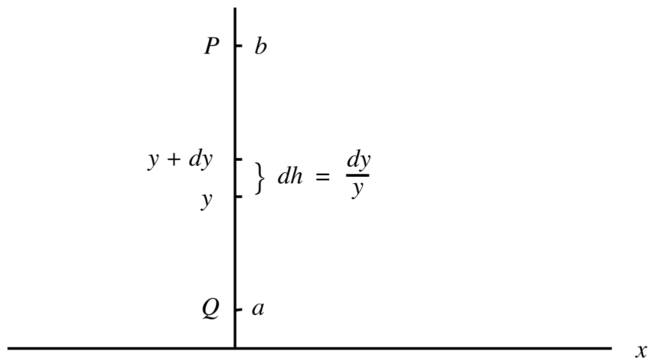

Figure 1.17 Hyperbolic length along the y-axis.

(infinitesimally) small vertical line segment at height y above the x-axis, then its hyperbolic length is dy/y (see Fig. 1.17). Consequently the total hyperbolic length of the segment PQ is

$$\int_a^b \frac{dy}{y} \quad = \quad \ln b - \ln a \quad = \quad \ln \frac{b}{a}. \qquad (2).$$

In particular, if $a = 1$ and $b = e = 2.718...$, then the hyperbolic length of the y-axis between $P(0, 1)$ and $Q(0, e)$ is

$$\ln \frac{e}{1} \quad = \quad \ln e \quad = \quad 1.$$

Moreover, if the scale on the axes is changed by a factor of s, then $P = (0, s)$ and $Q = (0, es)$ and again their hyperbolic distance is

$$\ln \frac{es}{s} \quad = \quad \ln e \quad = \quad 1.$$

Thus, if P and Q are points with coordinates $(0, 1)$ and $(0, e)$ relative to some unit of the Cartesian coordinate system, then the line segment joining P and Q has hyperbolic length 1 regardless of the scale that is actually used. Moreover, this Euclidean line segment PQ, which has hyperbolic length 1, is also a hyperbolic geodesic in contrast with the aforementioned segment AB which also has hyperbolic length 1 but is not a hyperbolic geodesic. Consequently, this geodesic can be taken as the *natural unit*, or *absolute unit* of length, of hyperbolic geometry.

Both spherical and hyperbolic geometry look very different from Euclidean geometry. Nevertheless, it is well known that on a large sphere, a small portion of the surface may be practically indistinguishable from a piece of a plane. This resemblance accounts for the fact that people first thought that the world was flat and small children still do so today. The same confusion could occur in the hyperbolic plane. If the portion of the hyperbolic plane that is subject to the direct experience and observation of its inhabitants is sufficiently small, their geometry would appear to them as practically indistinguishable from that of the Euclidean plane. This affinity between the hyperbolic and Euclidean planes is a topic that will be revisited many times in the subsequent discussion. The explanation of how the trigonometry of a small portion of the hyperbolic plane may be confusable with Euclidean trigonometry can be found in the references. At this point it will be demonstrated that, just like the Euclidean plane and in contrast with the sphere, hyperbolic geometry extends indefinitely in all directions. In other words, the inhabitants of the hyperbolic plane have no reason to suspect that part of their

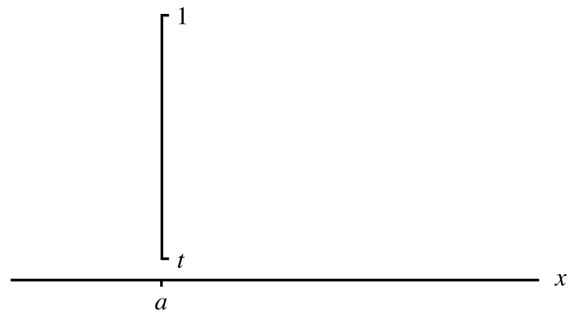

Figure 1.18 The hyperbolic plane extends indefinitely in all directions.

universe is missing. To see this note that by Eq'n (2) the hyperbolic distance from the point $(a, 1)$ to the point (a, t) in Figure 1.18 is

$$\ln \frac{1}{t} \quad = \quad - \ln t .$$

Hence, if travel in the direction of the x-axis is simulated by letting t approach 0, then this quantity diverges to $-(-\infty) = \infty$. In other words, for the hyperbolic people the x-axis lies infinitely far away.

HYPERBOLIC DISTANCE.

The hyperbolic distance between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ can be determined by means of the following formulas:

i) If $x_1 = x_2$ then the hyperbolic distance from A to B is $\left|\ln \frac{y_1}{y_2}\right|$;

ii) If $x_1 \neq x_2$, and (c, o) is the center of the geodesic segment that connects (x_1, y_1) and (x_2, y_2), and r is its radius, then the hyperbolic distance from A to

B is $\left|\ln \frac{(x_1 - c - r)y_2}{(x_2 - c - r)y_1}\right|$.

EXAMPLE 1.2.8. The hyperbolic distance between the points $(5, 4)$ and $(5,7)$ is $|\ln(4/7)| \approx 0.56$.

EXAMPLE 1.2.9. To find the hyperbolic distance between $A(8, 4)$ and $B(0, 8)$ note that the line segment AB has slope $\frac{8-4}{0-8} = -\frac{1}{2}$ and midpoint $M(4, 6)$ (see Fig. 1.19). Hence the perpendicular bisector of AB has equation $y - 6 = 2(x - 4)$ and is easily seen to intersect the x-axis in the point $C(1, 0)$. Thus, $c = 1$ and $r = \sqrt{(0-1)^2 + (8-0)^2} = \sqrt{65}$. Hence the required distance is $\left|\ln \frac{(8-1-\sqrt{65})8}{(0-1-\sqrt{65})4}\right| \approx 1.45\ldots$.

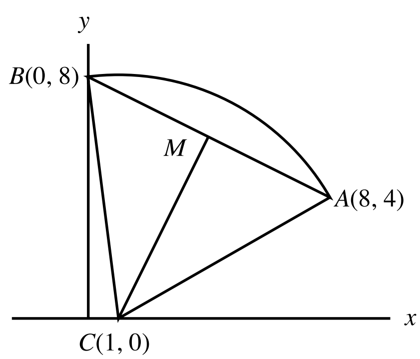

Figure 1.19 Computing the length of a hyperbolic geodesic.

EXERCISES 1.2

1. Let ABC be a hyperbolic triangle with a right angle at C.
   - a) $\sinh a = \sin\alpha \sinh c$
   - b) $\tanh a = \tan\alpha \sinh b$
   - c) $\tanh a = \cos\beta \tanh c$
   - d) $\cosh c = \cosh a \cosh b$
   - e) $\cos\alpha = \sin\beta \cosh a$
   - f) $\sinh b = \sin\beta \sinh c$
   - g) $\tanh b = \tan\beta \sinh a$
   - h) $\tanh b = \cos\alpha \tanh c$
   - i) $\cosh c = \cot\alpha \cot\beta$
   - j) $\cos\beta = \sin\alpha \cosh b$.
2. Solve the hyperbolic triangle with angles
   - a) $60^o, 50^o, 40^o$
   - b) $50^o, 50^o, 50^o$
   - c) $20^o, 50^o, 70^o$
   - d) θ, θ, θ, where $0^o < \theta < 60^o$
3. Solve the hyperbolic triangle with sides

a) 1, 1, 1 b) 2, 3, 4 c) 1/2, 1/2, 1/2
d) $d, d, d,$ where $0 < d$ e) .2, .3, .4 f) .02, .03, .04

4. Solve the hyperbolic triangle with
   a) $a = .5,\ \beta = 60^{o},\ \gamma = 40^{o}$ b) $b = .5, \alpha = 40^{o}, \gamma = 50^{o}$
   c) $a = 2, \beta = \gamma = 40^{o}$ d) $a = 10, \beta = \gamma = 40^{o}$
   e) $a = 1, \beta = \gamma = 60^{o}$ f) $a = .1, \beta = \gamma = 60^{o}$
   g) $a = .2, \beta = \gamma = 100^{o}$
5. For which values of a does there exist a hyperbolic triangle with $\beta = \gamma = 60^{o}$?
6. Solve the hyperbolic triangle with
   a) $a = 2, b = 1\ ,\ \gamma = 30^{o}$ b) $b = .5, c = 1.2, \alpha = 120^{o}$
   c) $a = 2, b = 1\ ,\ \gamma = 45^{o}$ d) $b = .5, c = 1.2, \alpha = 120^{o}$
7. Evaluate the limits of the angles of the hyperbolic triangles below both as $x \to 0$ and as $x \to \infty$
   a) $a = b = c = x$ b) $a = b = x,\ c = 2x$
   c) $a = x, b = c = 2x$
8. Does the SSS congruence theorem hold for hyperbolic triangles? Justify your answer.
9. Does the SAS congruence theorem hold for hyperbolic triangles? Justify your answer.
10. Does the ASA congruence theorem hold for hyperbolic triangles? Justify your answer.
11. Does the SAA congruence theorem hold for hyperbolic triangles? Justify your answer.
12. Explain why the AAA congruence theorem holds for hyperbolic triangles.
13. Find the hyperbolic distances between each pair of the three points $A(0, 6), B(10, 4), C(10, 16)$.
14. Explain why the hyperbolic length of every Euclidean line segment that is parallel to the x-axis is independent of the unit of the underlying Cartesian coordinate system.
15. Explain why the hyperbolic length of every Euclidean line segment that is parallel to the y-axis is independent of the unit of the underlying Cartesian coordinate system.
16. Explain why

$$\int_a^b \frac{\sqrt{1 + (f')^2}}{f}\, dx$$

is a reasonable formula for the hyperbolic length of a differentiable curve defined by $y = f(x)$, $a \le x \le b$.

17(C). Write a script that takes two distinct points as input and yields the hyperbolic geodesic joining them, as well as its hyperbolic length, as output.

18(C). Write a script that takes three distinct points as input and yields a sketch of the hyperbolic triangle they form, as well as its solution, as output. (Recall that $\cosh^{-1} x = \ln(x + \sqrt{x^2 + 1}\ )$.)

3. Other Geometries

All school children learn about the geometry of the plane. As was seen in the previous two sections, there are other geometries which, broadly speaking, can be classified into two categories.

One way to obtain a new geometry is to distort the plane. A piece of paper can be rolled into either a cylinder or a cone. A film of soap can assume many other shapes, including that of a sphere. The best known of this type of geometries is that of the sphere, some of whose properties were presented in Section 1. Of course, the distortion of surfaces may or may not result in distortions of lengths of curves. Rolling a piece of paper into a cone has no such effect - the straight lines on the paper are merely twisted into spirals (or circles) but their lengths are unaffected. On the other hand when a flat soap film waves in the air the lengths of the curves on it are continuously altered. It took mathematicians several centuries to realize, sometime around 1850, that this notion of distortion of lengths and distances could be, and should be, studied independently of the shape distortion that induced it. The geometries that are obtained by changing the way distance is measured in the plane are called *Riemannian geometries* and hyperbolic geometry is their best known and studied instance. Riemannian geometry has found many applications in science, the most spectacular of these being the theory of relativity. Hyperbolic geometry is still the subject of much contemporary research and has had many surprising applications to other mathematical disciplines.

Every Riemannian geometry has geodesics which are defined as the shortest curves joining two points. Such geodesics will form triangles and these triangles will have interior angles. These angles, in turn, provide a means for quantifying the distortion, or curvature, of a geometry. If ABC is a triangle of a geometry, with interior angles of radian measures α, β, γ, then the expression $\alpha + \beta + \gamma - \pi$ is called its *total* *curvature*. Accordingly, every triangle of the Euclidean plane has total curvature 0. It is therefore reasonable to interpret this quantity as a measure of the extent to which that triangle differs from a Euclidean triangle. By this definition, the total curvature of a spherical triangle is always positive and so the sphere is said to be positively curved. Hyperbolic geometry, on the other hand, is negatively curved. It was Gauss who formally defined this notion and pointed out its central role in the study of geometry.

This chapter concludes with the discussion of yet another specific geometry which, while also arising from an esoteric way of measuring distance, is not, for reasons that cannot be explained here, a Riemannian geometry. *Taxicab geometry* was first defined in 1973 but, unlike spherical and hyperbolic geometry, has not been integrated into the mathematical mainstream. Nevertheless, is has proven useful as a pedagogical tool that sheds a light on Euclidean geometry and also provides students with an elementarily defined mathematical territory they can explore on their own. Like hyperbolic geometry, *taxicab geometry* takes the Euclidean plane as its starting point and redefines distance. The *taxicab distance* between the points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$d_t(P, Q) = |x_1 - x_2| + |y_1 - y_2| .$$

Thus, the taxicab distance between $P = (0, 0)$ and $Q = (1,1)$ is

$$d_t(P, Q) = |0 - 1| + |0 - 1| = 2$$

whereas the Euclidean distance $d_E(P, Q)$ between them is

$$\sqrt{(0 - 1)^2 + (0 - 1)^2} = \sqrt{2} .$$

Similarly, the taxicab distance between (2, -3) and (3, 5) is 1 + 8 = 9 and their Euclidean distance is $\sqrt{1 + 64} = \sqrt{65}$. This geometry receives its name from the fact that it models the way a taxicab driver would think of distances in a city all of whose blocks are perfect squares.

It is clear that taxicab distances agree with Euclidean distances along both horizontal and vertical straight lines. If p is any other straight line, with inclination θ from the positive x-axis, then the taxicab distances along p are different from, but still proportional to, the Euclidean distances. As indicated by Figure 1.20,

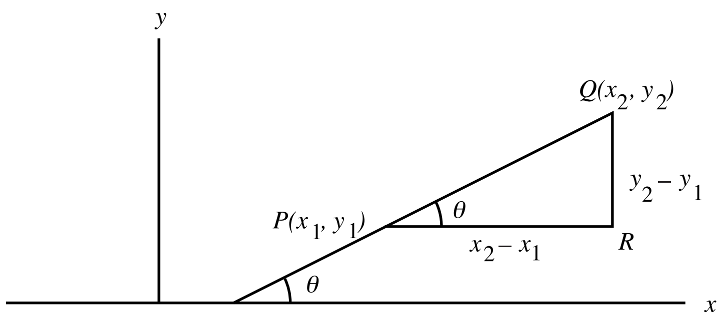

Figure 1.20

$$d_t(P, Q) = |x_2 - x_1| + |y_2 - y_1| = |\cos\theta|\, d_E(P, Q) + |\sin\theta|\, d_E(P, Q)$$

$$= (|\cos\theta| + |\sin\theta|)\, d_E(P, Q) .$$

It follows that along any fixed straight line taxicab distances behave very much like Euclidean distances. In particular, the geometrical notion of betweenness can still be expressed numerically.

PROPOSITION 1.3.1. *If the distinct points P, Q, R are collinear, then Q is between P and R if and only if*

$$d_t(P, Q) + d_t(Q, R) = d_t(P, R).$$

[]

There are many other similarities between the taxicab and Euclidean geometries and these are relegated to the exercises in the subsequent chapters. Some of these differences are qualitative. For example, note that in Figure 1.20

$$d_t(P, Q) = d_t(P, R) + d_t(R, Q).$$

In other words, the line segment PQ is not the only shortest path joining P and Q. In fact, if γ is any polygonal path joining P and Q all of whose segments have non-negative slope, then the taxicab length of γ still equals $d_t(P, Q)$ (Exercise 6). Nevertheless, no path joining P and Q has taxicab length shorter than $d_t(P, Q)$ (Exercise 9) and so it is not unreasonable to agree to regard the Euclidean straight lines as the straight lines of taxicab geometry. Not surprisingly, the taxicab measure of angles agrees with their Euclidean measure. There is, however, no consensus yet on how areas should be measured in this outlandish geometry.

Only one more striking difference between the taxicab and Euclidean geometries will be mentioned here. The SAS congruence theorem does not hold for taxicab geometry. For the two triangles in Figure 1.21 we have $d_t(A, B) = d_t(D, E) = 2 = d_t(A, C) = d_t(D, F)$ and $\angle BAC = \angle EDF = 90^o$ and yet $d_t(B, C) = 2 \neq 4 = d_t(E, F)$.

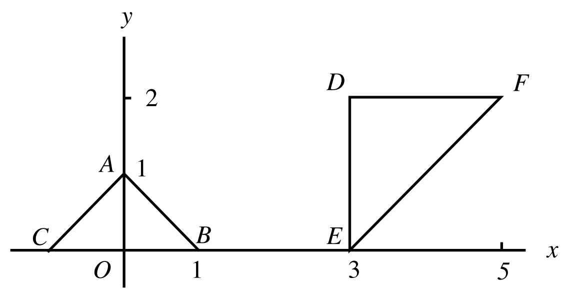

Figure 1.21 Two almost congruent triangles.

Maxi geometry, yet another variant of the Cartesian plane, is defined in Exercise 4 below. Some of its properties are the subject of Exercise 4.

EXERCISES 1.3

1. Compute the total curvature of the triangles in Exercise 1.1.2.

2. Compute the total curvature of the triangles in Exercise 1.2.2.

3. A *metric* is a function $f(P, Q)$ of pairs of points such that for any points $P, Q, R,$
   a) $f(P, Q) \geq 0$ and equality holds if and only if P and Q are identical points;
   b) $f(P, Q) = f(Q, P);$
   c) $f(P, Q) + f(Q, R) \geq f(Q, R);$
   Show that the taxicab distance is a metric.

4. The *maxi geometry* is defined on the Cartesian plane by redefining the distance between its points. The *maxi distance* between $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$d_m(P, Q) \quad = \quad \text{Maximum of} \{ |x_1 - x_2| , \ |y_1 - y_2| \} .$$

   In other words, the maxi distance is the larger of the horizontal and vertical Euclidean distances between P and Q. (The straight lines of maxi geometry are, by definition, the Euclidean straight lines, and the taxicab measures of angles are also taken to be identical with their Euclidean measures.)
   a) Show that the maxi distance is a metric.
   b) Show that if P, Q, R are collinear, then Q is between P and R if and only if $d_m(P, Q) + d_m(Q, R) = d_m(P, R).$

5. Determine the taxicab perimeter and curvature of the triangles with the following vertices:
   a) (0, 0), (0, 3), (3, 4); b) (-1, -2), (2, 3), (0, 6).

6. Suppose γ is a polygonal path joining P and Q such that all of its segments have nonnegative slopes. Show that the taxicab length of γ equals $d_t(P, Q)$. Is this also true if all of the segments have nonpositive slopes?

7. Explain why $\int_a^b (1 + |f'|)dx$ is a reasonable formula for the taxicab length of a differentiable curve defined by $y = f(x)$, $a \le x \le b$.

8. Let $y = f(x)$, $a \le x \le b$ be a monotone (either increasing or decreasing) differentiable function. Show that if $P = (a, f(a))$ and $Q = (b, f(b))$ then the taxicab length of the curve defined by f is $d_t(P, Q)$.

9. Explain why no path joining P and Q has taxicab length shorter that $d_t(P, Q)$.

10. Determine the maxi perimeter and total curvature of the triangles with the following vertices:
a) (0, 0), (0, 3), (3, 4); b) (-1, -2), (2, 3), (0, 6).

11. Find a maxi geometry analog for a) Exercise 6; b) Exercise 7; c) Exercise 8; d) Exercise 9.

CHAPTER REVIEW EXERCISES

1. Compute the perimeter of the triangle formed by joining the midpoints of an equilateral triangle all of whose sides have length $a = 1$ in
a) Euclidean geometry; b) spherical geometry; c) hyperbolic geometry.

2. Repeat Exercise 1 for $a = 5$.

3. Repeat Exercise 1 for $a = 10$.

4. Repeat Exercise 1 for $a = 0.1$.

5. Compute the areas of all the triangles in a) Exercise 1; b) Exercise 2; c) Exercise 4.

6. Compute the total curvature of all the triangles in a) Exercise 1; b) Exercise 2; c) Exercise 4.

7. Are the following statements true or false? Justify your answers.
   a) There is a spherical triangle with angles $\pi/2, \pi/3, \pi/6$.
   b) There is a hyperbolic triangle with angles $\pi/2, \pi/3, \pi/6$.
   c) There is a taxicab triangle with angles $\pi/2, \pi/3, \pi/6$.
   d) If two spherical triangles have angles $\pi/2, \pi/2, \pi/2$, then they are congruent.
   e) If two hyperbolic triangles have angles $\pi/4, \pi/4, \pi/4$, then they are congruent.
   f) If two taxicab triangles have angles $\pi/3, \pi/3, \pi/3$, then they are congruent.
   g) Euclidean, spherical, hyperbolic, taxicab, maxi, are all the geometries there are.
   h) Given any two points of spherical geometry, there is a unique geodesic that joins them.
   i) Given any two points of hyperbolic geometry, there is a unique geodesic that joins them.
   j) Given any two points of taxicab geometry, there is a unique geodesic that joins them.
   k) On a sphere of radius 1 there is a triangle of area 4.
   l) In hyperbolic geometry there is a triangle of area 4.

m) Every proposition that is valid in spherical geometry is false in hyperbolic geometry.

n) Every proposition that is valid in hyperbolic geometry is false in spherical geometry.